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Which Topological Spaces X are Determined by the Positive Unit Elements of C(X)

A. R. Aliabad and M. Motamedi

Dept. Math. Chamran University, Ahvaz, Iran
aliabady_r@scu.ac.ir, motamedi_m@scu.ac.ir

Abstract. Let $C(X)$ be the ring of real valued continuous functions on a Tychonoff space X and $U^+(X)$ the set of all positive units of $C(X)$. Supposing X and Y are compact connected topological spaces, we prove that there exists an ordered group isomorphism $\varphi : U^+(X) \longrightarrow U^+(Y)$ which is identity on $\mathbb{R}^+$ if and only if $X \simeq Y$.

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1 Introduction

Throughout this paper, all topological spaces X that we consider are Tychonoff and $C(X)$ ($C^*(X)$) stands for the ring of continuous (bounded continuous) real valued functions on a topological space X. Supposing $f \in C(X)$, we denote the set $f^{-1}\{0\}$ by $Z(f)$, its complement by $Coz(f)$ and the collection of all zero-sets in X by $Z(X)$. Whenever I is an ideal in $C(X)$, then we put $Z(I) = \{Z(f) : f \in I\}$ and $\bigcap Z(I) = \bigcap_{f\in I} Z(f)$. If $\cap Z(I)$ is nonempty, I is said to be fixed; else, free. The fixed maximal ideals are the sets $M^p(X) = \{f \in C(X) : p \in Z(f)\}$ where $p \in X$, and free maximal ideals of $C(X)$ are of the form $M^p(X) = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ where βX is the Stone-Čech compactification of X and $p \in \beta X \setminus X$.

A topological space X is said to be a P-space (F-space) if $O^p(X) = \{f \in C(X) : \ p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$ is a maximal ideal (prime ideal) in $C(X)$ for every $p \in \beta X$. In addition, X is called an almost P-space if $\mathrm{int}_X Z(f) \neq \varnothing$ for every

$f \in C(X)$. For undefined terms and notions, see [9].

We denote the group of units of a ring R by $U(R)$. Suppose that G is a group, then by $H \leq G$ we mean that H is a subgroup of G, $Max(G)$ denotes the set of all maximal subgroup of G and $T(G)$ denotes the torsion subgroup of G. For the sake of simplicity, $U(C(X))$ and $T(U(C(X)))$ will be denoted by $U(X)$ and $T(X)$, respectively. The set $\{f \in C(X) : \ f(x) > 0, \ \forall x \in X\}$ is denoted by $U^+(X)$. Clearly $U^+(X)$ is a subgroup of $U(X)$. Supposing $f \in U(X)$, we denote the set $f^{-1}\{1\}$ by $e(f)$ and its complement by $Coe(f)$. One can easily see that $\{e(f) : \ f \in U(X)\} = Z(X)$.

Suppose that X is a partially ordered set and $a, b \in X$, then $\sup\{a, b\}$ ($\inf\{a, b\}$), if it exists, is denoted by $a \vee b$ ($a \wedge b$). If X is a partially ordered group, then we have $a \wedge b = (a^{-1} \vee b^{-1})^{-1}$ and $a \vee b = (a^{-1} \wedge b^{-1})^{-1}$ (if either sides exists). A lattice group (ring) is a partially ordered group (ring) in which $a \vee b$ exists for every $a, b \in X$; clearly, in this case, $a \wedge b$ exists for every $a, b \in X$. Also, if G is a lattice group, then for every $a, b, c \in G$ we have $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$, $a(b \wedge c) = ab \wedge ac$ and $(b \wedge c)a = ba \wedge ca$, see [6]. If f, g are two real functions on a set X, then by $f \ll g$ we mean $f(x) < g(x)$ for every $x \in X$. In this paper, all groups are assumed to be abelian.

In Section 2, we obtain some general facts about $U^+(X)$ as a subgroup of $U(X)$. In Sections 3, we introduce and study some concepts which help us to obtain the following main result.

"Suppose that X and Y are compact connected topological spaces. There exists an ordered group isomorphism $\varphi : \ U^+(X) \longrightarrow U^+(Y)$ which is identity on $\mathbb{R}^+$ if and only if $X \simeq Y$."

2 Preliminary results

In the following proposition we are summarize some results concerning the paper that are proved in [2]. Recall that the Frattini subgroup of a group G is the intersection of all maximal subgroup of G and denoted by $\Phi(G)$.

