Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 31, 1533 - 1540

Fixed Point Theorems of Multivalued Mappings in Cone Metric Spaces

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Abstract

Let P be a subset of a Banach space E and P is normal and regular cone on E, we prove the existence of the fixed point for multivalued maps in cone metric spaces and these theorems generalize the recent results of varies authors,

Mathematics Subject Classification: Primary 47H10; Secondary 54H25

Keywords: Cone metric space, multivalued mappings, Fixed point

1 Introduction and preliminaries:

In recent years, several authors(see[1-5]) have studied the strong convergence to a fixed point with contractive constant in cone metric spaces. Seong Hoon Cho and Mi Sun Kim [5]have proved certain fixed point theorems by using Multivalued mapping in the setting of contractive constant in metric spaces. We first recall definitions and known results that are needed in the sequel. Let E be a Banach space and a subset P of E is said to be a cone if it satisfies the following conditions,

(i) $P \neq \emptyset$ and P is closed;

(ii) $ax + by \in P$ for all $x, y \in P$ and a,b are non-negative real numbers;

(iii) $P \cap (-P) = \emptyset$.

The partial ordering $\leq$ with respect to the cone P by $x \leq y$ if and only if $y - x \in$ P. If $y - x \in$ interior of P, then it is denoted by $x \ll y$.The cone P is said to be a Normal if a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The cone P is called regular if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

Definition 1.1. *Let X be a non-empty set, and suppose the mapping $d : X \times X \longrightarrow E$ is said to be a Cone metric space if it satisfies*

(i) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$.

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(iii) $d(x, y) = d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Example 1.2. *Let* $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}, X = R$ *and* $d : X \times X \longrightarrow E$ *defined by*

$$d(x, y) = (| x - y |, \alpha | x - y |)$$

where $\alpha \geq 0$ *is a constant. Then* (X, d) *is a cone metric space[1].*

Definition 1.3. *Let (X ,d) be a cone metric space,* $x \in X$ *and* $\{x_n\}$ *a sequence in X. Then*

(i) $\{x_n\}$ *converges to* x *whenever for every* $c \in E$ *with* $0 \ll c$ *there is a natural number* N *such that* $d(x_n, x) \ll c$ *for all* $n \geq N$.

(ii) $\{x_n\}$ *is a cauchy sequence whenever for every* $c \in E$ *with* $0 \ll c$ *there is a natural number* N *such that* $d(x_n, x_m) \ll c$ *for all* $n, m \geq N$.

Definition 1.4. *Let* (X, d) *is said to be a complete cone metric space, if every cauchy sequence is convergent in* X.

Let (X, d) be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subset of X. Let $H(., .)$ be the Hausdorff distance on $CB(X)$. That is, for $A, B \in CB(X)$,

$$H(A, B) = max\{sup_{a \in A} d(a, B), sup_{b \in B} d(A, b)\}$$

where $d(a, B) = inf\{d(a, b) : b \in B\}$ is the distance from the point a to the subset B. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in T(X)$.

2 Main results

Theorem 2.1. *Let (X, d) be a complete cone metric space and the mapping $T : \mathbf{X} \to \mathbf{CB}(\mathbf{X})$ be multivalued map satisfying for each $x, y \in X$, $H(Tx,Ty) \leq a[d(x,Tx) + d(y,Ty)] + b[d(x,Ty) + d(Tx,y)]$ for all $x, y \in X$, and $a + b < \frac{1}{2}$, $a, b \in [0, \frac{1}{2})$. Then T has a fixed point in X.*

Proof. For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\
&\leq a[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] + b[d(x_n, Tx_{n-1}) + d(Tx_n, x_{n-1})] \\
&\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_n, x_n) + d(x_{n+1}, x_{n-1})] \\
&\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_{n+1}, x_{n-1})] \\
&\leq a[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + b[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\
&\leq (a + b)[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\
d(x_{n+1}, x_n) &\leq Ld(x_n, x_{n-1}) \qquad where \quad L = \frac{(a + b)}{(1 - (a + b))} \\
d(x_{n+1}, x_n) &\leq L^n d(x_1, x_0)
\end{aligned}$$

