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Serre Duality for Artin-Schelter Regular *K*-Categories

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Abstract. In this paper we continue the study of Artin Schelter regular categories, initiated by the authors in [7], [8], having in mind applications to the representation theory of finite dimensional algebras. We will prove the existence of Serre duality for these categories along the lines of [5]. As an application we will get that for a finite dimensional algebra the quotient category of the category of functors of a stable Auslander-Reiten component module the category of functors of finite length will be a category with Serre duality and homological dimension one.

In [5] it was implicitly assumed that all graded simple have the same projective dimension n and that there exists a fixed integer m such that for all graded simple generated in degree zero the n-transpose defined in [7] is generated in degree m. We will see here that this condition is not needed.

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In this paper we continue the study of Artin Schelter regular categories, initiated by the authors in [7, 8], having in mind applications to the representation theory of finite dimensional algebras. We will prove the existence of Serre duality for these categories along the lines of [5]. As an application we will get that for a finite dimensional algebra the category of functors of a stable Auslander-Reiten component module the functors of finite length satisfies Serre duality. In the case this category is of type $\mathbb{A}_\infty$, $\mathbb{A}_\infty^\infty$ or $\mathbb{D}_\infty$ we get a noetherian category with Serre duality and homological dimension one.

The paper will be divided in three sections, in the first one we recall basic definitions and we prove some homological properties of Artin-Schelter regular categories which will led us to a categorical version of the local cohomology theorem of commutative algebra. The second section will be dedicated to prove a corresponding version of Serre duality. In the last section we give the application to finite dimensional algebras.

1. A local cohomology theorem for Artin-Schelter regular categories

In this paper we will use freely the notions developed in [6, 7, 8]. We recall the definition of a graded category.

Definition 1.1. Let K be a field, $\mathcal{C}$ an additive K -category, we say $\mathcal{C}$ is graded if for each pair of objects, C and D we have a decomposition

$$\mathrm{Hom}_{\mathcal{C}}(C, D) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(C, D)_i$$

as Z- graded K -vector spaces, and if $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')_i$ and $g \in \mathrm{Hom}_{\mathcal{C}}(C', D)_j$, then $gf \in \mathrm{Hom}_{\mathcal{C}}(C, D)_{i+j}$. In particular the identity maps are in degree zero.

Example 1.2. Let $\Lambda = \oplus_{i\geq 0}\Lambda_i$ be a positively graded K-algebra. For a graded module M, denote by $M[i]$ the shifted module defined by $M[i]_j = M_{i+j}$. Denote by $\mathrm{Gr}(\Lambda)_0$ the category of graded modules and degree zero maps, and by $\mathrm{Gr}(\Lambda)$ the category of graded modules and maps given by

$$\mathrm{Hom}_{\mathrm{Gr}(\Lambda)}(M, N) = \oplus_{i\in Z} \mathrm{Hom}_{\mathrm{Gr}(\Lambda)_0}(M, N[i]).$$

Then $\mathrm{Gr}(\Lambda)$ is a graded category.

Example 1.3. Let $\mathcal{C}$ be an additive K-category and denote by rad the radical of $\mathcal{C}$ which we know is given by

$$\begin{aligned}\mathrm{rad}(C, D) &= \{f \in \mathrm{Hom}_{\mathcal{C}}(C, D) \mid gf \in \mathrm{rad}\,\mathrm{End}_{\mathcal{C}}(C) \text{ for all } g \in \mathrm{Hom}_{\mathcal{C}}(D, C)\}\\ &= \{f \in \mathrm{Hom}_{\mathcal{C}}(C, D) \mid fh \in \mathrm{rad}\,\mathrm{End}_{\mathcal{C}}(D) \text{ for all } h \in \mathrm{Hom}_{\mathcal{C}}(D, C)\}\end{aligned}$$

Hence, $\mathrm{rad}\,\mathcal{C} = \mathrm{rad}\,\mathcal{C}^{\mathrm{op}}$. Define inductively $\mathrm{rad}^n = \mathrm{rad}\cdot\mathrm{rad}^{n-1}$.

Then the associated category $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ has the same objects as $\mathcal{C}$ and maps given by

$$\mathrm{Hom}_{\mathcal{A}_{\mathrm{gr}}(\mathcal{C})}(C, D) = \oplus_{i\geq 0} \mathrm{rad}^i(C, D)/\mathrm{rad}^{i+1}(C, D).$$

Example 1.4. Let $\mathcal{C}$ be an abelian K-category. The Yoneda or Ext category $E(\mathcal{C})$ has the same objects as $\mathcal{C}$ and maps given by

$$\mathrm{Hom}_{E(\mathcal{C})}(A, B) = \oplus_{k\geq 0} \mathrm{Ext}^k_{\mathcal{C}}(A, B).$$

By K we will denote a fixed base field and by $\mathcal{C}$ a positively graded Krull-Schmidt K-category, generated in degrees zero and one, that is, the Jacobson radical $\mathrm{rad}(-,-)$ satisfies $\mathrm{rad}(-,-) = \oplus_{i>0} \mathrm{Hom}_{\mathcal{C}}(-,-)_i$. We assume further $\mathcal{C}$ is locally finite, this means that for each i and each pair of objects X and Y we have $\dim_K \mathrm{Hom}_{\mathcal{C}}(X,Y)_i < \infty$.

Denote by $\mathrm{Gr}(\mathcal{C})$ the category of contravariant graded functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Gr}K$, from $\mathcal{C}$ to the category of graded K-vector spaces. We saw in [6] that under these conditions there exists a contravariant functor $D\colon \mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$ given by

$$D(F)_i(X) = \mathrm{Hom}_K(F(X)_i, K)$$

and there exists a natural monomorphism $\eta\colon F \to D^2(F)$. Moreover, if we denote by $\mathrm{lfGr}(\mathcal{C})$, $\mathrm{lfGr}(\mathcal{C}^{\mathrm{op}})$ the full subcategories of locally finite functors of $\mathrm{Gr}(\mathcal{C})$ and $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$, respectively, then $D\colon \mathrm{lfGr}(\mathcal{C}) \to \mathrm{lfGr}(\mathcal{C}^{\mathrm{op}})$ is a duality and D sends finitely generated functors, to finitely cogenerated functors. projective to injective, simple functors to simple, and it preserve functors of finite length.

We denote by $\mathrm{Hom}_{\mathcal{C}}(Y,X)_i$ the vector space consisting of all maps in degree i and by $\mathrm{Hom}_{\mathcal{C}}(Y,X) = \oplus_{i\in Z} \mathrm{Hom}_{\mathcal{C}}(Y,Z)_i$, the graded vector space of all maps in all degrees, sometimes we will write $(-,X)$ instead of $\mathrm{Hom}_{\mathcal{C}}(-,X)$. In a similar way, $\mathrm{Ext}^k_{\mathcal{C}}(Y,X)_i$ is the vector space consisting of all k-extensions in degree i and $\mathrm{Ext}^k_{\mathcal{C}}(Y,X) = \oplus_{i\in Z} \mathrm{Ext}^k_{\mathcal{C}}(Y,Z)_i$ is the graded vector space of all k-extensions in all degrees.

Definition 1.5. Throughout this paper an *Artin-Schelter regular category* will be a graded K-category satisfying the above conditions and in addition

i) There is a positive integer n such that all simple functors S have projective dimension n.
ii) For each simple functor S and each object X in $\mathcal{C}$ and $\mathrm{Ext}^i_{\mathcal{C}}(S,(-,X)) = 0$ for $0 \leq i \leq n-1$.
iii) There is a bijection between the simple functors in $\mathrm{Gr}(\mathcal{C})$ and the simple functors in $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}})$ given by $S \to \mathrm{Ext}^n_{\mathcal{C}}(S,(-,-))$.

For the applications we have in mind, we are mostly interesting in $\mathcal{C}$ being the associated graded category of a stable component of the Auslander-Reiten quiver of a finite dimensional algebra.

Theorem 1.6. *Let $\mathcal{C}$ be an Artin-Schelter regular K-category, M a $\mathcal{C}$-functor of finite length. By $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{P}_{\mathcal{C}^{\mathrm{op}}}$ we will denote the categories of projective $\mathcal{C}$-functors, ($\mathcal{C}^{\mathrm{op}}$), respectively. Then there exists a functor $\sigma'\colon \mathcal{P}_{\mathcal{C}} \to \mathcal{P}_{\mathcal{C}^{\mathrm{op}}}$ and a natural isomorphism:* $\mathrm{Ext}^n_C(M,(-,X)) \simeq \mathrm{Hom}_C(M, D(\sigma'(-,X)))$ *for all $(-,X)$ in $\mathcal{P}_{\mathcal{C}}$.*

Proof. The proof will be divided in several steps.

Step 1: Assume X is indecomposable. We have in $\mathrm{Gr}(\mathcal{C})$ a minimal injective coresolution of $(-,X)$:

$$0 \to (-,X) \to I_0 \to I_1 \to \ldots I_{n-1} \to I_n \to 0$$

Let S_C be a simple corresponding to an indecomposable object $C \in \mathcal{C}$. By hypothesis, $\operatorname{Ext}^i_C(S,(-,X)) = 0$ for $0 \leq i \leq n-1$. Then

$$\operatorname{Hom}_C(S_C,(-,X)) = \operatorname{Hom}_C(S_C, I_0) = 0,$$

hence,

$$\operatorname{Hom}_C(S_C,\Omega^{-1}(-,X)) = \operatorname{Ext}^1_C(S,(-,X)) = 0.$$

By induction, $\operatorname{Hom}_C(S_C,\Omega^{-i}(-,X)) = 0$ for $0 \leq i \leq n-1$.