Proposition 2.1. Let X be a topological space, then

(a) For every topological space X we have $|U^+(X)| = |U(X)| = |C(X)| = |C^*(X)|$;

(b) $T(X) = \{f \in C(X) : \ f^2 = 1\}$ and it is a subgroup of $U(X)$;

(c) $U(X)$ is the direct product of $U^+(X)$ and $T(X)$;

(d) If $K \in Max(U(X))$, then $U^+(X) \subseteq K$;

(e) $K \in Max(U(X))$ if and only if there exists $H \in Max(T(X))$ such that $K = U^+(X)H$;

(f) $U^+(X)$ is a maximal subgroup of $U(X)$ if and only if $T(X) = \{-1, 1\}$, equivalently X is connected;

(g) If $K \leq U(X)$, then $K \in Max(U(X))$ if and only if $|U(X)/K| = 2$;

(h) $\Phi(T(X)) = \{1\}$;

(i) $\Phi(U(X)) = U^+(X)$.

The following remark that is also applied in [2] is useful for the next section and helps us to find an example of two compact topological spaces X and Y such that $U(X) \simeq U(Y)$ but $X \not\simeq Y$.

Remark 2.2. $U^+(X)$ $(T(X))$ is a $\mathbb{Q}$-vector space ($\mathbb{Z}_2$-vector space) via $(c, f) \longrightarrow f^c$. Clearly, $\varphi : U^+(X) \longrightarrow Y^+(Y)$ $(\varphi : T(X) \longrightarrow T(Y))$ is a group homomorphism if and only if it is a vector space homomorphism. Let V be a vector space over a field F and S be a base for V. If $|F|$ and $|S|$ are finite, then $V \simeq F^{|S|}$ and so $|V| = |F|^{|S|}$. Also if $|F|$ or $|S|$ is infinite, then $|V| = \max\{|F|, |S|\}$. Therefore, if V, W are F-vector spaces and $|V| = |W|$, then $V \simeq W$ whenever one of the following holds.

(a) F is finite.

(b) $|F| < |V|$.

Example 2.3. Even if $U(X) \simeq U(Y)$ and X, Y are compact spaces, then one cannot conclude that $X \simeq Y$. To see this, suppose that $X = [0, 1]$ and $Y = \mathbb{R}^*$, where $\mathbb{R}^*$ is the one point compactification of $\mathbb{R}$. Clearly, $|U^+(X)| = |U^+(Y)| = \aleph_1$ and hence by Remark 2.2, $U^+(X) \simeq U^+(Y)$ as $\mathbb{Q}$-vector spaces and hence as abelian groups. On the other hand the connectedness of X and Y implies $T(X) \simeq T(Y) \simeq \{-1, 1\}$. Therefore, by part (c) of Proposition 2.1, $U(X) \simeq U(Y)$ while $X \not\simeq Y$.

Difinition 2.4. Let $(G, +)$ is an abelian group. G is called a divisible group if for every $a \in G$ and every $n \in \mathbb{N}$ there exists $b \in G$ such that $nb = a$. Also, G is reduced if G has not any divisible subgroup.

It is well known that if $(G, +)$ is an abelian group, then there exists a largest divisible subgroup D and a reduced subgroup E of G such that $G = D \oplus E$, see [10]. Clearly, D is unique and so we denote it by D_G. We denote by E_G the set of all complement sumand of D_G. One can easily see that every two elements of E_G are isomorphic.

Proposition 2.5. Let X be a topological space. Then the following statements hold.

(a) $D_{U(X)} = U^+(X)$.

(b) $E_{U(X)} = \{T(X)\}$.

Proof. (a). Clearly, $U^+(X)$ is a divisible subgroup of $U(X)$. Now, suppose that H is a divisible subgroup of G and $f \in H$. Hence there exists $g \in H$ such that $g^2 = f$ and consequently $f \in U^+(X)$.

(b). It is obvious that $T(X) \in E_{U(X)}$. Now, suppose that $E \in E_{U(X)}$. Since $E \simeq T(X)$, every element of E is of order 2. Thus, $E \subseteq T(X)$ and it follows that $E = T(X)$.

The Souslin number of an infinite topological space X is denoted by $s(X)$ and is the smallest cardinal number $\mathbf{m}$ such that for every family $\mathcal{U}$ of mutually disjoint non empty open subsets of X we have $|\mathcal{U}| \leq \mathbf{m}$, see [7].

Proposition 2.6. $s(X) \leq \dim(U^+(X))$ for every topological space X.

