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Roman Bondage and Roman Reinforcement Numbers of a Graphs

Karam Ebadi

Department of Studies in Mathematics
University of Mysore, Manasagangotri
Mysore-5700 06, Karnataka, India
Karam_Ebadi@yahoo.com

L. PushpaLatha

Department of Mathematics
Yuvaraja's College, Mysore, India
pushpakrishna@yahoo.com

Abstract

A Roman dominating function on a graph $G = (E, V)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G. In this paper, we initiate the study of Roman bondage and Roman reinforcement numbers in a graph. The Roman bondage number b_R for a graph G is the cardinality of the smallest number of edges $F \subset E(G)$ such that $\gamma_R(G - F) > \gamma_R(G)$. The Roman reinforcement number r_R for a graph G is the cardinality of the smallest number of edges $F \subset E(\bar{G})$ such that $\gamma_R(G + F) < \gamma_R(G)$. In this paper, exact values of b_R and r_R are found for some classes of graphs and sharp bounds are found for trees. Some general bounds are also given.

Keywords: Roman domination, bondage number, Roman bondage numbers

1 Introduction

In the past few years several papers have addressed the question of how the domination parameter of a graph is affected by edge or vertex addition or deletion. This is significant since dominating vertices can be thought of as transmitters serving a vast array of communication sites. The failure of some links may render the set of transmitters to be nondominating, ie, communications to one or more sites may be snapped. In particular, suppose that a saboteur does not know which sites in the network act as transmitters, but does know that the set of such sites corresponds to a minimum dominating set in the related graph. What is the fewest number of links he must sever, so that at least one additional transmitter would be required in order that communications with all sites be possible? With this in mind, Fink et al. [7] initiated the study of bondage number for a graph G, where the bondage number $b(G)$ was defined to be the cardinality of the smallest number of edges $F \subset E(G)$ such that $\gamma(G-F) > \gamma(G)$, where $\gamma(G)$ denoted the domination number of G. In [7], sharp bounds were obtained for $b(G)$ and exact values were determined for several classes of graphs. In [8], Hartnell and Rall give other bounds for $b(G)$ and disprove a conjecture made in [7]. Also Hartnell and Rall [9] characterize trees whose bondage number is 2. To study how edge addition affected the domination number of a graph, reinforcement numbers was introduced by Mynhardt and Kok [10]. The reinforcement $r(G)$ is the cardinality of the smallest number of edges $F \subset E(\bar{G})$ such that $\gamma(G-F) < \gamma(G)$. If $\gamma(G) = 1$ then define $r(G) = 0$.

Other authors have studied the change to the domination number of a graph under edge (vertex) removal or addition. Domination alteration sets in the case where the vertices or edges were removed was first studied by Bauer, Harary, Neiminen and Suffel [2]. Vertex domination critical graphs, ie, graphs for which $\gamma(G-v) < \gamma(G)$ for every vertex v of G, was investigated by Brigham, Chinn and Dutton in [3]. Further in [11], Sumner and Blitch studied domination critical graphs; graphs G for which $\gamma(G-e) < \gamma(G)$ for every edge $e \in E(\bar{G})$. Graphs G for which $\gamma(G-e) > \gamma(G)$ for every edge $e \in E(G)$ were considered by Acharya and Walikar in [1].

Similar problems related to the addition or deletion of vertices and edges and their effect on other graphical parameters have been studied. (See [3] and [5]).

The concept of Roman dominating function (RDF) was introduced by E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi and S. T. Hedetniem in [4]. (See also [6]). A Roman dominating function on a graph $G = (E, V)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G. We define the Roman bondage number b_R of G to be the cardinality of the smallest number of edges $F \subset E(G)$ such that $\gamma_R(G-F) > \gamma_R(G)$. Similar to the reinforcement number is the concept of the Roman reinforcement number r_R which is the minimum cardinality of a set of edges $F \subset E(\bar{G})$ such that $\gamma_R(G+F) < \gamma_R(G)$. If $\gamma_R(G) = 2$ then we define $r_R(G) = 0$. This characteristic of Roman dominating function is not exhibited by the ordinary dominating sets and hence the problem of Roman bondage for a graph is considerably harder than that of bondage.