By the long homology sequence, the exact sequence

$$0 \to \Omega^{-1}(-,X) \to I_{n-1} \to I_n \to 0$$

induces an exact sequence.

$$0 \to \operatorname{Hom}_C(S_C,\Omega^{-n+1}(-,X)) \to \operatorname{Hom}_C(S_C, I_{n-1}) \to \\ \operatorname{Hom}_C(S_C, I_n) \to \operatorname{Ext}^n_C(S_c,(-,X)) \to 0$$

and

$$\operatorname{Hom}_C(S_C,\Omega^{-n+1}(-,X)) = \operatorname{Hom}_C(S_C, I_{n-1}) = 0.$$

It follows that $\operatorname{Hom}_C(S_C, I_n) = \operatorname{Ext}^n_C(S_c,(-,X)) \simeq \sigma(S_C)(X)[m_X]$, where σ is the bijection on the simple functors and $[m_X]$ is a shift of the simple functor $\sigma(S_C)$ depending on X. The bijection $S_C \to \sigma(S_C)$ induces a bijection $\sigma\colon \operatorname{ind}_{\mathcal{C}} \to \operatorname{ind}_{\mathcal{C}^{\mathrm{op}}}$ given by $\sigma(S_C) = \overline{S}_{\sigma(C)}$

Let σ' be the inverse of σ. Then we have: $\operatorname{Hom}_C(S_C, I_n) = \begin{cases} 0, & \text{if } X \neq \sigma C \\ K, & \text{if } X = \sigma C \end{cases}$

It follows that $\operatorname{Hom}_C(S_C, I_n) = \overline{S}_{\sigma' X}(C)[m_X]$.

Extend the definition of σ' to projective objects by letting

$$\sigma'(-X) = (\sigma' X-)[-m_X].$$

Therefore $I_n = D(\sigma' X, -)[m_X] \oplus I'_n$, where I'_n has zero socle.

It follows $\operatorname{Ext}^n_C(S_C,(-,X)) \simeq \operatorname{Hom}_C(S_C, D(\sigma'(-,X)))$.

Once we know the structure of I_n it follows, by induction, that for all functors of finite length M,

$$\operatorname{Ext}^n_C(M,(-,X)) \simeq \operatorname{Hom}_C(M, D(\sigma'(-,X))).$$

This completes the first step.

Step 2: We will prove that the map on projective objects σ' extends to a functor $\sigma'\colon \mathcal{P}_{\mathcal{C}} \to \mathcal{P}_{\mathcal{C}^{\mathrm{op}}}$.

Let $f\colon X \to Y$ be a map in degree k. We have a commutative exact diagram: *)

$$\begin{tikzcd}
0 \arrow[r] & (-,X) \arrow[r] \arrow[d, "(-,f)"] & I_0 \arrow[r] \arrow[d, "f_0"] & I_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \arrow[r] & I_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & D(\sigma'(-,X)) \oplus I'_n \arrow[r] \arrow[d, "f_n"] & 0 \\
0 \arrow[r] & (-,Y)[k] \arrow[r] & J_0[k] \arrow[r] & J_1[k] \arrow[r] & \cdots \arrow[r] & J_{n-1}[k] \arrow[r] & D(\sigma'(-,Y)) \oplus J'_n[k] \arrow[r] & 0
\end{tikzcd}$$

Since J_n' has no socle, $f_n = \Omega^{-n}(-,f)$ has triangular form: $f_n = \left(\begin{smallmatrix} \psi & \nu \\ 0 & \rho \end{smallmatrix}\right)$. The map ψ does not depend on the lifting. Let f_0',f_1', f_2', ... ,$f_n' = \left(\begin{smallmatrix} \psi' & \nu' \\ 0 & \rho' \end{smallmatrix}\right)$ be another lifting of $(-,f)$. Then f_0-f_0', f_1-f_1', f_2-f_2', ... , f_n-f_n' are homotopic to zero and $\left(\begin{smallmatrix} \psi-\psi' & \nu-\nu' \\ 0 & \rho-\rho' \end{smallmatrix}\right)$ factors through $J_{n-1}[k]$. Since $J_{n-1}[k]$ has no socle, it follows $\psi-\psi' = 0$. By duality, there exists a map $\varphi\colon \sigma'(-,Y)[-k] \to \sigma'(-,X)$ such that $D(\varphi) = \psi$. It is clear that the assignment $f \mapsto \varphi$ is functorial.

This induces a functor $\sigma'\colon \mathcal{P}_{\mathcal{C}} \to \mathcal{P}_{\mathcal{C}^{\mathrm{op}}}$ as follows.

On objects $\sigma'(-X) = (\sigma'X-)[-m_X]$ and given a map $(-,f)\colon (-X) \to (-,Y)[k]$, that is, $f\colon X \to Y$ a map in degree k, and $\sigma'(-f) = \varphi\colon \sigma'(-Y)[-k] \to \sigma'(-X)$. By Yoneda's Lemma, there exists a map $\sigma'(f)\colon \sigma'(X) \to \sigma'(Y)$ in degree $m_Y - m_X + k$, such that $(\sigma' f,-) = \varphi$.

Before continuing with the proof we need the following definition.

Definition 1.7. Let $\mathcal{C}$ be a small graded K-category and $\tau\colon obj\mathcal{C} \to \mathbb{Z}$, $\tau(X) = m_X$ a function from the objects of $\mathcal{C}$ to the integers, we call τ the twist and a category with twist $(\mathcal{C},\tau)$ to the category with the same objects as $\mathcal{C}$ and maps $\mathrm{Hom}_{(\mathcal{C},\tau)}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)[m_Y - m_X]$.

Step 3: We have induced a graded functor $\sigma'\colon \operatorname{ind}\mathcal{C} \to (\operatorname{ind}\mathcal{C},\tau), \operatorname{ind}\mathcal{C}$ the full subcategory of $\mathcal{C}$ consisting of indecomposable objects. We prove next that with this functor the isomorphisms given in Step 1, become natural equivalences.

Let $x \in \mathrm{Ext}^n_{\mathcal{C}}(M,(-,X))$ be a n-fold extension. We have an induced map of exact sequences: $**$)

$$\begin{tikzcd}
0 \arrow[r] & (-,X) \arrow[r] \arrow[d, equal] & E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\binom{\eta}{0}"] & 0 \\
0 \arrow[r] & (-,X) \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & \cdots \arrow[r] & I_{n-1} \arrow[r] & D(\sigma'X,-) \oplus I_n'[m_X] \arrow[r] & 0
\end{tikzcd}$$

where $\eta\colon M \to D(\sigma'X,-)[m_X]$ is a natural transformation.

Let $f\colon X \to Y$ be a map. As above we have the exact commutative diagram $*$). But we can also take the big pushout to obtain an exact commutative diagram: $\triangle$)

$$\begin{tikzcd}
x\colon 0 \arrow[r] & (-,X) \arrow[r] \arrow[d, "(-,f)"] & E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
y\colon 0 \arrow[r] & (-,Y) \arrow[r] & L_0 \arrow[r] & L_1 \arrow[r] & \cdots \arrow[r] & L_{n-1} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

y induces an exact commutative diagram: △△)

$$\begin{tikzcd}
0 \arrow[r] & (-,Y)[k] \arrow[r] \arrow[d, equal] & L_0 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & L_{n-1} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\binom{\gamma}{0}"] & 0 \\
0 \arrow[r] & (-,Y)[k] \arrow[r] & J_0[k] \arrow[r] & J_1[k] \arrow[r] & \cdots \arrow[r] & J_{n-1}[k] \arrow[r] & D(\sigma'Y,-) \oplus J'_n[m_Y+k] \arrow[r] & 0
\end{tikzcd}$$

composing the diagrams $*$) and $**$) we obtain the following commutative exact diagram.

$$\begin{tikzcd}
0 \arrow[r] & (-,X) \arrow[r] \arrow[d, "(-,f)"] & E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\binom{D(\sigma' f)\eta}{0}"] & 0 \\
0 \arrow[r] & (-,Y)[k] \arrow[r] & J_0[k] \arrow[r] & J_1[k] \arrow[r] & \cdots \arrow[r] & J_{n-1}[k] \arrow[r] & D(\sigma'Y,-) \oplus J'_n[m_Y+k] \arrow[r] & 0
\end{tikzcd}$$

Composing the diagrams △) and △△) we obtain the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & (-,X) \arrow[r] \arrow[d, "(-,f)"] & E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\binom{\gamma}{0}"] & 0 \\
0 \arrow[r] & (-,Y)[k] \arrow[r] & J_0[k] \arrow[r] & J_1[k] \arrow[r] & \cdots \arrow[r] & J_{n-1}[k] \arrow[r] & D(\sigma'Y,-) \oplus J'_n[m_Y+k] \arrow[r] & 0
\end{tikzcd}$$