Proof. Suppose that $\mathcal{U}$ is a family of mutually disjoint non empty open subsets of X. It is enough to prove that $|\mathcal{U}| \leq \dim(U^+(X))$. For every $U \in \mathcal{U}$ pick $x_U \in U$ and define $f_U : X \longrightarrow [1, 2]$ such that $f_U(x_U) = 2$ and $U^c \subseteq e(f_U)$. To complete the proof it suffices to show that the set $\{f_U : U \in \mathcal{U}\}$ is linearly independent. Suppose that $f_{U_1}^{r_1} \cdots f_{U_k}^{r_k} = 1$ where $r_1, \cdots, r_k \in \mathbb{Q}$. Then for every $1 \leq i \leq k$ we have

$$\forall i = 1, \cdots, k, \quad (f_{U_1}^{r_1} \cdots f_{U_k}^{r_k})(x_{U_i}) = f_{U_i}^{r_i}(x_{U_i}) = 2^{r_i} = 1$$

$$\therefore \quad \forall i = 1, \cdots, k, \quad r_i = 0.$$

In what follows we show that both the strict inequality and the equality can occur in the relation $s(X) \leq \dim(U^+(X))$. For let $X = \mathbb{R}$, then $s(\mathbb{R}) = \aleph_0 < \aleph_1 = \dim_{\mathbb{Q}} U^+(\mathbb{R})$. Now, assume that $|X| = \aleph_1$, $a \in X$ is the unique non isolated point of X and every neighborhood of a is co-countabe. Then one can easily see that $s(X) = \aleph_1 = |C(X)| = |U^+(X)| = \dim(U^+(X))$.

The following remark is in order.

Remark 2.7. We recall that if M is an R-module, then the Goldie dimension of M, denoted by $G\text{-}\dim M_R$, is the least cardinal which is greater or equal to the cardinality of every family of independent nonzero submodule of M. Considering $U^+(X)$ as a $\mathbb{Z}$-module, we see that $G\text{-}\dim(U^+(X)) = \dim_{\mathbb{Q}}(U^+(X))$. Therefore, the previous proposition can be restated as follows,

$$\text{“}s(X) \leq G - \dim(U^+(X)) \text{ for every topological space } X\text{”},$$

see [4] and [5] for more details.

3 When a topological space X is determined by positive unit elements of $C(X)$?

In this section, we deal only with $U^+(X)$ and try, as our main aim, to charactrize topological space X via $U^+(X)$. First of all, by the following example, we want to mention that if X, Y are two compact spaces such that X is connected and $U^+(X) \simeq U^+(Y)$, then one cannot even conclude that Y is connected. Therefore, it is reasonable to seek a stronger condition to charactrize X via $U^+(X)$.

Example 3.1. Let $X = [0,1]$ and $Y = \mathbb{N}^*$, where N^* is one point compactification of $\mathbb{N}$. It is clear that $|U^+(X)| = |U^+(Y)| = \aleph_1$ and thus, by Remark 2.2, $U^+(X) \simeq U^+(Y)$. Obviously, X is connected but Y is not.

Definition 3.2. Suppose that $H \leq G$, we let $\sqrt{H} = \{a \in G : \ a^n \in H \text{ for some } n \in \mathbb{N}\}$. It is clear that $\sqrt{H}$ is a subgroup of G containing H. Suppose that H is a subgroup of a group G and $n \in \mathbb{N}$, we let $H^n = \{x^n : \ x \in H\}$. Recall that H is a pure subgroup of G if and only if $H \cap G^n = H^n$ for all $n \in \mathbb{N}$.

Lemma 3.3. Let $H \leq U^+(X)$, then the following statements are equivalent.
(a) H is a pure subgroup of $U^+(X)$.
(b) $\sqrt{H} = H$.
(c) H is a $\mathbb{Q}$-subspace of $U^+(X)$.
(d) $\mathbb{Q}_f = \{f^r : \ r \in \mathbb{Q}\} \subseteq H$ for every $f \in H$.
(e) H is a direct summand of $U^+(X)$.
Proof. It is clear. ■

Definition 3.4. $U^+(X)$ with the natural ordering is a lattice group, since for every $f, g, h \in U^+(X)$ if $f \leq g$, then $fh \leq gh$ (we note that $U(X)$ is a group and a sublattice of $C(X)$ but is not a lattice group). Indeed, the positive part of this lattice group is $P_{U^+(X)} = \{f \in U^+(X) : \ f \geq 1\}$. For every $f \in U^+(X)$ we define

$$\|f\|_1 = f \vee \frac{1}{f} \quad , \quad f_+ = f \vee 1 \quad , \quad f_- = \frac{1}{f} \vee 1.$$

Moreover, we denote $\{x \in X : \ f(x) > 1\}$ and $\{x \in X : \ f(x) < 1\}$ by $pose(f)$ and $nege(f)$, respectively.

It can be easily seen that a group homomorphism $\varphi : U^+(X) \longrightarrow U^+(Y)$ is an ordered group homomorphism if and only if $\varphi(P_{U^+(X)}) \subseteq P_{U^+(Y)}$.

