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Fibonacci Sequences and the Winning Conditions for the Blackout Game

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Abstract

The blackout game(Lightout Game, Merlin Game, σ+Game) is a popular game on a squareboard. When we toggle a button with black or white color, it changes the color of itself and other buttons which have common edges. It is similar to the "Reversi(Othello) Game". With this rule, we can win the game when we have a squareboard with all same colors after some clicks. Here we show that the winning conditions for the general $m \times n$ blackout games are related with the determinant of a block triangular matrix generated by a given blackout game. The Fibonacci sequences are used to get the determinant of the block triangular matrix. We investigate some properties of a generalized Fibonacci sequences with a winnable condition for the blackout game. Also, we introduce a JAVA simulation tool that gives us winnable conditions for an arbitrary given $m \times n$ blackout game.

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1 Introduction

The blackout game has been studied in the literature [1, 11, 12, 13] extensively. When we toggle a square with a black or white color, it changes the color of itself and other buttons. With this rule, we win the game when a squareboard contains same colors after we click some of the buttons. This game of 3×3 squareboard has been popular in the electronic game machines and on the web [10, 17].

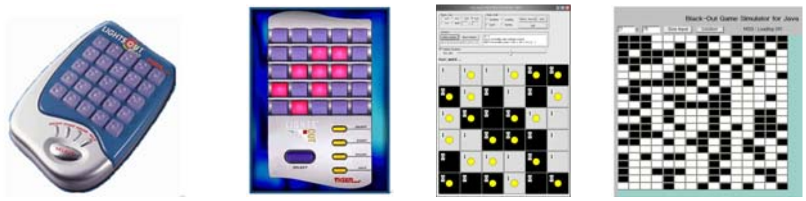


Figure 1: Various forms of the blackout games
http://matrix.skku.ac.kr/bljava/Test.html

The 3×3 game always has a winnable solution. But in general, our $m \times n$ blackout game on the squareboard may not have a winnable solution. The winnable conditions depend on the size of the game and the initial conditions. So, we have to investigate the conditions which makes this game winnable for given m and n. Many attempts have been made to find out the conditions which make the game winnable [1, 2]. We have given a linear algebraic solution for 3×3 and $n \times n$ in [13]. In this paper, we will give a linear algebraic proof for a winnable solution on the general rectangular size $m \times n$ blackout game. Now, we start with a couple of definitions which we use in this paper.

Definition 1.1 *[9] Let us assume $m \times n$ squareboard has squares with black and white colors in it and assign a numbering of each squares in it as follows.*

1	2	3	4	$\cdots$	n
$n+1$	$n+2$	$n+3$	$n+4$	$\cdots$	$2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(m-1)n+1$	$(m-1)n+2$	$(m-1)n+3$	$(m-1)n+4$	$\cdots$	mn

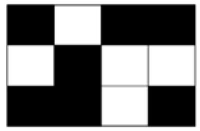
Figure 2: The numbering of the blackout game

Now we can consider a $m \times n$ squareboard as a $m \times n$ $(0,1)$-matrix B and its black square is 0 and its white square is 1. When the initial arbitrary *configuration with colors are given, we call the corresponding matrix B is an Initial Configuration Matrix (ICM).*

For example, the numbering of 3×4 blackout game and an ICM B are given in Figure 3.

1	2	3	4
5	6	7	8
9	10	11	12

,



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = B_{3\times 4}$$

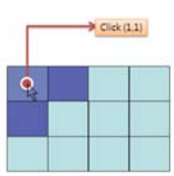
Figure 3: The numbering and an ICM B of size 3×4 blackout game.

When we toggle a button with a black or white color, it changes the colors of itself and other buttons which it shares common edges. This action can be represented by the corresponding $(0,1)$-matrix which is filled with 1 when their colors are changed. Now, we define a matrix that results in the action of clicking a button. When we take an action on a given initial configuration it can be considered as adding a new $(0,1)$-matrix to the ICM in mod 2 addition. We now define a $(0,1)$-matrix of each action by clicking a button as below.