By the uniqueness of the lifting we obtain: $\gamma = D(\sigma' f)\eta$. We have proved the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Ext}^n_{\mathcal{C}}(M,(-,X)) \arrow[r, equal] \arrow[d, "\operatorname{Ext}^n_{\mathcal{C}}(M,(-,f))"] & \operatorname{Hom}_{\mathcal{C}}(M, D(\sigma'(-,X))) \arrow[d, "\operatorname{Hom}_{\mathcal{C}}(M,D(\sigma'(-,f)))"] \\
\operatorname{Ext}^n_{\mathcal{C}}(M,(-,Y))[k] \arrow[r, equal] & \operatorname{Hom}_{\mathcal{C}}(M, D(\sigma'(-,Y))[k]
\end{tikzcd}$$

Let $\beta\colon N \to M$ be a natural transformation between functors of finite length. Taking the big pull back βx of x by β we obtain a diagram that we can glue with $**$) as follows.

$$\begin{tikzcd}
\beta x\colon 0 \arrow[r] & (-,X) \arrow[r] \arrow[d, equal] & F_0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & F_{n-1} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\beta"] & 0 \\
x\colon 0 \arrow[r] & (-,X) \arrow[r] \arrow[d, equal] & E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\binom{\eta}{0}"] & 0 \\
0 \arrow[r] & (-,X) \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & \cdots \arrow[r] & I_{n-1} \arrow[r] & D(\sigma'X,-) \oplus I'_n[m_X] \arrow[r] & 0
\end{tikzcd}$$

The composition of diagrams has in the last column the map $\binom{\eta\beta}{0}$. We obtain the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Ext}^n_{\mathcal{C}}(M,(-,X)) \arrow[r, equal] \arrow[d, "\mathrm{Ext}^n_{\mathcal{C}}(\beta,(-,X))"] & \mathrm{Hom}_{\mathcal{C}}(M,D(\sigma'(-,X))) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(\beta,D(\sigma'(-,X)))"] \\
\mathrm{Ext}^n_{\mathcal{C}}(N,(-,X)) \arrow[r, equal] & \mathrm{Hom}_{\mathcal{C}}(N,D(\sigma'(-,X))
\end{tikzcd}$$

Proving the naturality of the isomorphism in Step 1. In a similar way we consider covariant functors of finite length $N\colon \mathcal{C} \to \mathrm{Gr}K$ and obtain a natural equivalence

$$\mathrm{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(N,(-,X)) \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(N,D(\sigma(-,X)).$$

$\square$

Now we prove the following.

Proposition 1.8. *For any functor of finite length M there exists a natural equivalence $\rho_M\colon M \to M\sigma'\sigma$.*

Proof. Since for every simple functor S_c the functor $\mathrm{Ext}^n_{\mathcal{C}}(S_c,(-,-))$ is simple, for any functor of finite length M, the functor $\mathrm{Ext}^n_{\mathcal{C}}(M,(-,-))$ is of finite length. It follows for any functor of finite length M,

$$\mathrm{Ext}^i_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Ext}^n_{\mathcal{C}}(M,(-,-),(-,-))) = 0$$

for $0 \le i \le n-1$ and $\mathrm{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Ext}^n_{\mathcal{C}}(M,(-,-),(-,-))) \simeq M$.

The functor $\sigma\colon \mathcal{P}_{\mathcal{C}^{\mathrm{op}}} \to \mathcal{P}_{\mathcal{C}}$ is an equivalence. The isomorphism $\mathrm{Ext}^n_{\mathcal{C}}(S_c,(-,X)) \simeq \overline{S}_{\sigma'C}[m_X](X)$ induces an isomorphism.

$$\begin{aligned}\mathrm{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Ext}^n_{\mathcal{C}}(S_C,(-,-)),(-,-)) &\simeq \mathrm{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\overline{S}_{\sigma'C}[m_X],(-,-)) \\ &\simeq S_{\sigma\sigma'C}[m_X][m'_X] \simeq S_C\end{aligned}$$

It follows that the twist $\tau'\colon \mathrm{obj}\,\mathcal{C} \to \mathbb{Z}$ corresponding to σ is $-\tau$, where $\tau\colon \mathrm{obj}\,\mathcal{C} \to \mathbb{Z}$ is the twist corresponding to σ'.

We have functors $\sigma'\colon \mathrm{ind}\,\mathcal{C} \to (\mathrm{ind}\,\mathcal{C},\tau)$ and $\sigma\colon \mathrm{ind}\,\mathcal{C} \to (\mathrm{ind}\,\mathcal{C},\tau')$, composing them we obtain a functor $\sigma\sigma'\colon \mathrm{ind}\,\mathcal{C} \to \mathrm{ind}\,\mathcal{C}$ and we like to prove that it is isomorphic to the identity.

For every functor M of finite length we have natural equivalences.

$$M \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Hom}_{\mathcal{C}}(M,D(\sigma'(-,*)),D(\sigma(-,-)).$$

Evaluating in an indecomposable object X gives

$$\begin{aligned}
M(X) &\simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Hom}_{\mathcal{C}}(M, D(\sigma'(-,*)), D(\sigma X,-))[-m_X]) \\
&\simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Hom}_{\mathcal{C}}(M, D(\sigma'(-,*)), D(\sigma X,-)))[-m_X] \\
&\simeq \mathrm{Hom}_{\mathcal{C}}((\sigma X,-), D\,\mathrm{Hom}_{\mathcal{C}}(M, D(\sigma'(-,*)),)[-m_X] \\
&\simeq D\,\mathrm{Hom}_{\mathcal{C}}(M, D((-,\sigma'\sigma X))[-m_X])[-m_X] \\
&\simeq D\,\mathrm{Hom}_{\mathcal{C}}(M, D((-,\sigma'\sigma X)))[m_X][-m_X] \\
&\simeq D\,\mathrm{Hom}_{\mathcal{C}}(M, D((-,\sigma'\sigma X)) \\
&\simeq D\,\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(((-,\sigma'\sigma X)), D(M)) \\
&\simeq D^2(M)(\sigma'\sigma X) \simeq M(\sigma'\sigma X)
\end{aligned}$$

Therefore we obtain that $M \simeq M\sigma'\sigma$. We have proved that for any functor of finite length M there is a natural equivalence $\rho_M\colon M \to M\sigma'\sigma$.

In a similar way, for any functor of finite length $N\colon \mathcal{C} \to \mathrm{Gr}K$, there is a natural equivalence $\rho'_N\colon N \to N\sigma\sigma'$. □

The above proposition can be strengthen as follows.

Proposition 1.9. *The family of maps $\rho = \{\rho_M\}$ give a natural isomorphism $\rho\colon 1 \to \sigma'\sigma$ in the category of functors of finite length.*

Proof. Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ and $0 \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to N \to 0$ be a minimal graded projective resolutions of the modules of finite length M and N, respectively and $\eta\colon M \to N$ a natural transformation. It induces the following commutative exact diagram.

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[d, "\eta_n"] & P_{n-1} \arrow[r] \arrow[d, "\eta_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\eta_1"] & P_0 \arrow[r] \arrow[d, "\eta_0"] & M \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Each P_i is of the form, $P_i = \mathrm{Hom}_{\mathcal{C}}(-, C_i)$. We denote by P_i^* to the projective $\mathrm{Hom}(\mathrm{Hom}_{\mathcal{C}}(-, C_i), \mathrm{Hom}_{\mathcal{C}}(-, C-)) \simeq \mathrm{Hom}_{\mathcal{C}}(C_i, -)$. After dualizing we obtain the following commutative exact diagram.

$$\begin{tikzcd}
0 \arrow[r] & Q_0^* \arrow[r] \arrow[d, "\eta_0^*"] & Q_1^* \arrow[r] \arrow[d, "\eta_1^*"] & \cdots \arrow[r] & Q_{n-1} \arrow[r] \arrow[d, "\eta_{n-1}^*"] & Q_n \arrow[r] \arrow[d, "\eta_n^*"] & \mathrm{Ext}^n_{\mathcal{C}}(N, (-,-)) \arrow[r] \arrow[d, "\mathrm{Ext}^n_{\mathcal{C}}(\eta,(-,-))"] & 0 \\
0 \arrow[r] & P_0^* \arrow[r] & P_1^* \arrow[r] & \cdots \arrow[r] & P_{n-1}^* \arrow[r] & P_n^* \arrow[r] & \mathrm{Ext}^n_{\mathcal{C}}(M, (-,-)) \arrow[r] & 0
\end{tikzcd}$$

Dualizing again we obtain the following commutative exact diagram.

$$\begin{tikzcd}
0 \arrow[r] & P_n \arrow[r] \arrow[d, "\eta_n"] & P_{n-1} \arrow[r] \arrow[d, "\eta_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\eta_1"] & P_0 \arrow[r] \arrow[d, "\eta_0"] & \mathrm{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Ext}^n_{\mathcal{C}}(M,(-,-)),(-,-)) \arrow[r] \arrow[d, "\mathrm{Ext}^n(\mathrm{Ext}^n(\eta,(-,-)),(-,-))"] & 0 \\
0 \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & \mathrm{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\mathrm{Ext}^n_{\mathcal{C}}(N,(-,-)),(-,-)) \arrow[r] & 0
\end{tikzcd}$$

Hence we have isomorphisms θ making the diagram below commute.

