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About Diophantine Equations, an Analytic Approach

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Abstract

Our purpose in this paper is to show how much diophantine equations are rich in analytic applications. Effectively, those equations allow to build amazing sequences, series and numbers. The question of the proof of some theorems (not necessarily the famous ones) remains of course, we will see it in this communication. We will make also an allusion to the well-known Fermat numbers (2^{2^n}) and generalize them. We will see how the problem of the mathematical proof is actual and how it can be solved using amazing sequences and series. This analytic approach will conduct until the basis of arithmetics and logical mathematics, it constitutes also an algebraic approach of the tenth problem of Hilbert and more generally of Godel theorem.

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1 Introduction

We begin with Fermat equation and change the data of the problem. In stead of an only equation, we define two equations. Those laters conduct to define sequences ans series. We calculate their limits and prove that the initial equations has no solutions. Finally, we translate the same argument to other diophantine equations. The process is the same and conduct to an impossibility.

2 The sequences

Let us begin with Fermat equation, that is

$$U^n = X^n + Y^n, \qquad GCD(X,Y) = 1$$

We will consider in this paper two equivalent equations. More precisely, let us pose

$$u = U^{2n}, \quad x = U^n X^n, \quad y = U^n Y^n, \quad z = X^n Y^n$$

A simple calculation leads to

$$u = U^{2n} = U^n(X^n + Y^n) = x + y \qquad (1)$$

and

$$\frac{1}{z} = \frac{1}{X^n Y^n} = \frac{U^{2n}}{U^n X^n U^n Y^n} = \frac{u}{xy} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y} \qquad (2)$$

We deduce that if U, X, Y are integers verifying Fermat equation, then u, x, y, z as defined verify simultaneously the equations (1) and (2)

Lemma 1

$$u = x + y \qquad (1), \quad \frac{1}{z} = \frac{1}{x} + \frac{1}{y} \qquad (2)$$

We conclude that

$$z = X^n Y^n, \quad u = (X^n + Y^n)^2, \quad x = X^n(X^n + Y^n), \quad y = Y^n(X^n + Y^n)$$

If we pose

$$x_1 = x, \quad y_1 = y$$

for all integers x_1, y_1, there exists z_1 satisfying

$$\frac{1}{z_1} = \frac{1}{x_1} + \frac{1}{y_1}$$

and

$$z_1 = \frac{xy}{x+y} = z$$

and

$$(x_1+y_1)z_1 = x_1y_1, \quad x_1(y_1-z_1) = z_1y_1, \quad y_2 = y_1 - z_1 = \frac{z_1y_1}{x_1}$$

and

$$y_1(x_1-z_1) = z_1x_1, \quad x_2 = x_1 - z_1 = \frac{z_1x_1}{y_1}, \quad x_2y_2 = z_1^2$$

which means that

$$x_1 = x_2 + z_1 = x_2 + \sqrt{x_2y_2}, \quad y_1 = y_2 + z_1 = y_2 + \sqrt{x_2y_2}$$

and

$$u_1 = u = (x_1+y_1) = (\sqrt{x_2}+\sqrt{y_2})^2 > x_2 + y_2 > 0$$

(x_1+y_1) is an integer

$$x_1 = \sqrt{x_2}(\sqrt{x_2}+\sqrt{y_2}) > x_2 > 0$$

x_1 is an integer

$$y_1 = \sqrt{y_2}(\sqrt{x_2}+\sqrt{y_2}) > y_2 > 0$$

y_1 is an integer

$$z_1 = \frac{x_1y_1}{x_1+y_1} = X^nY^n = \sqrt{x_2y_2} > z_2 = \frac{x_2y_2}{x_2+y_2} > 0$$

z_2 is rational because for all rationals x_2, y_2, there exists z_2 satisfying the following equation

$$\frac{1}{z_2} = \frac{1}{x_2} + \frac{1}{y_2}$$

Using the same argument, we have

$$\frac{1}{z_i} = \frac{1}{x_i} + \frac{1}{y_i}, \forall i \in N^*$$

$$(x_i+y_i)z_i = x_iy_i, \quad x_i(y_i-z_i) = z_iy_i, \quad y_{i+1} = y_i - z_i = \frac{z_iy_i}{x_i}$$