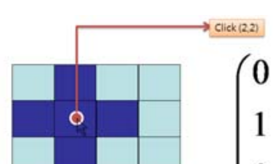
Definition 1.2 *[9] Let $M_k \in M_{m\times n}(R)$ be a $m \times n$ matrix. (where $1 \leq i \leq mn$). M_k is a $(0,1)$-matrix whose k's numbered position, top and bottom positions and left and right positions are 1 and all the rest are 0. i.e. it can be written as remaining $m_{ij} = 0$.*

$$M_k = [m_{ij}]_{m\times n} \text{ where } k = n(i-1)+j$$
$$\text{with } \begin{cases} m_{ij} = m_{i-1,j} = m_{i+1,j} = m_{i,j-1} = m_{i,j+1} = 1 \\ \text{otherwise} \quad m_{ij} = 0. \end{cases}$$

For example, two matrices M_1 and M_6 of the size 3×4 and its corresponding matrix of action at (1,1) position(numbered as 1st) and (2,2) position(numbered as 6th) are given in Figure 4.



$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 4: Resulting matrices of a 3×4 blackout game after a click

The process of playing the blackout game can be considered as an addition of a linear combination of M_k's and the ICM B to make a matrix of all 1's or all 0's. It can be written in the following form to complete the game.

$$B + c_1 M_1 + c_2 M_2 + \cdots + c_{mn} M_{mn} = O \tag{1}$$

or

$$B + c_1 M_1 + c_2 M_2 + \cdots + c_{mn} M_{mn} = J \tag{2}$$

(where O is a zero matrix and J is a matrix with all entries 1.)

To make it simple, we can consider the matrices B, J and O as $mn \times 1$ column vectors. For example, the ICM $B_{3\times 4}$ in Figure 3 can be considered as the following column vector.

$$(0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0) = [0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0]^T \tag{3}$$

This consideration can be taken to all M_k's. We write the matrix M_k as a column vector $\mathbf{m}_k$ and the matrix B as a column vector $\mathbf{b}$ and the matrices J and O as column vectors $\mathbf{j}$ and $\mathbf{0}$ respectively. Then the mathematical modeling of the blackout game can be written as finding a solution for a simple linear system of equations. We give the statement of the following theorem for later uses.

Theorem 1.3 *[9] Let the vectors $\mathbf{m}_k$ be a column vector of the following matrix A.*

$$A = \begin{bmatrix} \mathbf{m}_1 & | & \mathbf{m}_2 & | & \cdots & | & \mathbf{m}_{mn} \end{bmatrix} \tag{4}$$

And the matrix B can be written as a column vector $\mathbf{b}$ and the matrices J and O as column vectors $\mathbf{j}$ and $\mathbf{0}$ respectively. The previous equations (1.1) and (1.2) can be written as

$$A\mathbf{x} = -\mathbf{b} \tag{5}$$

or

$$A\mathbf{x} = \mathbf{j} - \mathbf{b} \tag{6}$$

where $\mathbf{x} = (c_1, c_2, \ldots, c_{mn})$.

Therefore, if we find a solution for the linear system of equations (1.5)(or (1.6)), then we can give a solution for the given blackout game of any size. If we can find a solution of the linear system of equations, we can click only squares which are chosen to make all squares with same color and it means we win this lighout game from a given configuration. In this case, we say this blackout game is winnable and winnable conditions depend on the consistency of the linear system of equations. Now we investigate the properties of the matrix A. In this paper, we analyze the matrix A to give answers for the winnable conditions of the $m \times n$ blackout game.

2 Block Tridiagonal Matrices

Looking at the 3×4 blackout game, we can have twelve 12×1 vectors of $\mathbf{m}_k$ and a 12×12 matrix A whose columns $\mathbf{m}_k$ are made from M_k as following.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{8}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{9}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{10}$$

This matrix A is a block tridiagonal matrix as shown in Figure 5. In this case, we have three tridiagonal matrices of size 4×4 in its main diagonal. Since the game is a blackout game of size 3×4, the number of rows means the number of tridiagonal matrices in its main diagonal and its number of columns means the size of the tridiagonal matrix. Furthermore we generalize our observation to make a general rectangular size $m \times n$ blackout game.