$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[d, "\eta"] & \operatorname{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\operatorname{Ext}^n_{\mathcal{C}}(M,(-,-)),(-,-)) \arrow[d, "\operatorname{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\operatorname{Ext}^n_{\mathcal{C}}(\eta,(-,-)),(-,-))"] \\ N \arrow[r, "\theta"] & \operatorname{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\operatorname{Ext}^n_{\mathcal{C}}(N,(-,-)),(-,-)) \end{tikzcd}$$

But the isomorphisms $\operatorname{Ext}^n_{\mathcal{C}^{\mathrm{op}}}(\operatorname{Ext}^n_{\mathcal{C}}(M,(-,-)),(-,-)) \simeq M\sigma'\sigma$ are natural. Therefore we obtain that $\rho\colon 1 \to \sigma'\sigma$ is a natural isomorphism in the category of functors of finite length. □

Remark 1.10. Similarly, $\rho'\colon 1 \to \sigma\sigma'$ is a natural isomorphism in the category of functors of finite length.

We use the fact that graded categories are complete, as claimed in the following result. The proof is a natural extension of the one given in [5], and we leave it to the reader.

Lemma 1.11. *Let $\mathcal{C}$ be a positively graded K-category and M a bounded below functor. Then there is a natural isomorphism: $\varprojlim M/M_{\geq k} \simeq M$.*

Corollary 1.12. *Let $\mathcal{C}$ be an Artin-Schelter regular category. Then the isomorphism $\rho\colon 1 \to \sigma'\sigma$ given in previous proposition extends to an isomorphism in the category of locally finite bounded below functors.*

Proof. The isomorphisms $\theta_j\colon M/M_{\geq j} \to M/M_{\geq j}\sigma'\sigma$ obtained in the proposition are natural, by the above lemma, they induce isomorphisms in the inverse limits: $\theta\colon M \to M\sigma'\sigma$. □

Proposition 1.13. *The functors*

$$\sigma\colon \operatorname{ind}\mathcal{C} \to (\operatorname{ind}\mathcal{C}, \tau')$$

and

$$\sigma'\colon \operatorname{ind}\mathcal{C} \to (\operatorname{ind}\mathcal{C}, \tau),$$

induce inverse equivalences

$$\operatorname{ind}\mathcal{C} \xrightarrow{\sigma} (\operatorname{ind}\mathcal{C}, \tau') \xrightarrow{\sigma'} \operatorname{ind}\mathcal{C}$$

and

$$\operatorname{ind}\mathcal{C} \xrightarrow{\sigma'} (\operatorname{ind}\mathcal{C}, \tau) \xrightarrow{\sigma} \operatorname{ind}\mathcal{C}.$$

Proof. By the above corollary, for any indecomposable module X we have a natural isomorphism: $\theta\colon (-,X) \to (-X)\sigma'\sigma$, where θ is a degree zero map.

Then $\theta_X\colon (X,X) \to (\sigma\sigma' X, X)$ sends 1_X to a map $\theta_X(1_X)\colon \sigma'\sigma X \to X$, but since we are assuming $\operatorname{rad}(-,-) = \oplus_{i\geq 1}\operatorname{Hom}_{\mathcal{C}}(-,-)_i$ and $\sigma\sigma' X$ is indecomposable, it follows $\theta_X(1_X) = u$ is an isomorphism.

Let $f\colon X \to Y$ be a degree i map. We have the following commutative diagram.

$$\begin{tikzcd}
(X,X) \arrow[r, "\theta_X"] \arrow[d, "{(f,X)}"] & (\sigma\sigma' X, X) \arrow[d, "{(\sigma\sigma' f, X)}"] \\
(X,Y) \arrow[r, "\theta_Y"] & (\sigma\sigma' Y, X)
\end{tikzcd}$$

Then $\theta_Y(f) = u\sigma'\sigma(f)$, hence $\sigma'\sigma(f) = 0$ implies $\theta_Y(f) = 0$, and θ_Y an isomorphism implies $f = 0$. We have proved $\sigma'\sigma$ is faithful.

Let $g\colon \sigma'\sigma Y \to \sigma'\sigma X$ be a map. Taking the composition ug and using the fact θ_Y is onto, we find a map $f\colon Y \to X$ with $\theta_Y(f) = ug = u\sigma'\sigma(f)$. But u an isomorphism implies $g = \sigma'\sigma(f)$. We have proved $\sigma'\sigma$ is an equivalence. □

We need the following.

Lemma 1.14. *Let $\mathcal{C}$ be any category and M a functor of projective dimension n, and assume M has a minimal projective resolution consisting of finitely generated projective functors. Then there is a natural isomorphism*

$$\operatorname{Ext}^n_{\mathcal{C}}(M, L) \simeq \operatorname{Ext}^n_{\mathcal{C}}(M, (-,-)) \otimes_{\mathcal{C}} L.$$

Proof. By dimension shift, $\operatorname{Ext}^n_{C}(M, L) \simeq \operatorname{Ext}^n_{\mathcal{C}}(\Omega^{n-1}M, L)$. Let

$$0 \to P_n \to P_{n-1} \to \Omega^{n-1}M \to 0$$

be a minimal projective presentation of $\Omega^{n-1}M$. The long homology sequence induces the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{C}}(\Omega^{n-1}M, L) \to \operatorname{Hom}_{\mathcal{C}}(P_{n-1}, L) \to \operatorname{Hom}_{\mathcal{C}}(P_n, L) \to \operatorname{Ext}^n_{\mathcal{C}}(M, L) \to 0$$

But $P_{n-1} = \operatorname{Hom}_{\mathcal{C}}(-, C_{n-1})$ and $P_n = \operatorname{Hom}_{\mathcal{C}}(-, C_n)$. Then we have natural isomorphisms

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(P_{n-1}, L) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathcal{C}}(P_n, L) \arrow[d, equal] \\
L(C_{n-1}) \arrow[r] \arrow[d, equal] & L(C_n) \arrow[d, equal] \\
\operatorname{Hom}_{\mathcal{C}}(C_{n-1}, -) \otimes_{\mathcal{C}} L \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(C_n, -) \otimes_{\mathcal{C}} L
\end{tikzcd}$$

We get an isomorphism of exact sequences:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(P_{n-1}, L) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathcal{C}}(P_n, L) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^n_{\mathcal{C}}(M, L) \arrow[r] \arrow[d, equal] & 0 \\
P^*_{n-1} \otimes_{\mathcal{C}} L \arrow[r] & P^*_n \otimes_{\mathcal{C}} L \arrow[r] & \operatorname{Ext}^n_{\mathcal{C}}(M, (-,-)) \otimes_{\mathcal{C}} L \arrow[r] & 0
\end{tikzcd}$$

and the claim follows. □

We can prove our first important theorem

Theorem 1.15 (Local Cohomology). *Let $\mathcal{C}$ be an Artin Schelter regular category of global dimension n. Then for any locally bounded below functor $M\colon \mathrm{Gr}\mathcal{C}^{\mathrm{op}} \to \mathrm{Gr}K$ and any graded functor $L\colon \mathrm{Gr}\mathcal{C}^{\mathrm{op}} \to \mathrm{Gr}K$ there exists natural isomorphisms*

$$\mathrm{Hom}_K(\varinjlim \mathrm{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, L), K) \simeq \mathrm{Ext}^i_{\mathcal{C}}(L, M\sigma')[\tau].$$

For $0 \leq i < n$, $\sigma'\colon \operatorname{ind}\mathcal{C}^{\mathrm{op}} \to (\operatorname{ind}\mathcal{C}^{\mathrm{op}}, \tau)$ is the isomorphism induced by σ' in the opposite category and τ is the twist.

Proof. By Lemma 1.14, there are isomorphisms:

$$\mathrm{Ext}^n_{\mathcal{C}}(M/M_{\geq k}, L) \simeq \mathrm{Ext}^n_{\mathcal{C}}(M/M_{\geq k}, (-,-)) \underset{\mathcal{C}}{\otimes} L$$

and

$$\begin{aligned}\mathrm{Ext}^n_{\mathcal{C}}(M/M_{\geq k}, (-,X)) &\simeq \mathrm{Hom}_{\mathcal{C}}(M/M_{\geq k}, D(\sigma'(-,X))\\ &\simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}((\sigma'(-,X), D(M/M_{\geq k}))\\ &\simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(((-,\sigma'X)[-m_X], D(M/M_{\geq k}))\\ &\simeq D(M/M_{\geq k}(\sigma'X))[m_X]\end{aligned}$$

Therefore we have that $\mathrm{Ext}^n_{\mathcal{C}}(M/M_{\geq k}, L) \simeq D(M/M_{\geq k}\sigma') \otimes_{\mathcal{C}} L[\tau']$, with τ' the twist. Then we have an isomorphism

$$\varinjlim{}_k \mathrm{Ext}^n_{\mathcal{C}}(M/M_{\geq k}, L) \simeq \varinjlim{}_k D(M/M_{\geq k}\sigma') \otimes_{\mathcal{C}} L[\tau'].$$

From this isomorphism it follows there exists a chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_K(\varinjlim{}_k \mathrm{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, L), K) &\simeq \mathrm{Hom}_{\mathcal{C}}(L[\tau'], \mathrm{Hom}_K(\varinjlim{}_k D(M/M_{\geq k}\sigma'), K)\\ &\simeq \mathrm{Hom}_{\mathcal{C}}(L[\tau'], \varprojlim{}_k D^2(M/M_{\geq k}\sigma'))\\ &\simeq \mathrm{Hom}_{\mathcal{C}}(L[\tau'], \varprojlim{}_k (M/M_{\geq k}\sigma'))\\ &\simeq \mathrm{Hom}_{\mathcal{C}}(L[\tau'], M\sigma'))\\ &\simeq \mathrm{Hom}_{\mathcal{C}}(L, M\sigma'[\tau])\end{aligned}$$

We have proved the claim for $n = 0$.