For $n > m$ we have

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\
&\leq [L^{n-1} + L^{n-2} + ... + L^m]d(x_1, x_0) \\
&\leq \frac{L^m}{(1 - L)} d(x_1, x_0)
\end{aligned}$$

Let $0 \ll c$ be given, choose a natural number N_1 such that $\frac{L^m}{(1-L)} d(x_1, x_0) \ll c$ for all $m \geq N_1$ this implies $d(x_n, x_m) \ll c$. For $n > m$, $\{x_n\}$ is a cauchy sequence in (X, d) is a complete cone metric space, there exists $p \in$X such that $x_n \to p$. Choose a natural number N_2 such that $d(x_n, p) \ll \frac{c(1-L)}{3}$, for all

$n \geq N_2$. Hence for $n \geq N_2$ we have $d(x_n, P) \ll \frac{c(1-k)}{3}$ where $k = a + b$

$$\begin{aligned}
d(Tp, p) &\leq H(Tx_n, Tp) + d(Tx_n, p) \\
&\leq a[d(x_n, Tx_n) + d(p, Tp)] + b[d(x_n, Tp) + d(Tx_n, p)] + d(x_{n+1}, p) \\
&\leq a[d(x_n, x_{n+1}) + d(p, Tp)] + b[d(x_n, Tp) + d(x_{n+1}, p)] + d(x_{n+1}, p) \\
&\leq a[d(x_n, x_{n+1}) + d(p, Tp)] + b[d(x_n, p) + d(p, Tp) + d(x_{n+1}, p)] + d(x_{n+1}, p) \\
(1-k)d(Tp, p) &\leq kd(x_n, p) + kd(x_{n+1}, p) + d(x_{n+1}, p) \\
&\leq d(x_n, p) + d(x_{n+1}, p) + d(x_{n+1}, p) \\
d(Tp, p) &\leq \frac{[d(x_n, p) + d(x_{n+1}, p) + d(x_{n+1}, p)]}{(1-k)} \\
d(Tp, p) &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \\
d(Tp, p) &\ll c
\end{aligned}$$

for all $n \geq N_2$, $d(Tp, p) \ll \frac{c}{m}$ for all $m \geq 1$, we get $\frac{c}{m} - d(Tp, p) \in P$, and as $m \to \infty$ we get $\frac{c}{m} \to 0$ and P is closed $-d(Tp, p) \in P$, but $d(Tp, p) \in P$, $\therefore d(Tp, p) = 0$. and so $p \in Tp$. ∎

Corollary 2.1. *Let (X, d) be a complete cone metric space and the mapping $T : \mathbf{X} \to \mathbf{CB}(\mathbf{X})$ be multivalued map satisfies condition*

$$d(Tx, Ty) \leq a(d(Tx, y) + d(x, Ty))$$

for all $x, y \in X$, where $a \in [0, \frac{1}{2})$ is a constant. Then T has a fixed point in X.

Proof. The proof of the corollary immediately follows by putting $b = 0$ in the previous theorem. ∎

Theorem 2.2. *Let (X, d) be a complete cone metric space and the mapping $T : \mathbf{X} \to \mathbf{CB}(\mathbf{X})$ be multivalued map satisfy the condition*

$$H(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all $x, y \in X$ and $r \in [0, 1)$. Then T has a fixed point in X.

Proof. For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\
&\leq r \max[d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})] \\
&\leq r \max[d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)] \\
&\leq rd(x_{n-1}, x_n) \\
&\leq r^n d(x_1, x_0)
\end{aligned}$$

For $n > m$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\ &\leq [r^{n-1} + r^{n-2} + ... + r^m] d(x_1, x_0) \\ &\leq \frac{r^m}{(1-r)} d(x_1, x_0) \end{aligned}$$