2 Exact values for some Classes of Graphs

We begin our investigation of these parameters by making a few observations, which facilitate the calculation of $b_R(G)$ and $r_R(G)$ with respect to $b(G)$ and $r(G)$. Moreover, to find b_R and r_R for graphs G, we need to find the Roman domination number γ_R for them. To facilitate that endeavor, we first make a few Observations.

Proposition 1. [4] For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Observation 2. If G is a graph with $\gamma(G) = \gamma_R(G)$, then $b_R(G) \leq b(G)$ and $r(G) \leq r_R(G)$.

Proof. Suppose F is the set of edges of minimum cardinality such that $\gamma(G-F) > \gamma(G)$, ie, $|F| = b(G)$. We know that $\gamma(G) \leq \gamma_R(G)$ from Proposition 1. By assumption, $\gamma_R(G) = \gamma(G) < \gamma(G-F) \leq \gamma_R(G-F)$. Hence, $b_R(G) \leq b(G)$. Similarly consider the smallest number of edges $F \subset E(\bar{G})$ such that $\gamma_R(G-F) < \gamma_R(G)$. Then it can be checked that adding F to the edge set of G also decreases $\gamma(G)$. $\square$

Observation 3. Let f be any $\gamma_R(G)$ -function and let G have at least two vertices says u and v, such that $f(u) = 2$ and $f(v) = 1$, then $r_R(G) = 1$.

Proof. Let G' be obtained from G by joining v to u. Define $g : V \longrightarrow \{0, 1, 2\}$ by $g(v) = 0$ and $g(z) = f(z)$ for each $z \in V \backslash \{v\}$. Obviously, g is a RDF of G' with $w(g) < w(f)$. Hence $\gamma_R(G) = 1$. $\square$

Let $v \in S \subseteq V$. Vertex u is called a private neighbour of v with respect to S (denoted by u is an $S-pn$ of v) if $u \in N[v] - N[S-v]$. An $S-pn$ of v is external if it is a vertex of $V-S$. The set $pn(v, S) = N[v] - N[S-v]$ of all S-pns of v is called the private neighbourhood set of v with respect to S. The set S is said to be irredundant if for every $v \in S$, $pn(v, S) \neq \phi$.

Observation 4. Let G be a graph of order $n \geq 3$ and $e = uv \in E(\bar{G})$. If G' is obtained from G by joining u to v, then $\gamma_R(G') \leq \gamma_R(G)$.

Proof. Let f be a $\gamma_R(G)$-function. If $f(u) = 1$ and $f(v) = 1$ or $f(u) = 0$ and $f(v) = 0$ or $f(u) = 0$ and $f(v) = 1$ or $f(u) = 1$ and $f(v) = 0$, it is trivial that $\gamma_R(G') = \gamma_R(G)$. Let $f(u) = 2$ and $f(v) = 2$, if $|pn(u\, or\, v, V_2)| = 2$, then $\gamma_R(G') \leq \gamma_R(G)$, if $|pn(u\, or\, v, V_2)| > 2$, it is easy to see that $\gamma_R(G') = \gamma_R(G)$. Let $f(u) = 2$ and $f(v) = 1$. Then by Observation [3], $\gamma_R(G') \leq \gamma_R(G)$. In the last case, suppose $f(u) = 2$ and $f(v) = 0$ or conversely. Since $f(v) = 0$ and $uv \notin E(G)$, there exists $w \in V_2$ such that $f(w) = 2$ and w dominates v, if $|pn(w, V_2)| = 2$. Define $g : V \longrightarrow \{0, 1, 2\}$ by $g(w) = 1$ and $g(z) = f(z)$

for each $z \in V\backslash\{w\}$. Obviously, g is a RDF of G and $w(g) < w(f)$. If $|pn(w, V_2)| > 2$, it is easy to see that $\gamma_R(G') = \gamma_R(G)$. This completes the proof. □

Observation 5. [4] Let $f = (V_0, V_1, V_2)$ be any γ_R -function. Then V_2 is a γ-set of $H = G[V_0 \cup V_2]$.

Proposition 6. If G is a graph with $\gamma_R(G) \neq \gamma(G)$, then $r_R(G) \leq r(G)$.