Assume the result is true for $0 \leq i < n$. Then

$$\mathrm{Hom}_K(\varinjlim \mathrm{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, L), K) \simeq \mathrm{Ext}^i_{\mathcal{C}}(L, M\sigma')[\tau].$$

Let $0 \to \Omega L \to P \to L \to 0$ be exact with P, a not necessary, finitely generated projective. By the long homology sequence we have the following exact sequence

$$\begin{aligned}&\mathrm{Ext}^{n-i-1}_{\mathcal{C}}(M/M_{\geq k}, P) \to \mathrm{Ext}^{n-i-1}_{\mathcal{C}}(M/M_{\geq k}, L) \to\\ &\qquad\qquad \mathrm{Ext}^{n-i-1}_{\mathcal{C}}(M/M_{\geq k}, \Omega L) \to \mathrm{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, P)\end{aligned}$$

But $\operatorname{Ext}^{n-i-1}_{\mathcal{C}}(M/M_{\geq k}, P) = 0$, since $M/M_{\geq k}$ is of finite length. Taking limits and dualizing we obtain the following exact sequence

$$D(\varinjlim_k \operatorname{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, P)) \to D(\varinjlim_k \operatorname{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, \Omega L)) \to D(\varinjlim_k \operatorname{Ext}^{n-i-1}_{\mathcal{C}}(M/M_{\geq k}, L)) \to 0.$$

For each $i \geq 0$, we have the following exact sequence

$$\operatorname{Ext}^{i}_{\mathcal{C}}(P, M\sigma')[\tau] \to \operatorname{Ext}^{i}_{\mathcal{C}}(\Omega L, M\sigma')[\tau] \to \operatorname{Ext}^{i+1}_{\mathcal{C}}(L, M\sigma')[\tau] \to 0$$

By induction hypothesis we have natural isomorphisms

$$D(\varinjlim \operatorname{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, P)) \simeq \operatorname{Ext}^{i}_{\mathcal{C}}(P, M\sigma')[\tau]$$

and

$$D(\varinjlim \operatorname{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, \Omega L)) \simeq \operatorname{Ext}^{i}_{\mathcal{C}}(\Omega L, M\sigma')[\tau].$$

It follows we have natural isomorphisms

$$D(\varinjlim \operatorname{Ext}^{n-i}_{\mathcal{C}}(M/M_{\geq k}, L)) \simeq \operatorname{Ext}^{i}_{\mathcal{C}}(L, M\sigma')[\tau]$$

as claimed. □

The next section is dedicated to prove a version of Serre duality for Artin-Schelter regular categories.

2. Quotient Categories

By $\mathcal{C}$ we denote a positively graded locally finite Krull Schmidt K-category with radical $r(-,-) = \oplus_{i\geq 1} \operatorname{Hom}_{\mathcal{C}}(-,-)_i$.

As we remarked before, $\operatorname{Gr}(\mathcal{C})$ is abelian with enough projective and injective objects. We denote by $\operatorname{Fin}\mathcal{C}$ the full subcategory of $\operatorname{Gr}(\mathcal{C})$ consisting of all functors with finite minimal projective resolutions consisting of finitely generated projectives. We assume further that the simple functors are in $\operatorname{Fin}\mathcal{C}$. Observe that these conditions are satisfied, both in the Artin-Schelter regular and in the Koszul cases.

By Tors we denote the full subcategory of $\operatorname{Gr}(\mathcal{C})$ of all torsion functors, that is, functors F with $t(F) = F$ and $t(F) = \sum_{L\in\mathcal{L}} L$, where $\mathcal{L} = \{L \subset F \mid L \text{ of finite length}\}$.

We proved in [7], that t is an idempotent radical and it is easy to see Tors forms a Serre category, this means: given an exact sequence of functors $0 \to L \to M \to N \to 0$, the functor $M \in \operatorname{Tors}$ if and only if L and N are in Tors.

We consider the quotient category: $\operatorname{QGr}(\mathcal{C}) = \operatorname{Gr}(\mathcal{C})/\operatorname{Tors}$. The category has the same objects as $\operatorname{Gr}(\mathcal{C})$ and maps

$$\operatorname{Hom}_{\operatorname{QGr}(\mathcal{C})}(F, G) = \varinjlim_{(F',G')\in\mathcal{L}} \operatorname{Hom}_{\mathcal{C}}(F', G/G')$$

where $\mathcal{L} = \{(L, M) \mid L \subset F, M \subset G \text{ and } F/L, M \text{ torsion}\}$.

Let $\pi\colon \operatorname{Gr}(\mathcal{C}) \to \operatorname{QGr}(\mathcal{C})$ be the canonical projection. Then it is known (see [3, 9]) that $\operatorname{QGr}(\mathcal{C})$ is abelian with enough injectives and π is exact. In fact, if I is a torsion free injective, then $\pi(I)$ is injective. The Ext-functors are the derived functors of $\mathrm{Hom}_{\mathrm{QGr}(\mathcal{C})}(F,-)$.

The set $\mathcal{L}' = \{(F\prime, t(G) \mid F/F' \text{ is torsion}\}$ is cofinal in $\mathcal{L}$, then

$$\mathrm{Hom}_{\mathrm{QGr}(\mathcal{C})}(F,G) = \varinjlim{}_{(F',t(G))\in\mathcal{L}'} \mathrm{Hom}_{\mathcal{C}}(F', G/t(G)).$$

Lemma 2.1. *Let* $\mathrm{Gr}(\mathcal{C})$ *be a Krull Schmidt category and assume all simple objects are in* $\mathrm{Fin}\,\mathcal{C}$*. Then for any finitely generated functor* M *in* $\mathrm{Fin}\,\mathcal{C}$ *and any* k*,* $M_{\geq k}$ *is also in* $\mathrm{Fin}\,\mathcal{C}$*.*

Proof. We have the following exact commutative diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \Omega(M) \arrow[r] \arrow[d] & \Omega(M/M_{\geq k}) \arrow[r] \arrow[d] & M_{\geq k} \arrow[r] & 0 \\
 & & P \arrow[r, equal] \arrow[d] & P \arrow[d] & & \\
0 \arrow[r] & M_{\geq k} \arrow[r] & M \arrow[r] \arrow[d] & M/M_{\geq k} \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

The module $M_{\geq k}$ is finitely generated, by the Horseshoe Lemma, we have the following exact sequence

$$0 \to \Omega^2 M \to \Omega^2(M/M_{\geq k}) \oplus Q \to \Omega(M_{\geq k}) \to 0,$$

with Q a finitely generated projective, hence, $\Omega(M_{\geq k})$ is finitely generated. It follows by induction, $M_{\geq k}$ has a projective resolution consisting of finitely generated projective. □

Lemma 2.2. *Assume* $\mathcal{C}$ *satisfies the conditions above. Let* $M \in \mathrm{Gr}(\mathcal{C})$ *be a torsion functor and assume* N *is in* $\mathrm{Fin}\,\mathcal{C}$*. Then* $\varinjlim_n \mathrm{Hom}_{\mathcal{C}}(N_{\geq n}, M) = 0$*.*

Proof. Changing $M[k]$ for M, it will be enough to prove

$$\varinjlim \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, M) = 0.$$

By Lemma 2.1, any truncation $N_{\geq k}$ is in $\mathrm{Fin}\,\mathcal{C}$, on the other hand, $M = \varinjlim M_\alpha$, with M_α a functor of finite length.

Then

$$\mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, \varinjlim M_\alpha) = \varinjlim \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, M_\alpha).$$

Therefore we have that

$$\begin{aligned}
\varinjlim{}_n \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, M) &= \varinjlim{}_n \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, \varinjlim{}_\alpha M_\alpha) \\
&= \varinjlim{}_n \varinjlim{}_\alpha \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, M_\alpha) \\
&= \varinjlim{}_\alpha \varinjlim{}_n \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, M_\alpha) = 0
\end{aligned}$$

This completes the proof. □

Proposition 2.3. *Let $\mathcal{C}$ be a locally finite positively graded Krull-Schmidt category with radical $r(-.-) = \underset{i\geq 1}{\oplus} \operatorname{Hom}_{\mathcal{C}}(-.-)_i$. If N is finitely generated and M an arbitrary functor, then*

$$\operatorname{Hom}_{\mathrm{QGr}(\mathcal{C})}(\pi N, \pi M) = \varinjlim \operatorname{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, M/t(M))).$$

Furthermore, if we assume all graded simple are in $\operatorname{Fin}\mathcal{C}$, *then we have natural isomorphisms*

$$\varinjlim \operatorname{Ext}^t_{\mathcal{C}}(N_{\geq k}, M) = \varinjlim \operatorname{Ext}^t_{\mathcal{C}}(N_{\geq k}, M/t(M))$$

for all $t \geq 0$.