Proposition 3.5. The followings hold for any topological space X.
(a) For every $f \in U^+(X)$ we have

$$f = \frac{f_+}{f_-} \quad , \quad \|f\|_1 = f_+ f_-.$$

(b) $\|\frac{1}{f}\|_1 = \|f\|_1$ for every $f \in U^+(X)$.
(c) $\|fg\|_1 \leq \|f\|_1 \ \|g\|_1$ for every $f, g \in U^+(X)$.
(d) $\|fg\|_1 = \|f\|_1 \ \|g\|_1$ if and only if $pose(f) \cap nege(g) = pose(g) \cap nege(f) = \varnothing$.

(e) $\|f^r\|_1 = \|f\|_1^r$ for every $f \in U^+(X)$ and for every $r \in \mathbb{R}^+$.

(f) $e(f_+) \cap e(f_-) = e(\|f\|_1) = e(f)$ and $e(f_+) \cup e(f_-) = X$ for every $f \in U^+(X)$.

(g) $e(f) \cap e(g) = e(\|f\|_1) \cap e(\|g\|_1) = e(\|f\|_1\ \|g\|_1)$ for every $f, g \in U^+(X)$.

Proof. Since $U^+(X)$ is a lattice group, (a), (b) and (c) are clear.

(d $\Rightarrow$). Assume on the contrary that there exists $x \in pose(g) \cap nege(f)$. Then $f(x) < \frac{1}{f(x)}$ and $\frac{1}{f(x)g^2(x)} < \frac{1}{f(x)}$. Thus, $f(x) \vee \frac{1}{f(x)g^2(x)} < \frac{1}{f(x)}$ and hence we can write

$$f(x)g(x) \vee \frac{1}{f(x)g(x)} < g(x)\frac{1}{f(x)} = \|g\|_1(x)\ \|f\|_1(x)$$

$$\Rightarrow \quad \|fg\|_1(x) < \|f\|_1(x)\ \|g\|_1(x).$$

Therefore, $\|fg\|_1 \neq |f\|_1\ \|g\|_1$.

(d $\Leftarrow$). We have to show that $\|fg\|_1(x) = \|f\|_1(x)\|g\|_1(x)$ for every $x \in X$. To this end, let $x \in X$. Obviously, if $f(x) = 1$, then the statement is true, so let $f(x) > 1$. By assumption we have $g(x) \geq 1$ and it can be easily seen that the statement also holds in this case. The case $f(x) < 1$ will be proved similarly.

(e), (f) and (g) are obvious. ■

Suppose that H is a subgroup of $U^+(X)$ and $\mathcal{F}$ a z-filter on X. Naturally, we may define $e(H) = \{e(f) : f \in H\}$ and $e^{-1}(\mathcal{F}) = \{f \in U^+(X) : e(f) \in \mathcal{F}\}$. Clearly, $ee^{-1}(\mathcal{F}) = \mathcal{F}$; $H \subseteq e^{-1}e(H)$ and the equality may not hold here.

Proposition 3.6. If $H \leq U^+(X)$, then the following statements hold.

(a) Let $e^{-1}e(H)$ be a subgroup of $U^+(X)$. Then $e^{-1}e(H)$ is a proper subgroup of $U^+(X)$ if and only if $\varnothing \notin e(H)$.

(b) If $\mathcal{F}$ is a z-filter, then $e^{-1}(\mathcal{F})$ is a proper pure subgroup of $U^+(X)$.

(c) If $H \subseteq U^+(X)$, then $e^{-1}e(H)$ is a proper subgroup of $U^+(X)$ if and only if $e(H)$ is a z-filter on X.

Proof. (a). Suppose that $\varnothing \in e(H)$. Let $f \in U^+(X)$, we put $g = \frac{1}{2} \wedge \frac{f}{2}$, then $0 \leq g \leq \frac{1}{2}$ and $0 \ll g \ll f$. Hence $e(g) = \varnothing$ and $e(\frac{f}{g}) = \varnothing$. Thus $f = g\frac{f}{g} \in e^{-1}e(H)$. Therefore, $e^{-1}e(H) = U^+(X)$. The converse is trivial.

(b). Let $f, g \in e^{-1}(\mathcal{F})$, hence $e(f^r), e(g^s) \in \mathcal{F}$ for every $r, s \in \mathbb{Q}$ and since $e(f^r) \cap e(g^s) \subseteq e(f^r g^s)$, $f^r g^s \in e^{-1}(\mathcal{F})$.