Proof. If $V_1 \neq \varnothing$, then by Observation 3 $\gamma_R(G) = 1$ hence $r_R(G) \leq r(G)$. Let $V_1 = \varnothing$, then V_2 is a γ-set in a graph G by Observation 5, if $V_2 = \{v\}$, then by definition $r_R(G) = 0$. Let $v \in V_2$ and $pn(v, V_2)$ is a minimum cardinality set say F, then by joining all vertices of F to another vertex of V_2 say u, we get $\gamma(G - F) < \gamma(G)$ hence $r(G) = |F|$. Since each $w \in V_2$ has at least two private neighbours in $G[V_0 \cup V_2]$, let $v \neq x \in F$ so by joining all vertices of F except x, define $g = (W_0, W_1, W_2)$ by $W_0 = (V_0 \cup \{v\}) - \{x\}$, $W_1 = \{x\}$ and $W_2 = V_2 - \{v\}$. It follows that

$g(V) = 1 + 2n_2 - 2 = 2n_2 - 1 < f(V) = \gamma_R(G)$. Hence $r_R(G) = |F| - 1 < r(G)$. □

We begin our study of Roman bondage by considering for a graph G, if there exists t vertices $t \geq 1$ with degree $n - 1$. Note $\lceil x \rceil$ denotes the ceiling function of the number x.

Theorem 7. Given a graph G on $n \geq 3$ vertices , if there exists t vertices. $t \geq 1$, each with degree $n-1$, then $b_R(G) = \lceil t/2 \rceil$ and the Roman reinforcement number $r_R(G) = 0$.

Proof. Let $u_1, u_2, ..., u_t$ be the t vertices of degree $n-1$. Then clearly removal of fewer than $t/2$ edges results in a graph H having maximum degree $n - 1$. Hence $b_R(G) \geq \lceil t/2 \rceil$. Now we consider the following cases.

Case 1. If t is even, then the removal of $t/2$ independent edges $u_1u_2, u_3u_4, ..., u_{t-1}u_t$ results in a graph H' regular of degree $n - 2$. Hence $b_R(G) = t/2$.

Case 2. If t is odd, then the removal of $(t - 1)/2$ independent edges $u_1u_2, u_3u_4, ..., u_{t-2}u_{t-1}$ yields a graph H'' containing exactly one vertex u_t of degree $n - 1$. Thus by removing an edge incident with u_t, we obtain a graph H''' with maximum degree $n - 2$. Hence $b_R(G) = (t - 1)/2 + 1$.

Hence from cases (1) and (2) it follows that $b_R(G) = \lceil t/2 \rceil$. □

Corollary 1. For the complete graph K_n with $n \geq 3$ vertices, $b_R(K_n) = \lceil n/2 \rceil$.

Proof. By Theorem 7, $b_R(K_n) = \lceil n/2 \rceil$, since $t = n$. □

Corollary 2. For any wheel W_n with $n \geq 5$ vertices, $b_R(W_n) = 1$.

Proof. Clearly W_n contains exactly one vertex of degree $n - 1$. Hence by Theorem 7, $b_R(W_n) = \lceil 1/2 \rceil = 1$. □

We next consider paths P_n and cycles C_n on n vertices. By following the proof techniques in [7] along with the Theorem above, we find that b_R and r_R values for these graphs are the same as $b(G)$ values. Hence we omit the proof and observe.

Observation 8. [4] For paths P_n and cycles C_n,

$$\gamma_R(P_n) = \gamma_R(C_n) = \lceil 2n/3 \rceil.$$

Observation 9. For paths P_n and cycles C_n with $n \geq 3$

$$b_R(P_n) = \begin{cases} 2 & \text{if } n \equiv 2 \text{ (mod3)}, \\ 1 & \text{otherwise.} \end{cases}$$

$$b_R(C_n) = \begin{cases} 3 & \text{if } n \equiv 2 \text{ (mod3)}, \\ 2 & \text{otherwise.} \end{cases}$$

and

$$r_R(P_n) = r_R(C_n) = \begin{cases} 2 & \text{if } n \equiv 2 \text{ (mod3)}, \\ 1 & \text{otherwise.} \end{cases}$$

For the class of complete multipartite graphs $K_{m_1,\dots,m_n}$ there are three cases to consider.