$$\begin{bmatrix} 1&1&0&0&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&1&0&0&0&0&0&0\\ 0&1&1&1&0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&1&0&0&0&0\\ 1&0&0&0&1&1&0&0&1&0&0&0\\ 0&1&0&0&1&1&1&0&0&1&0&0\\ 0&0&1&0&0&1&1&1&0&0&1&0\\ 0&0&0&1&0&0&1&1&0&0&0&1\\ 0&0&0&0&1&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&0&0&1&1&1&0\\ 0&0&0&0&0&0&1&0&0&1&1&1\\ 0&0&0&0&0&0&0&1&0&0&1&1 \end{bmatrix}$$

Figure 5: Block structure of the corresponding matrix for the blackout game of size 3×4

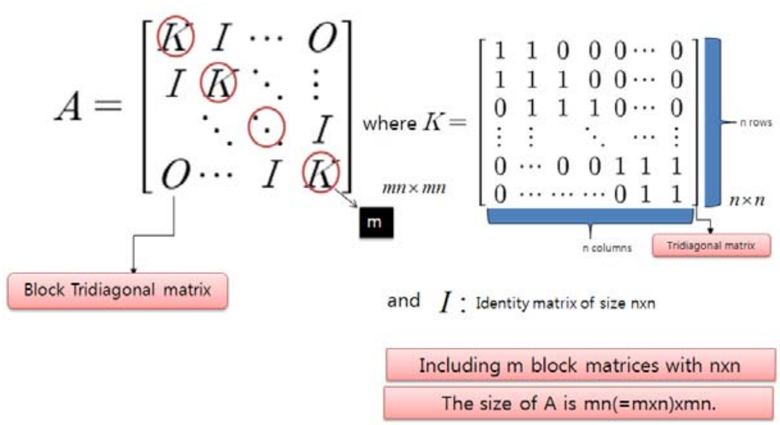


Figure 6: Block structure of the corresponding matrix for the blackout game of size $m \times n$

In general, an analysis of the existence of solutions on the linear system of equation $A\mathbf{x} = -\mathbf{b}$ or $A\mathbf{x} = \mathbf{j} - \mathbf{b}$ is based on the value $\det(A)$. If $\det(A)$ is not zero, each of the system has a unique solution. In the blackout game, this solution means the buttons that we click to make all buttons with the same color. In other word, a basic knowledge of linear algebra is enough to handle not only the 3×3 blackout game, but also the general rectangular size $m \times n$ blackout game.

Here we give the statement of another theorem.

Theorem 2.1 *[10] Suppose the matrix A is a block tridiagonal matrix corresponding to the general rectangular size $m \times n$ blackout game. If* $\det(A) \neq 0$, *the given game is winnable. And the column vector* $\mathbf{b}$ *is from the ICM B of the given initial condition,* $A^{-1}\mathbf{b}$ *gives the unique solution for the general rectangular size $m \times n$ blackout game.*

Even for the case of $\det(A) = 0$, we may find solutions for the general rectangular size $m \times n$ blackout game. But in this case, the existence of the solution depends on the vector $\mathbf{b}$ which is from the initial condition. The analysis on this case was dealt in our previous work [10].

3 Fibonacci sequence and tiling on the blackout game

In the previous section, we used the fact that the block triangular matrix generated by the blackout game has important information which is related with the decision of the winnability for the game. If the determinant of the block tridiagonal matrix is not zero, then there exists a unique solution to win the arbitrary given $m \times n$ blackout game. With these arguments, we gave a list of all the determinants of the block triangular matrices A taken from 1×1 blackout game to 24×24 blackout game in [9].

n

Row \ Col	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
3	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
4	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0
5	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
8	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
9	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	1	0	1	0	0	1	1	0	0
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
14	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0
15	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
19	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	1	0	1	0	0	1	1	0	0
20	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
21	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
24	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0

m

0 : det(A)=0 1 : det(A) is not 0 $A_{mn \times mn}$

Figure 7: Determinants Table from 1×1 blackout game to 24×24 blackout game

As we see in the table in Figure 7, there is an interesting tiling rule which contains two patterns. The first pattern is a tiling by 6×6 blocks that we will see in Figures 8 and 9, and the second pattern is a zero-filling rule at the position $5k-1$. (In Figure 9, some positions are filled with 2 for the distinction. Moreover, all determinant are also zero). In [9], authors mentioned without proofs this tiling can be applied for the large size blackout games. Now we give a mathematical proof for the same.