(c). Suppose that $e^{-1}e(H)$ is a proper subgroup of $U^+(X)$. Then by part (a), $\varnothing \notin e(H)$. Let $f, g \in e^{-1}e(H)$, thus $\|f\|_1\ \|g\|_1 \in e^{-1}e(H)$ and hence $e(f) \cap e(g) = e(\|f\|_1) \cap e(\|g\|_1) = e(\|f\|_1\ \|g\|_1) \in e(H)$. Now let $f \in e^{-1}e(H)$ and $e(f) \subseteq e(g)$. It is clear that $e(\|f\|_1\ \|g\|_1) = e(\|f\|_1) \cap e(\|g\|_1) = e(\|f\|_1)$ and so $\|f\|_1\ \|g\|_1 \in e^{-1}e(H)$. Thus, $\|g\|_1 = \frac{\|f\|_1\ \|g\|_1}{\|f\|_1} \in e^{-1}e(H)$ and therefore $g \in e^{-1}e(H)$ i.e., $e(g) \in e(H)$. By part (b), the converse is clear. ■

Definition 3.7. We say a subgroup H of $U^+(X)$ is an e-subgroup whenever $e^{-1}e(H) = H$.

Proposition 3.8. Let $H \leq U^+(X)$, then the following statements are equivalent.

(a) H is an e-subgroup.
(b) If $e(f) = e(g)$ and $f \in H$, then $g \in H$.
(c) If $e(f) \subseteq e(g)$ and $f \in H$, then $g \in H$.
(d) $H = U^+(X)$ or $e(H)$ is a z-filter.

Proof. By Proposition 3.6, it is clear. ■

Definition 3.9. We call a subgroup H of $U^+(X)$ closurely fixed whenever $\cap_{f \in H} \mathrm{cl}_{\beta X} e(f) \neq \varnothing$.

Obviously, every proper e-subgroup is closurely fixed. If a proper subgroup H of $U^+(X)$ is closed under addition, then H is never an e-subgroup. Indeed, it cannot be even a closurely fixed subgroup, since $e(1+1) = \varnothing$.

Proposition 3.10. A subgroup H of $U^+(X)$ is a maximal closurely fixed subgroup if and only if is of the form

$$H^p = \{f \in U^+(X) : \ p \in \mathrm{cl}_{\beta X} e(f)\}$$

where $p \in \beta X$.

Proof. ($\Rightarrow$). By assumption there exists $p \in \cap_{f \in H} \mathrm{cl}_{\beta X} e(f)$. It is clear that $H \leq H^p$ and hence $H = H^p$.

($\Leftarrow$). It is similarly simple. ■

Let X be a topological space, we denote by $\mathcal{M}^*$ the set of all subgroups of $U^+(X)$ that are of the form H^p where $p \in \beta X$. Given $f \in U^+(X)$ we define $\mathcal{M}^*(f) = \{M \in \mathcal{M}^* : \ f \in M\}$ and $\mathbf{M}^*_f = \cap \mathcal{M}^*(f)$.

Proposition 3.11. The set $\mathcal{M}^*(f)$ form a base of the closed sets for a topological space on $U^+(X)$ named Zarisky-like topology.

Proof. If $f, g \in U^+(X)$, then $e(f) \cup e(g) \in Z(X)$ and so there exists $h' \in C(X)$ such that $Z(h') = e(f) \cup e(g)$. Put $h = 1 + |h'|$, then $h \in U^+(X)$ and $e(f) \cup e(g) = e(h)$. Consequently, we have

$$\mathrm{cl}_{\beta X} e(f) \cup \mathrm{cl}_{\beta X} e(g) = \mathrm{cl}_{\beta X}(e(f) \cup e(g)) = \mathrm{cl}_{\beta X} e(h).$$

Now, it is enough to show that $\mathcal{M}^*(f) \cup \mathcal{M}^*(g) = \mathcal{M}^*(h)$. To prove this

$$H^p \in \mathcal{M}^*(f) \cup \mathcal{M}^*(g) \quad \Leftrightarrow \quad p \in \mathrm{cl}_{\beta X} e(f) \cup \mathrm{cl}_{\beta X} e(g) = \mathrm{cl}_{\beta X} e(h)$$

$$\Leftrightarrow \quad H^p \in \mathcal{M}^*(h). \quad ■$$

Proposition 3.12. Let X be a compact topological space, then the map $\varphi : X \longrightarrow \mathcal{M}^*$ defined by $\varphi(x) = H^x$ is a homeomorphism.

Proof. It is clear that φ is bijection and

$$\varphi(e(f)) = \{H^x : \ x \in e(f)\} = \{H^x : \ f \in H^x\} = \mathcal{M}^*(f).$$

Thus, φ maps a base of X to a base of $\mathcal{M}^*(X)$. ■

Proposition 3.13. Let X and Y be topological spaces, then the following statements hold.