and

$$y_i(x_i-z_i) = z_ix_i, \quad x_{i+1} = x_i - z_i = \frac{z_ix_i}{y_i}, \quad x_{i+1}y_{i+1} = z_i^2$$

which means that

$$x_i = x_{i+1} + z_i = x_{i+1} + \sqrt{x_{i+1}y_{i+1}} > x_{i+1} > 0$$

$$y_i = y_{i+1} + z_i = y_{i+1} + \sqrt{x_{i+1}y_{i+1}} > y_{i+1} > 0$$

and

$$u_i = (x_i+y_i) = (\sqrt{x_{i+1}}+\sqrt{y_{i+1}})^2 > x_{i+1} + y_{i+1} > 0$$

$$z_i = \sqrt{x_{i+1}y_{i+1}} > z_{i+1} = \frac{x_{i+1}y_{i+1}}{x_{i+1}+y_{i+1}} > 0$$

By a natural induction, we deduce the following result :

Lemma 2 For all $i \in N^*$, we have

$$x_i = x^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \qquad (H), \quad y_i = y^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \qquad (H')$$

x_i and y_i are as it follows

$$x_i = x^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1}, \quad y_i = y^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1}$$

$$\forall i > 1$$

But, $\forall a, b$

$$a - b = (a^{2^{i-1}} - b^{2^{i-1}}) \prod_{j=0}^{j=i-2} (a^{2^j} + b^{2^j})^{-1} \qquad (E)$$

$$x - y = (x^{2^{i-1}} - y^{2^{i-1}}) \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \qquad (E')$$

$$\prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j}) = \frac{(x^{2^{i-1}} - y^{2^{i-1}})}{x - y}$$

We can conclude that the expression of the sequences are, for

$$x \neq y, \quad x_i \neq y_i$$

Lemma 3

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y) = U^n \frac{X^{n2^{i-1}}}{X^{n2^{i-1}} - Y^{n2^{i-1}}}(X^n - Y^n)$$

and

$$y_i = \frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y) = U^n \frac{Y^{n2^{i-1}}}{X^{n2^{i-1}} - Y^{n2^{i-1}}}(X^n - Y^n)$$

$$u_i = x_i + y_i = U^n \frac{X^{n2^{i-1}} + Y^{n2^{i-1}}}{X^{n2^{i-1}} - Y^{n2^{i-1}}}(X^n - Y^n)$$

Lemma 4 The equations (1) and (2) have the constant

$$x_i - y_i = x - y$$

Lemma 5 The only solution of equations (1) and (2) is

$$xy(x-y)=0$$

A remark, before : we will not utilize conditions on X, Y, U, n in this proof, because the solution is related to the undecidability of some diophantine equations. Our proof is general to all diophantine equations and, in accord with Matiasevic theorem, we know that there is no algorithm to prove the impossibility of some diophantine equations. But, we give in this article one. The result is that our algorithm, which is correct, must conduct to an impossibility and this impossibility is that $xy(x-y)=0$ for all x, y. Therefore our approach is a proof of Matiasevic theorem and a resolution of tenth problem of Hilbert. It may also be a proof of Fermat theorem or Beal conjecture, etc... as we will see. We will give several proofs and utilize series.

3 The series

Proof of lemma 5 As

$$x_i=\frac{x^{2^{i-1}}}{x^{2^{i-1}}-y^{2^{i-1}}}(x-y), \quad y_i=\frac{y^{2^{i-1}}}{x^{2^{i-1}}-y^{2^{i-1}}}(x-y)$$

if

$$x>y\Rightarrow \lim_{i\longrightarrow\infty}(x_i)=x-y, \qquad \lim_{i\longrightarrow\infty}(y_i)=0$$

and

$$y>x\Rightarrow \lim_{i\longrightarrow\infty}(y_i)=y-x, \qquad \lim_{i\longrightarrow\infty}(x_i)=0$$

As we saw

$$\sqrt{x_iy_i}=y_{i-1}-y_i=x_{i-1}-x_i$$

then

$$x_i-x_{i+1}=\sqrt{x_{i+1}y_{i+1}}$$
$$x_{i-1}-x_i=\sqrt{x_iy_i}$$

...