Proposition 10. Let $G = K_{m_1,\dots,m_n}$ be the complete n-partite graph with $m_1 \leq m_2 \leq \dots \leq m_n$.

$$\gamma_R(G) = \begin{cases} 2 & \text{if } m_1 = 1, \\ 3 & \text{if } m_1 = 2, \\ 4 & \text{if } m_1 \geq 3. \end{cases}$$

Theorem 11.

$$r_R(G = K_{m_1,\dots,m_n}) = \begin{cases} 0 & \text{if } m_1 = 1, \\ 1 & \text{if } m_1 = 2, \\ m_1 - 2 & \text{if } m_1 \geq 3. \end{cases}$$

Proof. If $m_1 = 1$, then $r_R = 0$ by definition. In the case where $m_1 = 2$, by joining all vertices in the first partite set, γ_R for the resulting graph decreases to 2. In the third case $m_1 \geq 3$, we know that $\gamma_R = 4$. To make γ_R equal 3, we add $m_1 - 2$ edges in the first partite set as follows, let x and y be any vertices in the first partite set. Add all edges of the from (x, u) where u is a vertex in the first partite set other than x and y thereby making x adjacent to every vertex in the graph except y. Hence γ_R for the resulting graph decreases to 3. □

The analysis for $b_R(G)$ is considerably harder for multipartite sets and so we consider the complete bipartite graph $K_{m,n}$ where $1 \leq m \leq n$.

Theorem 12.

$$b_R(K_{m,n}) = \begin{cases} 1 & \text{if } m = 1 \text{ and } n \neq 1, \\ 5 & \text{if } m = n = 3, \\ m & \text{otherwise.} \end{cases}$$

Proof. Let $V = W_1 \cup W_2$ be the vertex set of $K_{m,n}$ where $|W_1| = m$,$|W_2| = n$ and $v \in W_2$. Let $m = 1$. By Theorem 7, $b_R(K_{m,n}) = 1$. If $m = n = 3$ and $u \in W_1$, then by removing all edges incident with u, we obtain a graph H containing two components K_1 and $K_{2,3}$. Hence

$\gamma_R(H) = \gamma_R(K_1) + \gamma_R(K_{2,3}) = 1 + 3 = \gamma_R(K_{3,3})$

To increase the Roman domination number, we need to delete the two edges adjacent to a vertex in the second partite set in $K_{2,3}$.

In the third case we first show that $b_R = 2$ if $n = m = 2$. For then we have a cycle on 4 vertices the Roman bondage of which is 2 from Observation 9. If $m \leq n$ and $n \neq 3$, delete m edges incident at a vertex in the second partite set to increase the Roman domination number. This completes the proof. □

3 Trees

We now consider the Roman bondage and Roman reinforcement numbers for a tree T. Define a support to be a vertex in a tree which is adjacent to an end-vertex.

Observation 13. Every tree T with $(n \geq 4)$ has at least one of the following conditions.

1. A support adjacent to at least 2 end-vertices.
2. A support adjacent to a support of degree 2.
3. A vertex adjacent to two supports of degree 2.
4. The support of a leaf and the vertex adjacent to the support are both of degree 2.

Proof. Root any tree T with more than 3 vertices at a root r and let l be an end-vertex farthest from r. Let s be the support of l. If s has degree greater than 2, then it must be adjacent to at least two end-vertices, or else we get Case 1. If s has degree 2 then consider the other vertex x adjacent to it. If degree $x = 2$, then we get Case 1. If degree of x is greater than 2, then it may have an end-vertex adjacent to it, achieving Case 2 or else it may be adjacent to a support of degree 2, thereby demonstrating Case 3. All other cases yield contradiction to the assumption that l is farthest from r. These cases are exhaustive. □