1	0	1	1	0	1
0	1	0	1	0	1
1	0	1	1	0	1
1	1	1	1	1	1
0	0	0	1	0	1
1	1	1	1	1	1

Figure 8: First pattern : The tiling of 6×6 block

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
3	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
4	1	1	1	2	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2
5	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
8	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
9	1	0	1	2	0	1	1	0	2	1	0	1	1	2	1	1	0	1	2	0	1	1	0	2
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
14	0	1	0	2	0	1	0	1	2	1	0	1	0	2	0	1	0	1	2	1	0	1	0	2
15	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
19	1	0	1	2	0	1	1	0	2	1	0	1	1	2	1	1	0	1	2	0	1	1	0	2
20	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
21	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
24	1	1	1	2	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2

0 : det(A)=0 1 : det(A) is not 0
2 : det(A)=0 and (5k-1)x(5l-1) case

Figure 9: Two tiling patterns in the table of determinant on A

These results are obtained by using the properties of the determinant for the block tridiagonal matrix. From the previous section, the block tridiagonal matrix form a formula given in equation (3.1) where K is a tridiagonal matrix whose position is on the main diagonal entries and I is an identity matrix.

$$A = \begin{bmatrix} K & I & O & \cdots & O \\ I & K & I & \cdots & O \\ O & \ddots & \ddots & \ddots & O \\ O & O & I & K & I \\ O & O & O & I & K \end{bmatrix}_{mn\times mn}, \quad K = \begin{bmatrix} 1 & 1 & O & \cdots & O \\ 1 & 1 & 1 & \cdots & O \\ O & \ddots & \ddots & \ddots & O \\ O & O & 1 & 1 & 1 \\ O & O & O & 1 & 1 \end{bmatrix}_{n\times n} \tag{11}$$

The following theorem gives us a formula to find a determinant of our block tridiagonal matrix.

Theorem 3.1 *[15] Let M be a block tridiagonal matrix as below.*

$$M = \begin{bmatrix} A_1 & B_1 & O & \cdots & O \\ C_1 & A_2 & B_2 & \cdots & O \\ O & \ddots & \ddots & \ddots & O \\ O & O & C_{m-2} & A_{m-1} & B_{m-1} \\ O & O & O & C_{m-1} & A_m \end{bmatrix} \text{ where } A_i, B_i \in M_n(\mathbf{R}). \tag{12}$$

Then the determinant of M can be defined as [1]

$$\det M = (-1)^{mn} \det(T_{11}) \det(B_1 \cdots B_{m-1}) \tag{13}$$

where T_{11} is the upper left block of size $n \times n$ in the following matrix T.

$$T = \begin{bmatrix} -A_m & -C_{m-1} \\ I_n & O \end{bmatrix} \begin{bmatrix} -B_{m-1}^{-1}A_{m-1} & -B_{m-1}^{-1}C_{m-2} \\ I_n & O \end{bmatrix} \cdots \begin{bmatrix} -B_1^{-1}A_1 & -B_1^{-1} \\ I_n & O \end{bmatrix} \tag{14}$$

Using the above theorem we now have what we need.

Theorem 3.2 *Let M be a block tridiagonal matrix as below.*

$$M = \begin{bmatrix} K & I & O & \cdots & O \\ I & K & I & \cdots & O \\ O & \ddots & \ddots & \ddots & O \\ O & O & I & K & I \\ O & O & O & I & K \end{bmatrix} \tag{15}$$

[1] In [15], m and n are changed. In this paper, we described that m and n are changed to fit our definition of the blackout game.

$$\det M = (-1)^{mn} \det(T_{11}) \tag{16}$$

where T_{11} is the upper left block of size $n \times n$ in the following matrix T.

$$T = \begin{bmatrix} -K & -I \\ I & O \end{bmatrix} \begin{bmatrix} -K & -I \\ I & O \end{bmatrix} \cdots \begin{bmatrix} -K & -I \\ I & O \end{bmatrix} \tag{17}$$

Therefore, T is the matrix $\begin{bmatrix} -K & -I \\ I & O \end{bmatrix}^m$.