Proof. Let N' be a subfunctor of N such that N/N' is torsion. Since N is finitely generated, N/N' is actually of finite length and $(N/N')_{\geq k} = 0$ for large enough k. This implies $N_{\geq k} \subset N'$ for large enough k. We have proved that the set $\{(N_{\geq k},\, t(M))\}$ is cofinal in $\mathcal{L}'$. Therefore we have that

$$\operatorname{Hom}_{\mathrm{QGr}(\mathcal{C})}(\pi N, \pi M) = \varinjlim \operatorname{Hom}_{\mathrm{Gr}(\mathcal{C})}(N_{\geq n}, M/t(M))).$$

We proved in Lemma 2.1, that for any k, the functor $N_{\geq k}$ is in $\operatorname{Fin}\mathcal{C}$. For each k we have an exact sequence of finitely generated objects

$$*) \quad 0 \to \Omega^t(N_{\geq k}) \to P^{(k)}_{t-1} \to ...P^{(k)}_1 \to P^{(k)}_0 \to N_{\geq k} \to 0.$$

For any $P^{(k)}_s$ and $j < k$, we have $(P^{(k)}_s)_j = 0$. Hence, for $j < k$ we have $(\Omega^t(N_{\geq k}))_j = 0$. The inclusion map $N_{\geq k} \to N_{\geq k-1}$ it induces maps of projective functors $P^{(k)}_s \to P^{(k-1)}_s$ and maps $\Omega^s(N_{\geq k}) \to \Omega^s(N_{\geq k-1})$. The short exact sequence

$$0 \to \Omega^t(N_{\geq k}) \to P^{(k)}_{t-1} \to \Omega^{t-1}(N_{\geq k}) \to 0$$

induces the following exact sequence of direct systems

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{C}}(\Omega^{t-1}(N_{\geq k}), t(M)) \to \operatorname{Hom}_{\mathcal{C}}(P^{(k)}_{t-1}, t(M)) \to \\ \operatorname{Hom}_{\mathcal{C}}(\Omega^t(N_{\geq k}), t(M)) \to \operatorname{Ext}^t_{\mathcal{C}}(N_{\geq k}, t(M)) \to 0 \end{aligned}$$

Since direct limit of directed system is exact, we obtain the epimorphism

$$\varinjlim{}_k \operatorname{Hom}_{\mathcal{C}}(\Omega^t(N_{\geq k}), t(M)) \to \varinjlim{}_k \operatorname{Ext}^t_{\mathcal{C}}(N_{\geq k}, t(M)) \to 0.$$

By an argument similar to the one used in Lemma 2.2,

$$\varinjlim{}_k \operatorname{Hom}_{\mathcal{C}}(\Omega^t(N_{\geq k}), t(M)) = 0.$$

It follows that $\varinjlim{}_k \operatorname{Ext}^t_{\mathcal{C}}(N_{\geq k}, t(M)) = 0$. Therefore we have that

$$\varinjlim{}_k \operatorname{Ext}^t_{\mathcal{C}}(N_{\geq k}, M) = \varinjlim{}_k \operatorname{Ext}^t_{\mathcal{C}}(N_{\geq k}, M/t(M)).$$

□

Proposition 2.4. *Let $\mathcal{C}$ be a positively graded Krull Schmidt locally finite K-category with radical $r(-,-) = \oplus_{i\geq 1} \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(-,-)_i$. Assume that all graded simple are in $\mathrm{Fin}\,\mathcal{C}$. Then given functors M, N in $\mathrm{Fin}\,\mathcal{C}$, we have an isomorphism $\pi M \simeq \pi N$ if and only if there exists some integer n such that $M_{\geq n} \simeq N_{\geq n}$.*

Proof. Since $\pi M \simeq \pi M_{\geq n}$ and $\pi N \simeq \pi N_{\geq n}$, then it is clear that $M_{\geq n} \simeq N_{\geq n}$ implies $\pi M \simeq \pi N$.

Assume $\pi M \simeq \pi N$. Then it is clear that there is a map $f\colon N_{\geq k} \to M$ such that $\pi(f)$ is an isomorphism. It follows that $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are torsion, $M/\,\mathrm{Im}\, f$ of finite length implies that for some integer ℓ the restriction $f\colon N_{\geq \ell} \to M_{\geq \ell}$ is an epimorphism. Since $N_{\geq \ell}$ and $M_{\geq \ell}$ are in $\mathrm{Fin}\,\mathcal{C}$ they are in particular finitely presented and it follows $\mathrm{Ker}\, f$ is of finite length. Then there exists some n such that $f\colon N_{\geq n} \to M_{\geq n}$ is an isomorphism. $\square$

Proposition 2.5. *Let $\mathcal{C}$ be a positively graded Krull-Schmidt locally finite K-category with radical $r(-,-) = \oplus_{i\geq 1} \mathrm{Hom}_{\mathrm{Gr}(\mathcal{C})}(-,-)_i$. Assume all graded simple are in $\mathrm{Fin}\,\mathcal{C}$. If L is a functor in $\mathrm{Fin}\,\mathcal{C}$ then for any functor M there are isomorphisms*

$$\mathrm{Ext}^k_{\mathrm{QGr}(\mathcal{C})}(\pi L, \pi M) = \varinjlim_t \mathrm{Ext}^k_{\mathcal{C}}(L_{\geq t}, M).$$

Proof. We prove the claim by induction on k. The case $k = 0$ has been already proved. Assume $\mathrm{Ext}^{k-1}_{\mathrm{QGr}(\mathcal{C})}(\pi L, \pi M) = \varinjlim_t \mathrm{Ext}^{k-1}_{\mathcal{C}}(L_{\geq t}, M)$. The injective envelope of a torsion free functor is torsion free. Let

$$*)0 \to M/t(M) \to I \to \Omega^{-1}(M/t(M)) \to 0$$

be exact with I the injective envelope of $M/t(M)$. Applying π we have the following exact sequence.

$$**) \qquad 0 \to \pi(M/t(M)) \to \pi(I) \to \pi(\Omega^{-1}(M/t(M))) \to 0.$$

It follows by [9] that $\pi(I)$ is injective. The exact sequence $**)$ induces by the long homology sequence the following exact sequence.

$$\mathrm{Ext}^{k-1}_{\mathrm{QGr}(\mathcal{C})}(\pi L, \pi I) \to \mathrm{Ext}^{k-1}_{\mathrm{QGr}(\mathcal{C})}(\pi L, \pi(\Omega^{-1}(M)/t(M))) \to \\ \mathrm{Ext}^{k}_{\mathrm{QGr}(\mathcal{C})}(\pi L, \pi(M/t(M))) \to 0.$$

Applying the long homology sequence to $*)$ and taking direct limits we obtain the exact sequence

$$\varinjlim_t \mathrm{Ext}^{k-1}_{\mathcal{C}}(L_{\geq t}, I) \to \varinjlim_t \mathrm{Ext}^{k-1}_{\mathcal{C}}(L_{\geq t}, \Omega^{-1}(M/t(M)) \to \\ \varinjlim_t \mathrm{Ext}^{k-1}_{\mathcal{C}}(L_{\geq t}, M/t(M)) \to 0.$$

By induction we have natural isomorphisms

$$\varinjlim_t \mathrm{Ext}^{k-1}_{\mathcal{C}}(L_{\geq t}, I) \simeq \mathrm{Ext}^{k-1}_{\mathrm{QGr}(\mathcal{C})}(\pi L, \pi I)$$

and

$$\varinjlim_t \mathrm{Ext}^{k-1}_{\mathcal{C}}(L_{\geq t}, \Omega^{-1}(M/t(M)) \simeq \mathrm{Ext}^{k-1}_{\mathrm{QGr}(\mathcal{C})}(\pi L, \pi(\Omega^{-1}(M)/t(M))).$$

It follows

$$\varinjlim_t \operatorname{Ext}^k_{\mathcal{C}}(L_{\geq t}, M/t(M)) \simeq \operatorname{Ext}^k_{\operatorname{QGr}(\mathcal{C})}(\pi L, \pi(M/t(M))).$$

But we have proved

$$\varinjlim_t \operatorname{Ext}^k_{\mathcal{C}}(L_{\geq t}, M/t(M)) \simeq \varinjlim_t \operatorname{Ext}^k_{\mathcal{C}}(L_{\geq t}, M)$$

and $\pi(M/t(M)) \simeq \pi(M)$. Therefore we have that

$$\varinjlim_t \operatorname{Ext}^k_{\mathcal{C}}(L_{\geq t}, M) \simeq \operatorname{Ext}^k_{\operatorname{QGr}(\mathcal{C})}(\pi L, \pi(M)).$$

□

Corollary 2.6. *Let $\mathcal{C}$ and L be as in the proposition and M a torsion functor. Then $\varinjlim_t \operatorname{Ext}^k_{\mathcal{C}}(L_{\geq t}, M) = 0$ for all $k \geq 0$.*

Lemma 2.7. *Let $\mathcal{C}$ be a positively graded Krull-Schmidt locally finite K-category with radical $\oplus_{i\geq 1} \operatorname{Hom}_{\operatorname{Gr}(\mathcal{C})}(-,-)_i$ and $\{M_\alpha\}_{\alpha\in I}, \{N_\alpha\}_{\alpha\in I}, \{L_\alpha\}_{\alpha\in I}$ direct systems of graded locally finite functors. Then given an exact sequence of systems*

$$0 \to L_\alpha \to M_\alpha \to N_\alpha \to 0,$$

taking inverse limits, the sequence

$$0 \to \varprojlim L_\alpha \to \varprojlim M_\alpha \to \varprojlim N_\alpha \to 0$$

is exact.