$$x_1-x_2=x-x_2=\sqrt{x_2y_2}$$

It implies the following series

$$\sum_{j=2}^{j=i+1}(\sqrt{x_jy_j})=x-x_2+x_2-x_3+...+x_i-x_{i+1}=x-x_{i+1}$$

and

$$\sum_{j=2}^{j=\infty}(\sqrt{x_jy_j})=\lim_{i\longrightarrow\infty}(x-x_{i+1})$$

Then, if $x > y$

$$\sum_{j=2}^{j=\infty}(\sqrt{x_j y_j}) = \lim_{i\longrightarrow\infty}(x - x_{i+1}) = x - (x - y) = y$$

and if $x < y$

$$\sum_{j=2}^{j=\infty}(\sqrt{x_j y_j}) = \lim_{i\longrightarrow\infty}(x - x_{i+1}) = x$$

Therefore

$$\sum_{j=2}^{j=i}((-1)^j\sqrt{x_j y_j}) = x - x_2 - (x_2 - x_3) + (x_3 - x_4) - ... + (-1)^i(x_{i-1} - x_i)$$

$$= x - 2x_2 + 2x_3 - ... + 2(-1)^{i-1}x_{i-1} + (-1)^{i+1}x_i$$

$$= 2\sum_{j=2}^{j=i-1}((-1)^{j+1}x_j) + x + (-1)^{i+1}x_i = 2\sum_{j=1}^{j=i}((-1)^{j+1}x_j) - x - (-1)^{i+1}x_i$$

$$= \sum_{j=2}^{j=i-1}((-1)^{j+1}x_j) + \sum_{j=1}^{j=i}((-1)^{j+1}x_j) = 2\sum_{j=2}^{j=i-1}((-1)^{j+1}y_j) + y + (-1)^{i+1}y_i$$

$$= 2\sum_{j=1}^{j=i}((-1)^{j+1}y_j) - y - (-1)^{i+1}y_i = \sum_{j=2}^{j=i-1}((-1)^{j+1}y_j) + \sum_{j=1}^{j=i}((-1)^{j+1}y_j)$$

It follows that

$$2\sum_{j=1}^{j=i}((-1)^{j+1}x_j) = \sum_{j=2}^{j=i}((-1)^j\sqrt{x_j y_j}) + x + (-1)^{i+1}x_i$$

and

$$2\sum_{j=1}^{j=i}((-1)^{j+1}y_j) = \sum_{j=2}^{j=i}((-1)^j\sqrt{x_j y_j}) + y + (-1)^{i+1}y_i$$

As we can not know the limit of $(-1)^{i+1}x_i$, we deduce that

$$\sum_{j=1}^{j=\infty}((-1)^j x_j)$$

can be not convergent. But

$$\sum_{j=2}^{j=\infty}((-1)^j\sqrt{x_j y_j})$$

is convergent. Also, knowing that y_i converges to zero in the infinity, we deduce that

$$\sum_{j=1}^{j=\infty} ((-1)^j y_j)$$

is convergent. So the limit of

$$\sum_{j=2}^{j=i} ((-1)^j \sqrt{x_j y_j}) = 2\sum_{j=1}^{j=i} ((-1)^{j+1} y_j) - y - (-1)^{i+1} y_i$$

$$= 2\sum_{j=1}^{j=i} ((-1)^{j+1} x_j) - x - (-1)^{i+1} x_i$$

exists and the series are convergent. It means that :

$$\lim_{i \longrightarrow \infty} (x_i) = x - y = 0$$

It is confirmed by the fact that the limit of the general term of the series is zero. Let us prove it. We will give two manners showing that $xy(x-y) = 0$. Let

$$\lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=m} (\sqrt{x_{2k} y_{2k}} e^{-\frac{2k}{\sqrt{2m}}})) = \lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=m} ((x_{2k-1} - x_{2k}) e^{-\frac{2k}{\sqrt{2m}}}))$$

$$= \lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=m} (x_{2k-1} e^{-\frac{2k}{\sqrt{2m}}}) - \sum_{k=1}^{k=m} (x_{2k} e^{-\frac{2k}{\sqrt{2m}}}))$$

$$= \lim_{m \longrightarrow \infty} (e^{-\frac{1}{\sqrt{2m}}} \sum_{k=1}^{k=m} (x_{2k-1} e^{-\frac{2k-1}{\sqrt{2m}}}) - \sum_{k=1}^{k=m} (x_{2k} e^{-\frac{2k}{\sqrt{2m}}}))$$