(a) $C(X) = U^+(X) - U^+(X)$.

(b) A group homomorphism $\varphi : \ U^+(X) \longrightarrow U^+(Y)$ can be extended to a ring homomorphism $\overline{\varphi} : \ C(X) \longrightarrow C(Y)$ if and only if φ is additive.

(c) If $\varphi : \ U^+(X) \longrightarrow U^+(Y)$ is a group isomorphism, additive and X, Y are compact, then X is homeomorphic to Y.

Proof. (a). Suppose that $f \in C(X)$. If we put $f_1 = 1 + (f \vee 0)$ and $f_2 = 1 + (-f \vee 0)$, then clearly $f_1, f_2 \in U^+(X)$ and $f = f_1 - f_2$.

(b $\Rightarrow$). It is evident.

(b $\Leftarrow$). Let $f \in C(X)$. By (a) there exists $f_1, f_2 \in U^+(X)$ such that $f = f_1 - f_2$. Now, we define $\overline{\varphi}(f) = \varphi(f_1) - \varphi(f_2)$. Clearly, $\overline{\varphi}$ is well defined. For let $f = f_1 - f_2 = f_1' - f_2'$, then $f_1 + f_2' = f_2 + f_1'$ and since φ is additive, $\varphi(f_1) + \varphi(f_2') = \varphi(f_2) + \varphi(f_1')$. Therefore, $\varphi(f_1) - \varphi(f_2) = \varphi(f_1') - \varphi(f_2')$. Moreover, we can easily see that $\overline{\varphi}$ is a ring homomorphism.

(c). If we show that $\overline{\varphi}$ is an isomorphism, then by Theorem 4.9 in [9] we are through. It is clear that $\overline{\varphi}$ is onto. Let $\overline{\varphi}(f) = 0$, then as in the proof of (a), $\varphi(1+(-f \vee 0)) = \varphi(1+(f \vee 0))$ and since φ is one-one, $1+(-f \vee 0) = 1+(f \vee 0)$ and thus $f = (f \vee 0) - (-f \vee 0) = 0$. ■

In [11], a ring R is called an S-ring if it is generated by its units. The part (a) of the previous proposition not only shows that $C(X)$ is an S-ring but it is indeed generated by $U^+(X)$.

Lemma 3.14. For every $f, g \in U^+(X)$ the following equality holds.

$$e(f+g) = \cup_{0<r<1} e(\frac{f}{r}) \cap e(\frac{g}{1-r}).$$

Proof. We can write

$$e(f+g) = \{x \in X : \ f(x) + g(x) = 1\}$$

$$= \{x \in X : \ \exists r \in (0,1), \quad f(x) = r, \quad g(x) = 1-r\}$$

$$= \{x \in X : \ \exists r \in (0,1), \quad x \in e(\frac{f}{r}) \cap e(\frac{g}{1-r})\} = \cup_{0<r<1} e(\frac{f}{r}) \cap e(\frac{g}{1-r}). \ ■$$

Proposition 3.15. Suppose that X and Y are compact topological spaces, $\varphi : U^+(X) \longrightarrow U^+(Y)$ is a group isomorphism such that $\varphi(\mathcal{M}^*) = \mathcal{M}^*(Y)$ and φ is identity on $\mathbb{R}^+$, then φ is additive and consequently $X \simeq Y$.

Proof. The proof will be divided into three steps.

Step1. Since $\varphi(\mathcal{M}^*(X)) = \mathcal{M}^*(Y)$, there exists a bijective map $\psi : X \longrightarrow Y$ such that $\varphi(H^x) = H^{\psi(x)}$. Now we show that

$$\forall f \in U^+(X), \quad \psi(e(f)) = e(\varphi(f)). \qquad (1)$$

To prove this claim, we proceed as follows.

$$\psi(x) \in \psi(e(f)) \quad \Leftrightarrow \quad x \in e(f) \quad \Leftrightarrow \quad f \in H^x$$

$$\Leftrightarrow \quad \varphi(f) \in \varphi(H^x) = H^{\psi(x)} \quad \Leftrightarrow \quad \psi(x) \in e(\varphi(f)).$$

Step2. For all $f, g \in U^+(X)$ we have

$$e(\varphi(f+g)) = e(\varphi(f) + \varphi(g)). \qquad (2)$$

To prove (2), let $f, g \in U^+(X)$, then by (1) and Lemma 3.14

$$\psi(x) \in e(\varphi(f+g)) \quad \Leftrightarrow \quad \psi(x) \in \psi(e(f+g))$$

$$\Leftrightarrow \quad x \in e(f+g) = \cup_{0<r<1} e(\frac{f}{r}) \cap e(\frac{g}{1-r})$$

$$\Leftrightarrow \quad \psi(x) \in \cup_{0<r<1} \psi(e(\frac{f}{r})) \cap \psi(e(\frac{g}{1-r})) = \cup_{0<r<1} e(\varphi(\frac{f}{r})) \cap e(\varphi(\frac{g}{1-r}))$$

$$= \cup_{0<r<1} e(\frac{\varphi(f)}{r}) \cap e(\frac{\varphi(g)}{1-r}) = e(\varphi(f) + \varphi(g)).$$

Note that, by assumption, $\varphi(rf) = r\varphi(f)$ for every $r \in \mathbb{R}^+$ and every $f \in U^+(X)$.