Proof. Dualize the sequence to obtain direct systems, hen take direct limits and dualize again. □

Theorem 2.8. *Let $\mathcal{C}$ be an Artin Schelter regular category of global dimension n. Let M, L be torsion free functors , $M, L \in \operatorname{Fin}\mathcal{C}$. Then for all integers $0 \leq i \leq n-1$, there exists natural homomorphisms*

$$\mu_{M,L} \colon \varinjlim_k \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L) \to \operatorname{Hom}_K(\varinjlim_\ell \operatorname{Ext}^{n-1-i}_{\mathcal{C}}(L_{\geq \ell}, M\sigma'[\tau], K)$$

and if $M = (-, X)$, then $\mu_{M,L}$ is an isomorphism.

Proof. We apply the long homology sequence to $0 \to M_{\geq k} \to M \to M/M_{\geq k} \to 0$ to obtain a connecting map $\operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L) \to \operatorname{Ext}^{i+1}_{\mathcal{C}}(M/M_{\geq k}, L)$. Taking direct limits we obtain a natural map

$$\varinjlim_k \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L) \to \varinjlim_k \operatorname{Ext}^{i+1}_{\mathcal{C}}(M/M_{\geq k}, L).$$

By local cohomology we have a natural isomorphism

$$\varinjlim_k \operatorname{Ext}^{i+1}_{\mathcal{C}}(M/M_{\geq k}, L) \simeq \operatorname{Hom}_K(\operatorname{Ext}^{n-1-i}_{\mathcal{C}}(L, M\sigma'[\tau], K).$$

Substituting $L_{\geq \ell}$ for L we obtain maps

$$\varinjlim_k \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L_{\geq \ell}) \to \operatorname{Hom}_K(\operatorname{Ext}^{n-1-i}_{\mathcal{C}}(L_{\geq \ell}, M\sigma'[\tau], K).$$

Consider the exact sequence $0 \to L_{\geq \ell} \to L \to L/L_{\geq \ell} \to 0$ and apply the long homology sequence to obtain the following exact sequence.

$$\operatorname{Ext}^{i-1}_{\mathcal{C}}(M_{\geq k}, L/L_{\geq \ell}) \to \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L_{\geq \ell}) \to \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L) \to \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L/L_{\geq \ell}).$$

Taking direct limits and using Corollary 2.6, we get an isomorphism

$$\varinjlim_k \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L_{\geq \ell}) \simeq \varinjlim_k \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L).$$

Composing with the above homomorphisms, we obtain a natural homomorphism

$$\varinjlim_k \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L) \to \operatorname{Hom}_K(\operatorname{Ext}^{n-1-i}_{\mathcal{C}}(L_{\geq \ell}, M\sigma'[\tau], K).$$

Finally, taking inverse limits over ℓ and the fact that the dual of a direct limit is an inverse limit we get a natural homomorphism

$$\mu_{M,L}\colon \varinjlim_k \operatorname{Ext}^i_{\mathcal{C}}(M_{\geq k}, L) \to \operatorname{Hom}_K(\varinjlim_\ell \operatorname{Ext}^{n-1-i}_{\mathcal{C}}(L_{\geq \ell}, M\sigma'[\tau], K).$$

It remains to prove that when $M = (-, X)$ the homomorphism becomes an isomorphism. Since L is torsion free, $\operatorname{Hom}_{\mathcal{C}}((-, X)/r^k(-, X), L) = 0$. The exact sequence

$$0 \to r^k(-, X) \to (-, X) \to (-, X)/r^k(-, X) \to 0$$

induces the following exact sequence

$$\begin{aligned}0 \to \operatorname{Hom}_{\mathcal{C}}((-, X), L) \to \operatorname{Hom}_{\mathcal{C}}(r^k(-, X), L) \to &\\ \operatorname{Ext}^1_{\mathcal{C}}((-, X)/r^k(-, X), L) \to 0&\end{aligned}$$

Changing L for $L_{\geq \ell}$ we obtain exact sequences

$$0 \to L_{\geq \ell}(X) \to \operatorname{Hom}_{\mathcal{C}}(r^k(-, X), L_{\geq \ell}) \to \operatorname{Ext}^1_{\mathcal{C}}((-, X)/r^k(-, X), L_{\geq \ell}) \to 0.$$

Taking direct limits and composing with the natural isomorphism

$$\varinjlim_k \operatorname{Ext}^1_{\mathcal{C}}((-, X)/r^k(-, X), L_{\geq \ell}) \simeq \operatorname{Hom}_K(\operatorname{Ext}^{n-1}_{\mathcal{C}}(L_{\geq \ell}, (-, X)\sigma'[\tau]), K),$$

we obtain the exact sequence

$$0 \to L_{\geq \ell}(X) \to \varinjlim_k \operatorname{Hom}_{\mathcal{C}}(r^k(-, X), L_{\geq \ell}) \to \mathcal{D}(\operatorname{Ext}^{n-1}_{\mathcal{C}}(L_{\geq \ell}, (-, X)\sigma'[\tau])) \to 0.$$

As above, there is a natural isomorphism

$$\varinjlim_k \operatorname{Hom}_{\mathcal{C}}(r^k(-, X), L_{\geq \ell}) \simeq \varinjlim_k \operatorname{Hom}_{\mathcal{C}}(r^k(-, X), L).$$

Hence we obtain the exact sequence

$$0 \to L_{\geq \ell}(X) \to \varinjlim_k \operatorname{Hom}_{\mathcal{C}}(r^k(-, X), L) \to \mathcal{D}(\operatorname{Ext}^{n-1}_{\mathcal{C}}(L_{\geq \ell}, (-, X)\sigma'[\tau])) \to 0.$$

Taking inverse limits and using the fact $\varprojlim_\ell L_{\geq \ell} \simeq \cap L_{\geq \ell} = 0$, we get that

$$\varinjlim_k \operatorname{Hom}_{\mathcal{C}}(r^k(-, X), L) \simeq \varprojlim_\ell \operatorname{Hom}_K(\operatorname{Ext}^{n-1}_{\mathcal{C}}(L_{\geq \ell}, (-, X)\sigma'[\tau]), K).$$

It follows that

$$\varinjlim_k \operatorname{Hom}_{\mathcal{C}}(r^k(-, X), L) \simeq \operatorname{Hom}_K(\varinjlim_\ell \operatorname{Ext}^{n-1}_{\mathcal{C}}(L_{\geq \ell}, (-, \sigma' X)[\tau]), K).$$

We have proved $\mu_{M,L}$ is an isomorphism for $i = 0$. We now prove that $\mu_{M,L}$ is an isomorphism for $i > 0$. We have the exact sequence

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{C}}((-, X), L) \to \operatorname{Ext}^i_{\mathcal{C}}(r^k(-, X), L) \to &\\ \operatorname{Ext}^{i+1}_{\mathcal{C}}((-, X)/r^k(-, X), L) \to \operatorname{Ext}^{i+1}_{\mathcal{C}}((-, X), L)&\end{aligned}$$

with $\operatorname{Ext}^i_{\mathcal{C}}((-,X),L) \simeq \operatorname{Ext}^{i+1}_{\mathcal{C}}((-,X),L) = 0$. Taking direct limits and using local cohomology we obtain the isomorphisms

$$\begin{aligned}\varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}(r^k(-,X),L) &\simeq \varinjlim{}_k \operatorname{Ext}^{i+1}_{\mathcal{C}}((-,X)/r^k(-,X),L)\\ &\simeq \operatorname{Hom}_K(\operatorname{Ext}^{n-i-1}_{\mathcal{C}}(L,(-,X)\sigma'[\tau]),K).\end{aligned}$$

Changing as above $L_{\geq \ell}$ for L, we obtain the isomorphisms

$$\varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}(r^k(-,X),L_{\geq\ell}) \simeq \operatorname{Hom}_K(\operatorname{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq\ell},(-,X)\sigma'[\tau]),K).$$

We obtain by a similar argument as above the exact sequence

$$\begin{aligned}\varinjlim{}_k \operatorname{Ext}^{i-1}_{\mathcal{C}}(r^k(-,X),L/L_{\geq\ell}) \to \varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}(r^k(-,X),L_{\geq\ell}) \to\\ \varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}(r^k(-,X),L) \to \varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}(r^k(-,X),L/L_{\geq\ell})\end{aligned}$$

But we proved

$$\varinjlim{}_k \operatorname{Ext}^{i-1}_{\mathcal{C}}(r^k(-,X),L/L_{\geq\ell}) = \varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}(r^k(-,X),L/L_{\geq\ell}) = 0.$$

It follows that

$$\varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}(r^k(-,X),L) \simeq \operatorname{Hom}_K(\operatorname{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq\ell},(-,X)\sigma'[\tau]),K).$$

Finally taking inverse limits over ℓ and commuting with the duality we obtain the natural isomorphisms

$$\varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}(r^k(-,X),L) \simeq \operatorname{Hom}_K(\varinjlim{}_\ell \operatorname{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq\ell},(-,X)\sigma'[\tau]),K).$$

□

Remark 2.9. Observe that if we assume $\mathcal{C}$ locally finite positively graded but not necessary generated in degree $0,1$, the above arguments would give natural isomorphisms

$$\varinjlim{}_k \operatorname{Ext}^i_{\mathcal{C}}((-,X)_{\geq k},L) \simeq \operatorname{Hom}_K(\varinjlim{}_\ell \operatorname{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq\ell},(-,X)\sigma'[\tau]),K).$$

We can generalize the previous theorem to get the following formula.