$$= \lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=m} (x_{2k-1} e^{-\frac{2k-1}{\sqrt{2m}}}) - \sum_{k=1}^{k=m} (x_{2k} e^{-\frac{2k}{\sqrt{2m}}}))$$

$$= \lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=2m} ((-1)^{k+1} x_k e^{-\frac{k}{\sqrt{2m}}}))$$

$$= \lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=m} ((y_{2k-1} - y_{2k}) e^{-\frac{2k}{\sqrt{2m}}})) = \lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=m} (y_{2k-1} e^{-\frac{2k}{\sqrt{2m}}}) - \sum_{k=1}^{k=m} (y_{2k} e^{-\frac{2k}{\sqrt{2m}}}))$$

$$= \lim_{m \longrightarrow \infty} (e^{-\frac{1}{\sqrt{2m}}} \sum_{k=1}^{k=m} (y_{2k-1} e^{-\frac{2k-1}{\sqrt{2m}}}) - \sum_{k=1}^{k=m} (y_{2k} e^{-\frac{2k}{\sqrt{2m}}}))$$

$$= \lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=m} (y_{2k-1} e^{-\frac{2k-1}{\sqrt{2m}}}) - \sum_{k=1}^{k=m} (y_{2k} e^{-\frac{2k}{\sqrt{2m}}})) = \lim_{m \longrightarrow \infty} (\sum_{k=1}^{k=2m} ((-1)^{k+1} y_k e^{-\frac{k}{\sqrt{2m}}}))$$

Consequently

$$\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=2m}((-1)^{k+1}(x_k-y_k)e^{-\frac{k}{\sqrt{2m}}}))=0$$

$$=\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=2m}((-1)^{k+1}(x-y)e^{-\frac{k}{\sqrt{2m}}}))=0=(x-y)\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=2m}((-1)^{k+1}e^{-\frac{k}{\sqrt{2m}}}))=0$$

$$=(x-y)\lim_{m\longrightarrow\infty}(e^{-\frac{1}{\sqrt{2m}}}\sum_{k=0}^{k=2m-1}((-1)^{k}e^{-\frac{k}{\sqrt{2m}}}))=0$$

$$=(x-y)\lim_{m\longrightarrow\infty}(\frac{1+(-1)^{2m-1}e^{-\sqrt{2m}}}{1+e^{-\frac{1}{\sqrt{2m}}}})=0=(x-y)\frac{1}{2}=0\Rightarrow x-y=0$$

the only solution is $xy(x-y)=0$, $xy=0$ if at least one of the sequences x_i or y_i is constant. For the second way, let

$$x_i(t)=\sum_{k=2}^{k=i-1}(x_ke^{-kt}),\quad y_i(t)=\sum_{k=2}^{k=i-1}(y_ke^{-kt})$$

$$u_i(t)=\sum_{k=2}^{k=i}(\sqrt{x_ky_k}e^{-kt})=\sum_{k=2}^{k=i-1}((x_{k-1}-x_k)e^{-kt})$$

$$=xe^{-2t}+x_2(-e^{-2t}+e^{-3t})+x_3(-e^{-3t}+e^{-4t})+...+x_{i-1}(-e^{-(i-1)t}+e^{-it})-x_ie^{-it}$$

$$=xe^{-2t}+x_2e^{-2t}(-1+e^{-t})+x_3e^{-3t}(-1+e^{-t})+...+x_{i-1}e^{-(i-1)t}(-1+e^{-t})-x_ie^{-it}$$

$$=xe^{-2t}-x_ie^{-it}+(-1+e^{-t})(x_2e^{-2t}+x_3e^{-3t}+...+x_ie^{-(i-1)t})=xe^{-2t}-x_ie^{-it}+(-1+e^{-t})x_i(t)$$

If $t=\frac{1}{\sqrt{i}}$ then

$$\frac{-2+3e^{-\frac{1}{\sqrt{i}}}}{2}x_i(\frac{1}{\sqrt{i}})=(-1+e^{-\frac{1}{\sqrt{i}}})x_i(\frac{1}{\sqrt{i}})+\frac{1}{2}e^{-\frac{1}{\sqrt{i}}}x_i(\frac{1}{\sqrt{i}})$$