Step3. Let $f, g \in U^+(X)$ be such that $\varphi(f+g) \neq \varphi(f) + \varphi(g)$ and seek a contradiction. By this assumption, there exists $y \in Y$ such that $\varphi(f+g)(y) = r$, $(\varphi(f) + \varphi(g))(y) = s$ and $r \neq s$, thus

$$(\varphi(\frac{f}{r} + \frac{f}{r}))(y) = \frac{1}{r}(\varphi(f+g))(y) = 1 \neq \frac{s}{r}$$

$$= \frac{1}{r}(\varphi(f) + \varphi(g))(y) = (\varphi(\frac{f}{r}) + \varphi(\frac{g}{r}))(y)$$

which implies $e(\varphi(\frac{f}{r} + \frac{g}{r})) \neq e(\varphi(\frac{f}{r}) + \varphi(\frac{g}{r}))$, a contradiction. ■

Corollary 3.16. Let X and Y be compact spaces and $\varphi : U^+(X) \longrightarrow U^+(Y)$ be a group isomorphism, then the following statements are equivalent.

(a) φ is identity on $\mathbb{R}^+$ and $\varphi(\mathcal{M}^*(X)) = \mathcal{M}^*(Y)$.

(b) $\varphi(1+f)=1+\varphi(f)$ for all $f\in U^+(X)$.
(c) φ is additive.

Proof. (a) $\Rightarrow$ (b) is obvious by Proposition 3.15 and (b) $\Rightarrow$ (c) is also clear.

(c) $\Rightarrow$ (a). As φ is additive, it can be extended to an isomorphism $\overline{\varphi}: C(X)\longrightarrow C(Y)$. Now let $f_1,\cdots,f_k\in H^x$, then

$$\sum_{i=1}^{k}|1-f_i(x)|=0 \quad\Rightarrow\quad \sum_{i=1}^{k}|1-f_i|\in M^x(X)$$

$$\Rightarrow\quad \sum_{i=1}^{k}|1-\varphi(f_i)|=\overline{\varphi}(\sum_{i=1}^{k}|1-f_i|)\in\overline{\varphi}(M^x(X)).$$

Hence $\sum_{i=1}^{k}|1-\varphi(f_i)|$ is a non unit element of $C(Y)$. Thus, there exists $y\in Y$ such that $\sum_{i=1}^{k}|1-\varphi(f_i)(y)|=0$ which implies $\cap_{i=1}^{k}Z(1-\varphi(f_i))\neq\varnothing$ and consequently $\cap_{i=1}^{k}e(f_i)\neq\varnothing$. Therefore, $\varphi(H^x)$ is fixed and since φ is an isomorphism $\varphi(H^x)\in\mathcal{M}^*(Y)$. ■

Proposition 3.17. Let $\varphi:\ U^+(X)\longrightarrow U^+(Y)$ be a group homomorphism. Then the following statements are equivalent.
(a) $\varphi(f\vee 1)=\varphi(f)\vee 1$, for all $f\in U^+(X)$.
(b) $f\vee g=1$ implies $\varphi(f)\vee\varphi(g)=1$, for all $f,g\in U^+(X)$.
(c) $\varphi(f\vee g)=\varphi(f)\vee\varphi(g)$, for all $f,g\in U^+(X)$.
(d) φ is a lattice homomorphism.
(e) $\varphi(f\wedge 1)=\varphi(f)\wedge 1$, for all $f\in U^+(X)$.
(f) $f\wedge g=1$ implies $\varphi(f)\wedge\varphi(g)=1$, for all $f,g\in U^+(X)$.
(g) $\varphi(f\wedge g)=\varphi(f)\wedge\varphi(g)$, for all $f,g\in U^+(X)$.
(h) $\varphi(\|f\|_1)=\|\varphi(f)\|_1$, for all $f\in U^+(X)$.
Moreover, if φ is an isomorphism, then the above statements are equivalent to φ being order preserving.