Let $0 \ll c$ be given, choose a natural number N_1 such that $\frac{r^m}{(1-r)} d(x_1, x_0) \ll c$ for all $m \geq N_1$ this implies $d(x_n, x_m) \ll c$. For $n > m$, $\{x_n\}$ is a cauchy sequence in (X, d) is a complete cone metric space, there exists $p \in X$ such that $x_n \to p$. Choose a natural number N_2 such that $d(x_n, p) \ll \frac{c}{3}$, for all $n \geq N_2$. Hence for $n \geq N_2$ we have $d(x_n, P) \ll \frac{c}{3}$

$$\begin{aligned} d(Tp, p) &\leq H(Tx_n, Tp) + d(Tx_n, p) \\ &\leq r \max[d(x_n, p), d(x_n, Tx_n), d(p, Tp)] + d(x_{n+1}, p) \\ &\leq r \max[d(x_n, p), d(x_n, x_{n+1}), d(p, Tp)] + d(x_{n+1}, p) \\ &\leq r \max[d(x_n, p), d(x_n, p) + d(p, x_{n+1}), d(p, Tp)] + d(x_{n+1}, p) \\ d(Tp, p) &\ll c \end{aligned}$$

for all $n \geq N_2$, $d(Tp, p) \ll \frac{c}{m}$ for all $m \geq 1$, we get $\frac{c}{m} - d(Tp, p) \in P$, and as $m \to \infty$ we get $\frac{c}{m} \to 0$ and P is closed $-d(Tp, p) \in P$, but $d(Tp, p) \in P$ $\Rightarrow d(Tp, p) = 0$, and so $p \in Tp$. ■

Corollary 2.2. *Let (X, d) be a complete cone metric space and the mapping $T : \mathbf{X} \to \mathbf{CB}(\mathbf{X})$ be multivalued mapping satisfy the condition*

$$H(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ where $k \in [0, 1)$ is a constant. Then T has a fixed point in X.

Proof. The proof of the corollary immediately follows by taking $d(x, y)$ as maximum value in the previous theorem ■

Note 2.3. *We prove the above theorems in the setting of P is a normal cone with normal constat K.*

Theorem 2.4. *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose the mapping $T : \mathbf{X} \to \mathbf{CB}(\mathbf{X})$ be multivalued mapping satisfying the condition*

$$H(Tx, Ty) \leq r \max[d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)]$$

for all $x, y \in X$, and $r \in [0, 1)$. Then T has a fixed point in X.

Proof. For every $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\
&\leq r \max[d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1}), d(Tx_n, x_{n-1})] \\
&\leq r \max[d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n+1}, x_{n-1})] \\
&\leq r \max[d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n-1})] \\
&\leq r \max[d(x_n, x_{n-1}), d(x_{n+1}, x_{n-1})]
\end{aligned}$$

Case(i) If $d(x_{n+1}, x_n) \leq rd(x_n, x_{n-1})$ then we get,$d(x_{n+1}, x_n) \leq r^n d(x_1, x_0)$. For $n > m$

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\
&\leq [r^{n-1} + r^{n-2} + ... + r^m] d(x_1, x_0) \\
&\leq \frac{r^m}{(1-r)} d(x_1, x_0)
\end{aligned}$$

We get $\|d(x_n, x_m)\| \leq K \frac{r^m}{(1-r)} \|d(x_1, x_0)\|$. $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a cauchy sequence. By the completeness of X, there is $p \in X$ such that $x_n \to p$ as $n \to \infty$.

$$\begin{aligned}
d(Tp, p) &\leq H(Tx_n, Tp) + d(Tx_n, p) \\
&\leq r \max[d(x_n, p), d(x_n, Tx_n), d(p, Tp), d(x_n, Tp), d(Tx_n, p)] + d(x_{n+1}, p) \\
&\leq r \max[d(x_n, p), d(x_n, x_{n+1}), d(p, Tp), d(x_n, Tp), d(x_{n+1}, p)] + d(x_{n+1}, p) \\
&\leq rd(p, Tp) \\
d(Tp, p) &= 0 \quad Hence \quad p \in Tp.
\end{aligned}$$