Hence

$$\frac{-2+3e^{-\frac{1}{\sqrt{i}}}}{3}x_i(\frac{1}{\sqrt{i}})=(-1+e^{-\frac{1}{\sqrt{i}}})x_i(\frac{1}{\sqrt{i}})+\frac{1}{3}x_i(\frac{1}{\sqrt{i}})$$

It follows

$$(-2+3e^{-\frac{1}{\sqrt{i}}})(\frac{1}{2}-\frac{1}{3})x_i(\frac{1}{\sqrt{i}})=x_i(\frac{1}{\sqrt{i}})(\frac{e^{-\frac{1}{\sqrt{i}}}}{2}-\frac{1}{3})$$

Thus

$$\lim_{i\longrightarrow\infty}((-2+3e^{-\frac{1}{\sqrt{i}}})(x_i(\frac{1}{\sqrt{i}}))(\frac{1}{6}))=\lim_{i\longrightarrow\infty}(x_i(\frac{1}{\sqrt{i}})(\frac{e^{-\frac{1}{\sqrt{i}}}}{2}-\frac{1}{3}))$$

Therefore

$$\lim_{i\longrightarrow\infty}((-2+3e^{-\frac{1}{\sqrt{i}}})(x_i(\frac{1}{\sqrt{i}}))(\frac{1}{6})) = \lim_{i\longrightarrow\infty}(x_i(\frac{1}{\sqrt{i}})(\frac{1}{2}-\frac{1}{3})) = \lim_{i\longrightarrow\infty}(x_i(\frac{1}{\sqrt{i}})(\frac{1}{6}))$$

And then

$$\lim_{i\longrightarrow\infty}((-2+3e^{-\frac{1}{\sqrt{i}}})x_i(\frac{1}{\sqrt{i}})) = \lim_{i\longrightarrow\infty}(e^{-\frac{1}{\sqrt{i}}}x_i(\frac{1}{\sqrt{i}}))$$

So

$$\lim_{i\longrightarrow\infty}(2(-1+e^{-\frac{1}{\sqrt{i}}})x_i(\frac{1}{\sqrt{i}})) = 0$$

Consequently

$$\lim_{i\longrightarrow\infty}(u_i(\frac{1}{\sqrt{i}})) = \lim_{i\longrightarrow\infty}(\sum_{k=2}^{k=i}(\sqrt{x_k y_k}e^{-\frac{k}{\sqrt{i}}})) = \lim_{i\longrightarrow\infty}(\sum_{k=2}^{k=i}(\sqrt{x_k y_k})) = y$$

$$= x + \lim_{i\longrightarrow\infty}((-1+e^{-\frac{1}{\sqrt{i}}})\sum_{k=2}^{k=i-1}(x_k e^{-\frac{k}{\sqrt{i}}}) - x_i e^{-\sqrt{i}}) = x$$

$$= x + \lim_{i\longrightarrow\infty}((-1+e^{-\frac{1}{\sqrt{i}}})x_i(\frac{1}{\sqrt{i}}) - x_i e^{-\sqrt{i}}) = x$$

$$= y + \lim_{i\longrightarrow\infty}((-1+e^{-\frac{1}{\sqrt{i}}})\sum_{k=2}^{k=i-1}(y_k e^{-\frac{k}{\sqrt{i}}}) - y_i e^{-\sqrt{i}}) = y$$

$$= y + \lim_{i\longrightarrow\infty}((-1+e^{-\frac{1}{\sqrt{i}}})y_i(\frac{1}{\sqrt{i}}) - y_i e^{-\sqrt{i}}) = y$$

That is

$$x - y = 0$$

To sum up, we have

$$x_i > x_{i+1}, \quad y_i > y_{i+1} \Rightarrow x = y$$

$$x_i = x_{i+1} = x_{i+1} + \sqrt{x_{i+1}y_{i+1}} \Rightarrow xy = 0$$

$$y_i = y_{i+1} = y_{i+1} + \sqrt{x_{i+1}y_{i+1}} \Rightarrow xy = 0$$

Which leeds to $xy(x-y) = 0$

Then

$$\sum_{k=2}^{k=\infty}((-1)^k\sqrt{x_k y_k}) = \sum_{k=2}^{k=\infty}((-1)^k x_k) = \sum_{k=2}^{k=\infty}((-1)^k y_k)$$

$$= x_2 - x_3 + x_4 - x_5 + ...$$

$$= 2\sum_{k=1}^{k=\infty}((-1)^{k+1}x_k) - x$$

$$= 2\sum_{k=1}^{k=\infty}((-1)^{k+1}y_k) - y$$

Hence

$$3(x_2 - x_3 + x_4 - x_5 + ...) = 3\sum_{k=2}^{k=\infty}((-1)^k x_k) = 3\sum_{k=2}^{k=\infty}((-1)^k y_k) = x = y$$