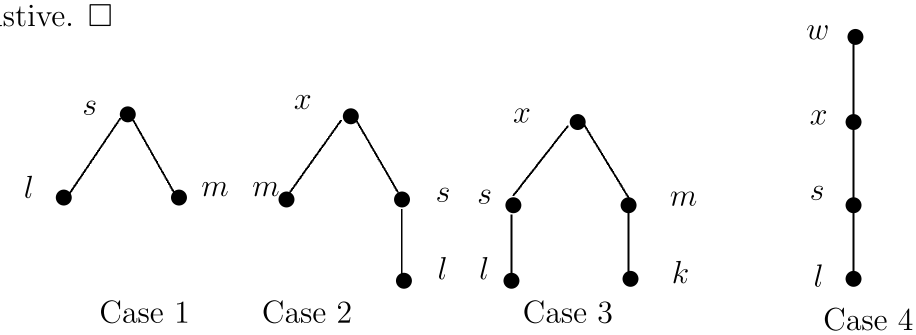

Figure 1

Theorem 14. $b_R(T) \leq 3$, for any tree T with $n \geq 3$.
Proof. First assume f is a $\gamma_R(T)$-function. If the tree has 3 or 4 vertices then it can be checked that $b_R(T) = 1$. So, consider a tree T with $|V(T)| \geq 5$. We want to show that removal of at most 3 edges is sufficient to increase the value of γ_R for any tree. We use Observation 13 to prove this result.

Case 1. Let l and m be end-vertices adjacent to s. We now consider two subcases.

Subcase 1. Suppose $deg\, s = 3$. First, let $f(s) = 2$ and let T' be obtained from T by removing $sm \in E(T)$. Define $g : V \longrightarrow \{0, 1, 2\}$ by $g(m) = 1$, if $g(z) = f(z)$ for each $z \in V \backslash \{m\}$. Obviously, g is a RDF of T' and if $w(g) > w(f)$, we are done. If $w(g) = w(f)$, then there exists w such that $g(w) = 2$ and adjacent to s. Then remove $ws \in E(T')$ to obtain T''. Define $g' : V \longrightarrow \{0, 1, 2\}$ by $g'(l) = 2$ and $g'(z) = g(z)$ for each $z \in V \backslash \{l\}$, obviously g' is a RDF of T'' for which $\gamma_R(G) \leq w(g) < w(g')$. Thus $b_R(T) \leq 2$.

Subcase 2. Suppose $deg\, s > 3$ and $e = sl \in E(T)$. Let T' is obtained from T by removing e. Define $g : V \longrightarrow \{0, 1, 2\}$ by $g(l) = 1$ and $g(z) = f(z)$ for each $z \in V \backslash \{l\}$. Obviously, g is a RDF of T' and $w(g) > w(f)$, (Since s is adjacent to at least 3 end-vertices). Thus $b_R(T) = 1$.

Case 2. Let a support x be adjacent to a support s of degree 2. We consider two subcases.

Subcase 1. Suppose $deg\, x = 3$ and first, let $f(x) = 2$. Let T' is obtained from T by removing $xs \in E(T)$. Define $g : V \longrightarrow \{0, 1, 2\}$ by $g(l) = 2$, if $g(z) = f(z)$ for each $z \in V \backslash \{l\}$, then if $w(g) > w(f)$, we are done. If not, then there exists $w \in V(T')$ adjacent to x such that $g(w) = 2$. Let T'' be obtained from T' by removing xw. Define $g' : V \longrightarrow \{0, 1, 2\}$ by $g'(m) = 2$ and $g(z) = g'(z)$ for each $z \in V \backslash \{m\}$. Obviously, g' is a RDF of T'' and $w(g') > w(g) \geq w(f)$. Thus $b_R(T) \leq 2$.

Subcase 2. Suppose $degx \geq 4$, then $f(x) = 2$. Let T' is obtained from T by removing mx. Define $g : V \longrightarrow \{0, 1, 2\}$ by $g(m) = 1$ and $g(z) = f(z)$ for each $z \in V \backslash \{m\}$. Obviously, g is a RDF of T' and $w(g) > w(f)$. In this case also $b_R(T) \leq 2$.

Case 3. Let a support x be adjacent to two supports say s and m such that $degs = degm = 2$. Let $deg\, x = 3$ and let T' be obtained from T by removing the edges xs, xm. Let g be a $\gamma_R(T')$-function. First let $g(x) = 1$. Then obviously g is a RDF of T' for which $w(f) < w(g)$. Let $g(x) = 0$, so there exists $w \in V(T')$ such that $g(w) = 2$ and adjacent to x. By removing xw, define $g' : V \longrightarrow \{0, 1, 2\}$ by $g'(x) = 1$ and $g'(z) = g(z)$ for each $z \in V \backslash \{x\}$. Obviously, g' is a RDF of T' for which $w(f) \leq w(g) < w(g')$. In this case $b_R(T) \leq 3$. If $deg\, x > 3$, then it is easy to see that $b_R(T) = 1$.