Proof. In theorem 3.1, let us put $A_i = K, B_i = I, C_i = I$ for each i then, we can get T, T_{11} and $\det(M)$.

The $n \times n$ size upper left blocks of T are getting complicated as m increases. We give values of T_{11} for different values m in Table 1.

m	T_{11}	Sum of absolute values of coefficients
$m = 1$	$-K$	1
$m = 2$	$K^2 - I$	$1 + \lvert -1 \rvert = 2$
$m = 3$	$-K^3 + 2K$	$\lvert -1 \rvert + 2 = 3$
$m = 4$	$K^4 - 3K^2 + I$	$1 + \lvert -3 \rvert + 1 = 5$
$m = 5$	$-K^5 + 4K^3 - 3K$	$\lvert -1 \rvert + 4 + \lvert -3 \rvert = 8$
$m = 6$	$K^6 - 5K^4 + 6K^2 - I$	$1 + \lvert -5 \rvert + 6 + \lvert -1 \rvert = 13$
$m = 7$	$-K^7 + 6K^5 - 10K^3 + 4K$	$\lvert -1 \rvert + 6 + \lvert -10 \rvert + 4 = 21$
$m = 8$	$K^8 - 7K^6 + 15K^4 - 10K^2 + I$	$1 + \lvert -7 \rvert + 15 + \lvert -10 \rvert + 1 = 34$
$\vdots$	$\vdots$	$\vdots$

Table 1: The upper left block of T, i.e. T_{11}

From the above Table 1, we see that the sum of all absolute values of coefficients in each matrix equation from a Fibonacci sequence. It is readily seen that the above results are generated from the block matrix $\begin{bmatrix} -K & -I \\ I & O \end{bmatrix}$. Here we find the Fibonacci sequence and the coefficients from a generalized Pascal's triangle. To find the generating function, we now consider equation (3.8) as below.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -3 & 0 & 4 & 0 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 6 & 0 & -5 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 4 & 0 & -10 & 0 & 6 & 0 & -1 & 0 & \cdots \\ 1 & 0 & -10 & 0 & 15 & 0 & -7 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} 1 \\ k \\ k^2 \\ k^3 \\ k^4 \\ k^5 \\ k^6 \\ k^7 \\ k^8 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ -k \\ k^2-1 \\ 2k-k^3 \\ -3k^2+k^4+1 \\ -3k+4k^3-k^5 \\ 6k^2-5k^4+k^6-1 \\ 4k-10k^3+6k^5-k^7 \\ -10k^2+15k^4-7k^6+k^8+1 \\ \vdots \end{bmatrix}. \tag{18}$$

We see this interpretation has a relationship with the concept of Riordan arrays [16]. Using the concept of the Riordan array, we find A-sequences from the two generating functions $\left(\frac{1}{1+z^2}, \frac{-z}{1+z^2}\right)$ and we have its relationship to the Riordan matrix as below.

$$\left(\frac{1}{1+z^2}, \frac{-z}{1+z^2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -3 & 0 & 4 & 0 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 6 & 0 & -5 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 4 & 0 & -10 & 0 & 6 & 0 & -1 & 0 & \cdots \\ 1 & 0 & -10 & 0 & 15 & 0 & -7 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{19}$$

Therefore the generating function for the general polynomial of the Table 1 can be written as

$$\frac{1}{1+z^2}\frac{1}{1-k\left(\frac{-z}{1+z^2}\right)} = \frac{1}{1+z^2+kz}. \tag{20}$$

We can conclude the general polynomial of the Table 1 is as below.

$$\frac{1}{1+z^2+kz} = 1 - kz + \left(k^2-1\right)z^2 + \left(2k-k^3\right)z^3 + \left(k^4-3k^2+1\right)z^4 + \left(4k^3-3k-k^5\right)z^5 + \cdots. \tag{21}$$

Thus, the existence of the solution for $m \times n$ blackout game depends on the determinant of T_{11} which is the determinant of the coefficient matrix polynomial of z^m in (3.11) where k is replaced by the matrix K in (3.1).