Then $x - y = 0$. This development is in fact a test of impossibility. The sequences and series are a consequence of Fermat equation and of other diophantine equations (as we will see). The question now is : why are there solutions for $n = 2$? The answer is in the previous formulas. It is important to note that for $n = 1$, there are trivial solutions. But, for $n = 1$, lemma 3 allows to write

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y) = U\frac{X^{2^{i-1}}}{X^{2^{i-1}} - Y^{2^{i-1}}}(X-Y) = U\frac{X^{22^{i-2}}}{X^{22^{i-2}} - Y^{22^{i-2}}}(X-Y)$$

and

$$y_i = \frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x-y) = U\frac{Y^{2^{i-1}}}{X^{2^{i-1}} - Y^{2^{i-1}}}(X-Y) = U\frac{Y^{22^{i-2}}}{X^{22^{i-2}} - Y^{22^{i-2}}}(X-Y)$$

$$u_i = x_i + y_i = U\frac{X^{22^{i-2}} + Y^{22^{i-2}}}{X^{22^{i-2}} - Y^{22^{i-2}}}(X - Y) = \frac{X^{22^{i-2}} + Y^{22^{i-2}}}{X^{22^{i-2}} - Y^{22^{i-2}}}(X^2 - Y^2)$$

It is the expression of u'_{i-1}, the u_{i-1} of the exponent 2. The case $n = 1$ conducts to the case $n = 2$ and as there are an infinity of solutions for $n = 1$, there will be an infinity of solutions for $n = 2$! and for $n = 4$

$$u_i = x_i + y_i = \frac{X^{42^{i-3}} + Y^{42^{i-3}}}{X^{42^{i-3}} - Y^{42^{i-3}}}U(X - Y)$$

it does not imply the case $n = 4$, because in this formula the exponent $i - 3$ does not guarantee the existence of the sequence for $i = 2$. So the case $n = 2$ is the only exception. The only solution for $n > 2$ is $xy(x - y) = 0$.

Another application of the sequences and series is Beal equation. It is

$$U^c = X^a + Y^b, \quad GCD(X, Y) = 1$$

If we take

$$u = U^{2c}, \quad x = U^c X^a, \quad y = U^c Y^b, \quad z = X^a Y^b$$

$$u = U^{2c} = U^c(X^a + Y^b) = x + y$$

$$\frac{1}{z} = \frac{1}{X^a Y^b} = \frac{U^{2c}}{U^c X^a U^c Y^b} = \frac{u}{xy} = \frac{x + y}{xy} = \frac{1}{x} + \frac{1}{y}$$

which is the result of lemma 1 and, with

$$U^c = X^a + Y^b$$

The solutions, as known, are

$$xy(x - y) = U^{2c}X^aY^b(U^cX^a - U^cY^b) = 0$$

Which are solutions for $a > 2$ and $b > 2$ and $c > 2$. Another application is the following equation

$$Y^n = X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i}$$

$$GCD(X_j, X_k) = 1; \forall j, k, j \neq k$$

We conjecture and can prove that this equation has no solution for $n > i(i-1)$ and $n_j > i(i-1)$, $\forall j \in \{1, 2, ..., i\}$. We can not know when there are solutions, as proved by Matiasevic. Let us build the generalized sequences for this equation. Now : Our goal is to prove that if X_i, $(i \geq 2)$, n_i, U, n, $GCD(X_1, X_2, ..., X_i) = 1$ are positive integers, then $X_a = X_b = 0; \forall a, b = 1, 2, ..., i$

$$n_k > i(i-1), \forall k = 1, 2, ..., i, n > i(i-1)$$

for an equation such the following (E)

$$X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i} = U^n$$

When $n \leq i(i-1), n_k \leq i(i-1), k = 1, 2, ..., i$ there are solutions, for example : $i = 2$ has