Case 4. In this case $f(x) = 2$ or $f(s) = 2$. First let $f(x) = 2$ and let T' be obtained from T by removing $xs \in E(T)$. Define $g : V \longrightarrow \{0, 1, 2\}$ by $g(l) = 2$ and $g(z) = f(z)$ for each $z \in V \backslash \{l\}$. Obviously, g is a RDF of T' for which $w(f) \leq w(g)$ (If $g(x) \leq 1$, then define $g : V \longrightarrow \{0,1,2\}$ by $g(s) = 2$, $g(l) = 0$ and $g(z) = f(z)$ for each $z \in V\backslash\{s,l\}$. Obviously, $w(g) < w(f)$, a contradiction). Let $f(s) = 2$ and T' be obtained from T by removing xw, xs. Define $g : V \longrightarrow \{0,1,2\}$ by $g(x) = 1$ and $g(z) = f(z)$ for each $z \in V\backslash\{x\}$. Obviously, $w(g) < w(f)$. This completes the proof. □

We observe that $b_R(T)$ can equal 3 only in the third case. If any of the other three cases are present in the tree T, then $b_R(T) \leq 2$. But some trees satisfying Case 3 and having no other case present in them have b_R equal to 3 while some others have 2. We give below some trees illustrating this behavior. It can be checked that the graph in Figure 2(A) has b_R equal to three. But deleting e and f in Figure 2(B) makes the RDF increase. Also in Figure 2(C), deleting the edge k is sufficient for increasing γ_R. The characterization of trees with $b_R = 3$ remains open.

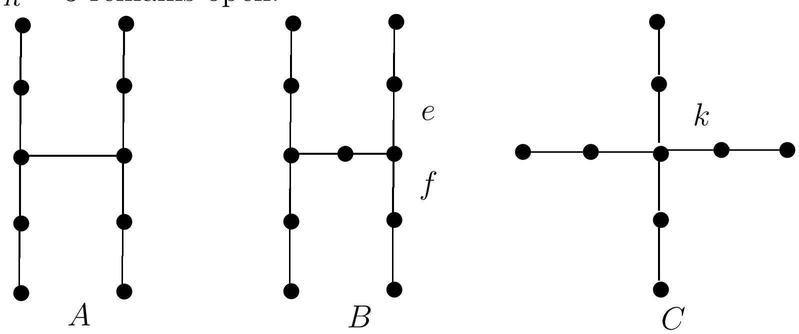

Figure 2: Tree satisfying Case 3 with $b_R = 2$ or 3

For arbitrary graphs G and H, we define the Cartesian product of G and H to be the graph $G\Box H$ with vertices $\{(u,v)|u \in G, v \in H\}$. Two vertices (u_1,v_1) and (u_2,v_2) are adjacent in $G\Box H$ if and only if one of the following is true; $u_1 = u_2$ and v_1 is adjacent to v_2 in H; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G. If $G = P_m$ and $H = P_n$, then the Cartesian product $G\Box H$ is called the $m \times n$ grid graph and is denoted $G_{m,n}$.

Proposition 15. [4] For the $2 \times n$ grid graph $G_{2,n}$, $\gamma_R(G_{2,n}) = n + 1$.

Theorem 16. For the $2 \times n$ grid graph $G_{2,n}$, $b_R(G_{2,n}) = 2$.

Proof. Note that if $f = (V_0,V_1,V_2)$ is an RDF for $G_{2,n}$, then any vertex in V_2 can dominate at most four vertices, while a vertex in V_1 can dominate only one. If $n \geq 2$ the removal of two edges from $G_{2,n}$ leaves a graph H consisting of two graphs K_2 and $G_{2,n-1}$. By proposition [15], $\gamma_R(G_{2,n}) = n + 1$. Then $\gamma_R(H) = \gamma_R(K_2) + \gamma_R(G_{2,n-1}) = 2 + (n-1) + 1 = 1 + (n+1) > \gamma_R(G_{2,n})$. Hence $b_R(G_{2,n}) = 2$. □

Theorem 17. For the $2 \times n$ grid graph $G_{2,n}$, $r_R(G_{2,n}) \leq 3$.