Proof. Since $U^+(X)$ is a lattice ordered group, it is enough to show that (h) $\Rightarrow$ (a). Let $f\in U^+(X)$, then

$$\varphi(f\vee 1)=\varphi(\sqrt{f}(\sqrt{f}\vee\frac{1}{\sqrt{f}}))=\varphi(\sqrt{f}\ \|\sqrt{f}\|_1)=(\sqrt{\varphi(f)}\ \|\sqrt{\varphi(f)}\|_1$$

$$\sqrt{\varphi(f)}(\sqrt{\varphi(f)}\vee\frac{1}{\sqrt{\varphi(f)}})=\varphi(f)\vee 1.$$

Now, assume that φ is an ordered group isomorphism. Let $f,g\in U^+(X)$, as φ is order preserving, obviously we have $\varphi(f),\varphi(g)\leq\varphi(f\vee g)$. Now, let $\varphi(f),\varphi(g)\leq\varphi(h)$, then we can write

$$\varphi(\frac{f}{h})\leq 1\quad\Rightarrow\quad\frac{f}{h}\leq\varphi^{-1}(1)=1\quad\Rightarrow\quad f\leq h.$$

Similarly $g \leq h$ holds. Thus, $f \vee g \leq h$ and hence $\varphi(f \vee g) \leq \varphi(h)$. Therefore, $\varphi(f \vee g) = \varphi(f) \vee \varphi(g)$ for every $f, g \in U^+(X)$. ■

Theorem 3.18. Suppose that X and Y are compact connected topological spaces. There exists an ordered group isomorphism $\varphi : U^+(X) \longrightarrow U^+(Y)$ which is identity on $\mathbb{R}^+$ if and only if $X \simeq Y$.

Proof. ($\Rightarrow$). By Proposition 3.13 and Corollary 3.16, it suffices to prove that $\varphi(\mathcal{M}^*(X)) = \mathcal{M}^*(Y)$. To this end, it is enough to prove that $\varphi(H^p)$ has finite intersection property. We first show that

$$e(f) \neq \varnothing \quad \Rightarrow \quad e(\varphi(f)) \neq \varnothing. \qquad (1)$$

Let $e(\varphi(f)) = \varnothing$ and seek a contradiction. As Y is connected, $\varphi(f) \ll 1$ or $\varphi(f) \gg 1$. So let $\varphi(f) \ll 1$. Since Y is compact, $\varphi(f) \leq r < 1$ for some $r \in \mathbb{R}^+$. Hence $f \leq \varphi^{-1}(r) = r$ and so $e(f) = \varnothing$, which is a contradiction. Now let $f_1, \cdots, f_n \in H^p$. It is obvious that $\|f_i\|_1 \in H^p$ for every $1 \leq i \leq n$. Thus, $\prod_{i=1}^n \|f_i\|_1 \in H^p$ and hence $e(\prod_{i=1}^n \|f_i\|_1) \neq \varnothing$. Therefore, by (1), it follows that $e(\varphi(\prod_{i=1}^n \|f_i\|_1)) \neq \varnothing$. Since φ is order preserving, by Proposition 3.17, it is a lattice isomorphism. So, we can write

$$\varnothing \neq e(\varphi(\prod_{i=1}^n \|f_i\|_1)) = e(\prod_{i=1}^n \|\varphi(f_i)\|_1) = \cap_{i=1}^n e(\|\varphi(f_i)\|_1) = \cap_{i=1}^n e(\varphi(f_i)).$$

($\Leftarrow$). It is clear. ■

Remark 3.19. In fact, most of the results in the context of $C(X)$ can be restated in terms of $U(X)$ or $U^+(X)$. Let us give some samples to confirm our claim. For all $p \in \beta X$ we define

$$H^{\circ p} = \{f \in U^+(X) : \quad p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} e(f)\}.$$

Also, an e-subgroup H of $U^+(X)$ is said to be e-prime if $e(H)$ is a prime z-filter. Suppose that X is a topological space. Then one can easily observe the following facts.

(a) For every $p \in \beta X$, we have $e(H^{\circ p}) = \mathcal{O}^p(X)$ where $\mathcal{O}^p(X) = \{Z \in Z(X) : \ p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z\}$;

(b) If H is an e-prime subgroup of $U^+(X)$, then there exists a unique $p \in \beta X$ such that $H^{\circ p} \subseteq H \subseteq H^p$;

(c) X is a P-space if and only if $H^{\circ p} = H^p$ for every $p \in \beta X$;

(d) X is an almost P-space if and only if $e(f) = \varnothing$ or $\mathrm{int}_{\beta X} e(f) \neq \varnothing$ for every $f \in U^+(X)$;

(e) X is an F-space if and only if $H^{\circ p}$ is e-prime subgroup of $U^+(X)$.

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