We have known a formula for the determinant of the submatrix K_i.

Theorem 3.3 *[5] Let K_i be a matrix given below*

$$K_i = \begin{bmatrix} 1 & 1 & O & \cdots & O \\ 1 & 1 & 1 & \cdots & O \\ O & \ddots & \ddots & \ddots & O \\ O & O & 1 & 1 & 1 \\ O & O & O & 1 & 1 \end{bmatrix}_{i\times i}. \tag{22}$$

Then K_i can be determined for each i as following.

- $i = 1$: $\det(K_1) = 1$
- $i = 2$: $\det(K_2) = 1 - 1 = 0$
- $\vdots$
- $i = t$: $\det(K_t) = \det(K_{t-1}) - \det(K_{t-2})$

From the above theorem, the determinant of the matrix K_i can be listed as in Figure 10. This shows a pattern by a group of six numbers $\{1, 0, -1, -1, 0, 1\}$ and this pattern is repeated. Any sequence $b_n = b_{n-1} - b_{n-2}$ can be written as $b_n = a_n b_0 + a_{n-1}(b_1 - b_0)$ where a_t is the determinant of the $t \times t$ matrix K with $m(i,j) = 1$ if $|i-j| \geq 1$ and 0 otherwise as we see in [3]. This sequence can be found using A010892 in [18].

So, this result agrees with the first pattern of 6×6 blocks which was shown in Figures 8 and 9.

Finally, we have developed a simulation tool for this $m \times n$ blackout game. This tool uses our result and algorithm, it can determine that the given blackout game is winnable or not and it shows us a (green) solution when the given game is winnable. This tool is made with JAVA language, we can check and use this tool in the following URL. (Figure 11)

t	det (K)	t	det (K)	t	det (K)
1	1	7	1	13	1
2	0	8	0	14	0
3	−1	9	−1	15	−1
4	−1	10	−1	16	−1
5	0	11	0	17	0
6	1	12	1	18	1

t	det (K)	t	det (K)	t	det (K)
19	1	25	1	31	1
20	0	26	0	32	0
21	−1	27	−1	33	−1
22	−1	28	−1	34	−1
23	0	29	0	35	0
24	1	30	1	36	1

. . .

Figure 10: The determinant of K_i

http://matrix.skku.ac.kr/bljava/Test.html

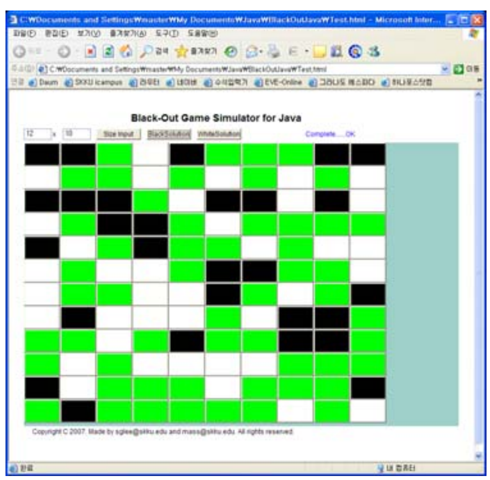


Figure 11: The simulation tool for the $m \times n$ blackout game

4 Conclusion

We have shown that the winnable conditions of the blackout game have a direct relation with the determinant of a block tridiagonal matrix. And the determinant of the block tridiagonal matrix can be solved by the properties of the generalized Fibonacci sequence. The blackout game which uses the linear algebraic modeling have some useful meanings in finding problems for the determinant of the block tridiagonal matrix.

The block tridiagonal matrix and its determinant has played an important role for analyzing $m \times n$ blackout game. A generalized Fibonacci sequence and Riordan array were also used to find out the determinant of the block tridiagonal matrix.

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