$$3^2 + 4^2 = 5^2$$

and $i = 3$ has

$$3^3 + 4^3 + 5^3 = 6^3$$

and

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

and $i = 4$ has

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

etc... We will suppose that Eq (E) is satisfied and that $GCD(X_k) = 1$, $k \in \{1, 2, ..., i\}; i \geq 2$, let

$$x_k = U^{(i-1)n}X_k^{n_k}, \quad k \in \{1, 2, ..., i\}$$

and

$$u = U^{in}, \quad v = X_1^{n_1}X_2^{n_2}...X_i^{n_i}$$

Lemma 6 x_k,$k = 1, 2, ...i$, u, v verify the following equalities

$$x_1 + x_2 + ... + x_i = U^{(i-1)n}(X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i}) = U^{in} = u \qquad (3)$$

$$\frac{1}{v} = \frac{1}{X_1^{n_1}X_2^{n_2}...X_i^{n_i}} = \frac{U^{i(i-1)n}}{U^{(i-1)n}X_1^{n_1}U^{(i-1)n}X_2^{n_2}...U^{(i-1)n}X_i^{n_i}} = \frac{u^{(i-1)}}{x_1x_2...x_i} \qquad (4)$$

Definition Let us define the following sequences

$$x_{k,0} = x_k, \quad u_0 = u, \quad v_0 = v$$

and

$$x_{k,1} = x_k^i(x_1 + x_2 + ... + x_i)^{-(i-1)}, \quad k \in \{1, 2, ..., i\}$$

which implies

$$u = x_1 + x_2 + ... + x_i = (x_{1,1}^{\frac{1}{i}} + x_{2,1}^{\frac{1}{i}} + ... + x_{i,1}^{\frac{1}{i}})^i > u_1 > 1$$

and

$$x_{k,0} = x_k = x_{k,1}^{\frac{1}{i}}(x_1 + x_2 + ... + x_i)^{\frac{(i-1)}{i}}$$

$$= x_{k,1}^{\frac{1}{i}}(x_{1,1}^{\frac{1}{i}} + x_{2,1}^{\frac{1}{i}} + ... + x_{i,1}^{\frac{1}{i}})^{(i-1)} > x_{k,1} > 0$$

and

$$v = \frac{x_{1,0}x_{2,0}...x_{i,0}}{u^{(i-1)}} = x_{1,1}^{\frac{1}{i}}x_{2,1}^{\frac{1}{i}}...x_{i,1}^{\frac{1}{i}} > v_1 = \frac{x_{1,1}x_{2,1}...x_{i,1}}{u_1^{(i-1)}} > 0$$

the reasoning is available until infinity. Then

$$u_j = x_{1,j} + x_{2,j} + ... + x_{i,j} = (x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^i > u_{j+1} > 1$$

and

$$x_{k,j} = x_{k,j+1}^{\frac{1}{i}}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^{(i-1)} > x_{k,j+1} > 0$$

and

$$v_j = \frac{x_{1,j}x_{2,j}...x_{i,j}}{u_j^{(i-1)}} = x_{1,j+1}^{\frac{1}{i}}x_{2,j+1}^{\frac{1}{i}}...x_{i,j+1}^{\frac{1}{i}} > v_{j+1} = \frac{x_{1,j+1}x_{2,j+1}...x_{i,j+1}}{u_{j+1}^{(i-1)}} > 0$$

$x_{k,j}$, v_j, u_j are positive $\forall j > 1$, $\forall k \in \{1, 2, ..., i\}$.

Lemma 7 The following equality (P) is satisfied :

$$x_{k,j} = x_k^{i^j}(\prod_{l=0}^{l=j-1} x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-(i-1)}$$