Case(ii) If $d(x_{n+1}, x_n) \leq rd(x_{n+1}, x_{n-1})$ then we get

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq r[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\
d(x_{n+1}, x_n) &\leq \frac{r}{1-r}[d(x_n, x_{n-1})] \\
d(x_{n+1}, x_n) &\leq [d(x_n, x_{n-1})] \quad where \quad h = \frac{r}{1-r} < 1.
\end{aligned}$$

For $n > m$

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\
&\leq [h^{n-1} + h^{n-2} + ... + h^m] d(x_1, x_0) \\
&\leq \frac{h^m}{(1-h)} d(x_1, x_0)
\end{aligned}$$

We get $\|d(x_n, x_m)\| \leq K\frac{h^m}{(1-h)}\|d(x_1, x_0)\|$. $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Hence $\{x_n\}$ is a cauchy sequence. By the completeness of X, there is $p \in X$ such that $x_n \to p$ as $n \to \infty$.

$$\begin{aligned} d(Tp,p) &\leq H(Tx_n,Tp) + d(Tx_n,p) \\ &\leq r\max[d(x_n,p), d(x_n,Tx_n), d(p,Tp), d(x_n,Tp), d(Tx_n,p)] + d(x_{n+1},p) \\ &\leq r\max[d(x_n,p), d(x_n,x_{n+1}), d(p,Tp), d(x_n,Tp), d(x_{n+1},p)] + d(x_{n+1},p) \\ &\leq rd(p,Tp) \\ d(Tp,p) &= 0 \quad Hence \quad p \in Tp. \end{aligned}$$

$$\begin{aligned} d(p,q) &\leq H(Tp,Tq) \\ &\leq r\max[d(p,q), d(p,Tp), d(q,Tq), d(p,Tq), d(Tp,q)] \\ &\leq r\max[d(p,q), d(p,p), d(q,q), d(p,q), d(p,q)] \\ &\leq r[d(p,q)] \end{aligned}$$

This is contradiction and hence T has a unique fixed point in X. ∎

Corollary 2.3. *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K. Suppose the mapping $T : \mathbf{X} \to \mathbf{CB(X)}$ be multivalued mapping satisfies the condition*

$$H(Tx,Ty) \leq r\max[d(x,y), d(x,Tx), d(y,Ty)]$$

for all $x, y \in X$, and $r \in [0, 1)$. Then T has a fixed point in X.

Proof. The proof of the corollary immediately follows since

$$\max[d(x,y), d(x,Tx), d(y,Ty)] \leq \max[d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(Tx,y)]$$

∎

Theorem 2.5. *Let (X, d) be a cone metric space and let S be the class of functions $\alpha : \mathbf{R}^+ \to [\mathbf{0}, \mathbf{1})$ satisfies $\alpha(t_n) \to 1 \implies t_n \to 0$ and T be a multivalued map on X with Tx is nonempty closed subset of X and a regular cone, for each $x \in X$*

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$$

for each $x, y \in X$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $n \geq 1$, $x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$.

$$\begin{aligned} d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\ &\leq \alpha(d(x_n, x_{n-1}))d(x_n, x_{n-1}) \end{aligned}$$

If $d(x_n, x_{n-1}) = 0$ then $\{d(x_{n+1}, x_n)\}$ is a monotonically decreasing and bounded below , as Tx is regular we have $\{d(x_{n+1}, x_n)\}$ is convergent. And if $d(x_n, x_{n-1}) > 0$ then $\frac{d(x_{n+1},x_n)}{d(x_n,x_{n-1})} \leq \alpha(d(x_n, x_{n-1})) \to 1$ as $n \to \infty$ and $\alpha \in S$, then we get $d(x_{n+1}, x_n)$ is a monotonically decreasing and bounded below , as Tx is regular we have $\{d(x_{n+1}, x_n)\}$ is convergent.

$$\begin{aligned} d(x_{n+1}, Tp) &\leq H(Tx_n, Tp) \\ &\leq \alpha(d(x_n, p))d(x_n, p) \end{aligned}$$

As $n \to \infty$ we have $d(Tp, p) = 0$ (or) $p \in T_p$ ■

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Received: November, 2009