Proof. Let $f = (V_0,V_1,V_2)$ be any γ_R-function. By proposition [15], $\gamma_R(G_{2,n}) = n + 1$ and any vertex in V_2 can dominate at most four vertices, if $|V_1| \neq 0$, then by Observation 3, $r_R(G) = 1$. Let $|V_1| = 0$ and $v \in V_2$, since any vertex in V_2 can dominate at most four vertices, by joining every vertex of $N(v)$ to another vertex of V_2, define $g = V \longrightarrow \{0,1,2\}$ by $g(v) = 1$ and $g(z) = f(z)$ for each $z \in V\backslash\{v\}$. Obviously, $w(g) < w(f)$. Hence $r_R(G_{2,n}) \leq 3$. □

Proposition 18. Let $f = (V_0, V_1, V_2)$ be a γ_R-function of a connected graph G. Let $uv, vw \in E(G)$ such that $f(u) = 2$, $f(v) = 0$ and $f(w) = 1$, then $b_R(G) \leq min\{deg\, u + deg\, v - 2\}$.

Proof. Let $e = uv$ and $uv, vw \in E(G)$ and $deg\, u + deg\, v$ is minimum with $f(u) = 2$, $f(v) = 0$ and $f(w) = 1$ (v and w are adjacent). Let $N(u) \cap V_2 = \{u_1, u_2, ..., u_t\}$ and $X = N(u)\backslash\{u_1, u_2, ..., u_t\} = \{v_1, v_2, ..., v_s\}$. Suppose G' is obtained from G by removing all edges incident to u and v except e. Let g be a $\gamma_R(G')$-function defined by $g(z) = f(z)$ for each $z \in V\backslash\{v_1, v_2, ..., v_s\}$. Then $\sum_{v_i \in X} g(v_i) > \sum_{v_i \in X} f(v_i)$, since there exists at least one $u \neq v \neq v_k \in pn(V_2, u)$. Otherwise, define $g : V \longrightarrow \{0,1,2\}$ by $g(v) = 2$, $g(u) = g(w) = 0$ and $g(z) = f(z)$ for each $z \in V\backslash\{v, u, w\}$, a contradiction. Hence $b_R(G) \leq min\{deg\, u + deg\, v - 2\}$. □

Proposition 19. i) If H is a subgraph of G such that $|H| \geq 3$, then $b_R(H) \leq b_R(G)$.

ii) If H is a subgraph of G such that $b_R(H) = 1$ and k is the number of edges removed to form H, then $1 \leq b_R(G) \leq k + 1$.

4 General Bounds

Theorem 20. For any graph G, if $|V_1| = 0$ and $|V_2| \geq 2$, then $r_R(G) \leq min\{deg\, u - |E| \mid u \in V_2\}$.

Where E is the set of edges incident with u from the vertices of V_2.

Proof. Suppose $deg\, u = |E| + |E'|$, where E is the set of edges incident with u from the vertices of V_2 and E' be the set of edges incident with u from the vertices of V_0, let $E = \varnothing$. Thus by joining every vertex of $N(u)$ to some other vertex of V_2 say w, define $g : V \longrightarrow \{0,1,2\}$ by $g(u) = 1$ and $g(z) = f(z)$ for each $z \in V\backslash\{u\}$. Hence $r_R(G) \leq min\{deg\, u \mid u \in V_2\}$.

let $E \neq \varnothing$ and $X = N(u)\backslash E = \{u_1, u_2, ..., u_t\}$. Let G' be obtained from G by joining every vertex of $X\backslash\{u_k\}$, $(1 \leq k \leq t)$ to some other vertex of V_2 say w. Define $g : V \longrightarrow \{0,1,2\}$ by $g(u_k) = 2$, $g(u) = 0$ and $g(z) = f(z)$ for each $z \in V\backslash\{u_k, u\}$. Hence $r_R(G) \leq min\{deg\, u - |E| \mid u \in V_2\}$. □

Corollary 3. For any graph G,

$r_R(G) \leq \Delta(G)$.

Proof. This follows from Theorem 20. □

Corollary 4. For any graph G,

$r_R(G) \leq n - 2$.

Proof. Since $\Delta(G) \leq n - 2$, by Theorem 21, $r_R(G) \leq n - 2$. □

Theorem 21. For any graph G with order n, $r_R(G) \leq n - 1 - \Delta(G)$.