Proof of lemma 7 for $j = 1$ it is verified by the definition of $x_{k,1}$, u_1 et v_1, we suppose (P) true and the expression of u_j implies, with (P), that

$$x_{k,j} = x_{k,j+1}^{\frac{1}{i}}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{3}} + ... + x_{i,j+1}^{\frac{1}{i}})^{(i-1)}$$

so

$$x_{k,j+1}^{\frac{1}{i}} = x_{k,j}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^{-(i-1)}$$

and

$$x_{k,j+1} = x_{k,j}^{i}(x_{1,j+1}^{\bar{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^{-(i-1)i} = x_{k,j}^{i}(x_{1,j} + x_{2,j} + ... + x_{i,j})^{-(i-1)}$$

$$= x_k^{i^{j+1}} \prod_{l=0}^{l=j-1} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-i(i-1)}(x_1^{i^j} + x_2^{i^j} + ... + x_i^{i^j})^{-(i-1)}.$$

$$\prod_{l=0}^{l=j-1} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{(i-1)^2}$$

$$= x_k^{i^{j+1}} \prod_{l=0}^{l=j} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-(i-1)}$$

Then

$$x_{k,j} = x_k^{i^j} \prod_{l=0}^{l=j-1} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-(i-1)}$$

Lemma 8 the solution for all $k \geq 2$ is

$$X_k = 0$$

Proof of lemma 8 We can prove the lemma 8 easily, effectively

$$u = U^{2n}, \quad x = U^n X_k^{n_k}, \quad y = U^n(U^n - X_k^{n_k}), \quad z = X_k^{n_k}(U^n - X_k^{n_k})$$

therefore

$$u = x + y, \quad \frac{1}{z} = \frac{1}{x} + \frac{1}{y}$$

it is lemma 1. Its solution is, as it is proven

$$U = X_k = 0; \forall k \in \{1, 2, ..., i\}$$

But, why they are solutions for $n > i(i-1)$, $n_k > i(i-1)$? Let us suppose that

$$n = n_k = i(i-1)$$

the formulas become

$$x_{k,j} = x_k^{i^j} \prod_{l=0}^{l=j-1} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-(i-1)}$$

$$= U^{ni^j} X_k^{n_k i^j} \prod_{l=0}^{l=j-1} (U^{ni^l} X_1^{n_1 i^l} + U^{ni^l} X_2^{n_2 i^l} + ... + U^{ni^l} X_i^{n_i i^l})^{-(i-1)}$$

$$= U^{i(i-1)i^j} X_k^{i(i-1)i^j} \prod_{l=0}^{l=j-1} (U^{i(i-1)i^l} X_1^{i(i-1)i^l} + ... + U^{i(i-1)i^l} X_i^{i(i-1)i^l})^{-(i-1)}$$

$$= U^{(i-1)i^{j+1}} X_k^{(i-1)i^{j+1}} \prod_{l=0}^{l=j-1} (U^{(i-1)i^{l+1}} x_1^{(i-1)i^{l+1}} + ... + U^{(i-1)i^{l+1}} X_i^{(i-1)i^{l+1}})^{-(i-1)}$$

It is the formula of the $x_{k,j+1}$ of the exponent $i-1$. If we suppose that there are solutions for the exponent $i-1$, there will be no solutions for an exponent greater than $i(i-1)$. Returning to Beal equation, we conclude that effectively, after the generalization, there is no solution for $a > 2, b > 2, c > 2$! Hence, we must make an attention to the initial change of the data. For example, let the following equation

$$kU^n = X^n + Y^n$$

for some k integer like 7, there are solutions, for other cases like 2, there are not. It is too easy to pose

$$u = (kU^n)^2, \quad x = kU^n X^n, \quad y = kU^n Y^n, \quad z = X^n Y^n$$

We can verify that lemma 1 is satisfied

$$u = kU^n(X^n + Y^n) = x + y$$

$$\frac{1}{z} = \frac{k^2 U^{2n}}{kU^n X^n kU^n Y^n} = \frac{u}{xy} = \frac{1}{x} + \frac{1}{y}$$

Those formulas do not imply at all the solutions of the sequences. The correct solution is to pose

$$u = U^{2n}, \quad x = U^n X^n, \quad y = U^n Y^n, \quad z = X^n Y^n$$

Like this, the correct equations are

$$u = U^{2n} = \frac{kU^n(X^n + y^n)}{k^2} = \frac{x+y}{k}$$

and

$$\frac{1}{z} = \frac{U^{2n}}{U^n X^n U^n Y^n} = \frac{u}{xy} = \frac{x+y}{kxy} = \frac{1}{kx} + \frac{1}{ky}$$

4 Conclusion

The conclusion is that sequences, series and numbers have several applications in all diophantine equations, we saw some of them and there are many others like Pilai equation, Smarandache equation, the Catalan equation, etc...

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