Proof. If $\Delta(G) = n - 1$, then by definition $r_R(G) = 0$, suppose $\Delta(G) < n - 1$ and v be the vertex such that $deg\, v = \Delta(G)$. Since $\Delta(G) < n - 1$, $\gamma_R(G) > 2$.

Then adding $p - \Delta(G) - 1$ edges to the vertex v, v will be adjacent to every vertex and this will reduce $\gamma_R(G)$. Since r_R is the smallest number of edges that achieves this, the result holds. □

Theorem 22. If G is any graph, then $r_R(G) = n - 1 - \Delta(G)$ if and only if $r_R(G) = 3$.

Proof. Let $f = (V_0, V_1, V_2)$ be any γ_R-function and $\gamma_R(G) = 3$, then the only way to reduce γ_R, is to reduce it to 2. This can be accomplished in a minimum way by making the max degree vertex to be adjacent to all the vertices, hence $r_R(G) = n-1-\Delta(G)$. To show the other direction, we show the contrapositive; of $\gamma_R(G) = 2$ or $\gamma_R(G) \geq 4$ then $r_R(G) < n - 1 - \Delta(G)$. If $\gamma_R(G) = 2$, then there exists a vertex of full degree whence $\Delta(G) = n - 1$ and so $r_R(G) = 0$. Suppose G is a graph where $\gamma_R(G) \geq 4$. Consider a set $V_1 \cup V_2$ with ≥ 2 vertices. First, let $|V_2| = 0$, then it is easy to see that $r_R(G) < n - 1 - \Delta(G)$. Let $|V_2| \neq 0$, then $|V_1| = 0$ or $|V_1| \neq 0$. Let $|V_1| \neq 0$, then by Observation 3 $r_R(G) = 1 < n - 1 - \Delta(G)$ (since $\gamma_R(G) \geq 4$ and $|V_1| \neq 0$, then $V_1 \cup V_2| \geq 3$). Let $|V_1| = 0$, Since $\gamma_R(G) \geq 4$, $|V_2| \geq 2$. Order these vertices in increasing order according to their degrees. Let $s_n \in V_2$ have the maximum degree among these vertices. Add in the edges F from s_n to the vertices adjacent to s_{n-1} except $s_{n-1} \neq x \in pn(s_{n-1}, V_2)$, which s_{n-1} dominates. Define $g : V \longrightarrow \{0, 1, 2\}$ by $g(s_{n-1}) = 0$, $g(x) = 1$ and $g(z) = f(z)$ for each $z \in V \setminus \{s_{n-1}, x\}$. Obviously, $w(g) < w(f)$. Thus $\gamma_R(G - F) < \gamma_R(G)$. Specifically s_n does not dominate x, thus $deg\ (s_n) < n - 1$. Hence $|F| < n - 1 - \Delta(G)$. □

Theorem 23. If G is any graph, then $r_R(G) = n - 2 - \Delta(G)$ if and only if $r_R(G) = 4$.

Proof. By following the proof techniques in Theorem 22. □

Theorem 24. $r_R(G) + \lfloor \gamma_R(G)/2 \rfloor \leq n - \Delta(G)$, for any graph G with order n.

Proof. We prove the result $\gamma_R(G) \leq 2n - 2\Delta(G) - 2r_R(G)$ for any graph G. If there exists a vertex with degree $n-1$, then $r_R = 0$ and $\gamma_R(G) = 2$, and the bounds hold. Now suppose $\gamma_R(G) \geq 3$, Add $r_R - 1$ edges to the vertex with maximum degree. Call that set F. This is possible since $\Delta(G) \leq n-1-r_R(G)$, from the previous result. Then $\gamma_R(G + F) = \gamma_R(G)$. And by Proposition 1, $\gamma_R(G) = \gamma_R(G+F) \leq 2\gamma(G+F) \leq 2n-2\Delta(G+F) = 2n-2\Delta(G)-2r_R-2$. This proves the result. □

Corollary 5. $r_R(G) \leq n - \Delta(G) - \lfloor \gamma_R(G)/2 \rfloor$, for any graph G.

We strongly believe the following to be true.

Conjecture 26. $b_R(G) \leq n - 1$, for any graph G.

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