Theorem 2.10. *Let $\mathcal{C}$ be an Artin Schelter regular category of global dimension n. Let M, L be torsion free functors, $M, L \in \operatorname{Fin}\mathcal{C}$. Then for all integers $0 \leq i \leq n-1$, there exists a natural isomorphisms*

$$\operatorname{Ext}^i_{\mathrm{QGr}(\mathcal{C})}(\pi M,\pi L) \simeq D(\operatorname{Ext}^{n-1-i}_{\mathrm{QGr}(\mathcal{C})}(\pi L,\pi M\sigma'[\tau]).$$

Proof. We prove the theorem by induction on the projective dimension of M, the case M being projective was proved in the previous theorem. So we may assume that M has projective dimension k, and that the claim has been proved for all functors of projective dimension less than k.

Consider the exact sequence $0 \to \Omega M \to P \to M \to 0$, with P the projective cover of M. For any integer s, we obtain the truncated exact sequence

$$0 \to \Omega M_{\geq s} \to P_{\geq s} \to M_{\geq s} \to 0,$$

which induces an exact sequence

$$\begin{aligned}&\mathrm{Ext}^i_{\mathcal{C}}(P_{\geq s}, L) \to \mathrm{Ext}^i_{\mathcal{C}}((\Omega M)_{\geq s}, L) \to \\ &\qquad \mathrm{Ext}^{i+1}_{\mathcal{C}}(M_{\geq s}, L) \to \mathrm{Ext}^{i+1}_{\mathcal{C}}(P_{\geq s}, L) \to \mathrm{Ext}^{i+1}_{\mathcal{C}}((\Omega M)_{\geq s}, L)\end{aligned}$$

Taking direct limits we obtain the exact sequence

$$\begin{aligned}&\varinjlim_s \mathrm{Ext}^i_{\mathcal{C}}(P_{\geq s}, L) \to \varinjlim_s \mathrm{Ext}^i_{\mathcal{C}}((\Omega M)_{\geq s}, L) \to \\ &\quad \varinjlim_s \mathrm{Ext}^{i+1}_{\mathcal{C}}(M_{\geq s}, L) \to \varinjlim_s \mathrm{Ext}^{i+1}_{\mathcal{C}}(P_{\geq s}, L) \to \varinjlim_s \mathrm{Ext}^{i+1}_{\mathcal{C}}((\Omega M)_{\geq s}, L)\end{aligned}$$

On the other hand, the exact sequence

$$0 \to \Omega M\sigma'[\tau] \to P\sigma'[\tau] \to M\sigma'[\tau] \to 0$$

induces for each k the exact sequence

$$\begin{aligned}&\to \mathrm{Ext}^{n-i-2}_{\mathcal{C}}\, L_{\geq k}, (\Omega M\sigma'[\tau]) \to \mathrm{Ext}^{n-i-2}_{\mathcal{C}}(L_{\geq k}, P\sigma'[\tau]) \to \\ &\qquad \mathrm{Ext}^{n-i-2}_{\mathcal{C}}(L_{\geq k}, M\sigma'[\tau]) \to\to \mathrm{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq k}, \Omega M\sigma'[\tau]) \to \\ &\qquad\qquad \mathrm{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq k}, P\sigma'[\tau]) \to\end{aligned}$$

Taking limits and dualizing we the exact sequence

$$\begin{aligned}&D(\varinjlim_k \mathrm{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq k}, P\sigma'[\tau])) \to D(\varinjlim_k \mathrm{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq k}, \Omega M\sigma'[\tau])) \to \\ &\quad \to D(\varinjlim_k \mathrm{Ext}^{n-i-2}_{\mathcal{C}}(L_{\geq k}, M\sigma'[\tau])) \to D(\varinjlim_k \mathrm{Ext}^{n-i-2}_{\mathcal{C}}(L_{\geq k}, P\sigma'[\tau])) \to \\ &\qquad \to D(\varinjlim_k \mathrm{Ext}^{n-i-2}_{\mathcal{C}}\, L_{\geq k}, (\Omega M\sigma'[\tau]))\end{aligned}$$

By induction hypothesis we have the following isomorphisms

$$\begin{aligned}\varinjlim_s \mathrm{Ext}^i_{\mathcal{C}}(P_{\geq s}, L) &\simeq D(\varinjlim_k \mathrm{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq k}, P\sigma'[\tau])), \varinjlim_s \mathrm{Ext}^i_{\mathcal{C}}((\Omega M)_{\geq s}, L) \\ &\simeq D(\varinjlim_k \mathrm{Ext}^{n-i-1}_{\mathcal{C}}(L_{\geq k}, \Omega M\sigma'[\tau])), \varinjlim_s \mathrm{Ext}^{i+1}_{\mathcal{C}}(P_{\geq s}, L) \\ &\simeq D(\varinjlim_k \mathrm{Ext}^{n-i-2}_{\mathcal{C}}(L_{\geq k}, P\sigma'[\tau])), \varinjlim_s \mathrm{Ext}^{i+1}_{\mathcal{C}}((\Omega M)_{\geq s}, L) \\ &\simeq D(\varinjlim_k \mathrm{Ext}^{n-i-2}_{\mathcal{C}}(L_{\geq k}, M\sigma'[\tau])).\end{aligned}$$

Using Proposition 2.8 and Five's Lemma, it follows that

$$\varinjlim_s \mathrm{Ext}^{i+1}_{\mathcal{C}}(M_{\geq s}, L) \simeq D(\varinjlim_k \mathrm{Ext}^{n-i-2}_{\mathcal{C}}(L_{\geq k}, M\sigma'[\tau])).$$

Therefore we have proved that

$$\mathrm{Ext}^{i+1}_{\mathrm{QGr}(\mathcal{C})}(\pi M, \pi L) \simeq D(\mathrm{Ext}^{n-2-i}_{\mathrm{QGr}(\mathcal{C})}(\pi L, \pi M\sigma'[\tau]).$$

$\square$

3. Applications to finite dimensional algebras

We recall the following notions and results from [7, 8]. Let A be a finite dimensional K-algebra, mod_A the category of finitely generated A-modules, and let $\mathcal{C}$ be a stable Auslander-Reiten component, $\mathcal{A}_{\mathrm{gr}}(\mathcal{C})$ is Artin Schelter regular of global dimension 2. Applying the above results we obtain generalizations of the results on preprojective algebras concerning local cohomology and Serre duality [5]. We have proved in [8] we also have similar results on noetherianess and Gelfand-Kirillov dimension [4, 1, 2]. We recall results from [8].

Theorem 3.1 ([8])**.** *Let $\mathcal{C}$ be a regular Auslander-Reiten component of the finite dimensional algebra Λ and $E(S(\mathcal{C}))$ the associated* Ext *category. Then the following statements are true:*

1) *$E(S(\mathcal{C}))$ is a Frobenius category of radical cube zero.*
2) *The categories $E(S(\mathcal{C}))/\operatorname{soc} E(S(\mathcal{C}))$, $\mathcal{C}^{\mathrm{op}}/r^2$ are equivalent and $\mathrm{Gr}(\mathcal{C}^{\mathrm{op}}/r^2)$ is stably equivalent to $\mathrm{Gr}(\mathcal{S})$, where $\mathrm{Gr}(\mathcal{S})$ decomposes as a product of sections $\mathrm{Gr}(\mathcal{S}) = \prod \mathrm{Gr}(\mathcal{S}_j) \times \mathrm{Gr}(\mathcal{S}_j^{\mathrm{op}})$ and each $\mathcal{S}_j$ is an hereditary category, such that $\mathcal{S}_j$ and $\mathcal{S}_i$ have the same quiver Q but $\mathcal{S}_j$ and $\mathcal{S}_j^{\mathrm{op}}$ have opposite quivers.*
3) *If the quiver Q of $\mathcal{S}_j$ is finite, then $\mathcal{S}_j$ is of infinite representation type.*

Definition 3.2. Let $\mathcal{C}$ be a graded K-category, a graded functor F such that the functors F_i defined by $F_i(X) = F(X)_{i,}$ have finite support. The growth of F is the function $\phi_F \colon Z \to Z$ given by $\phi_F(i) = \sum\limits_{X \in \sup pF} \dim_K F(X)_i$.

Definition 3.3. Define the Gelfand Kirillov dimension of F as

$$\mathrm{GKdim}(F) = \varlimsup_{n\to\infty} \log_n\Big(\sum_{k=0}^{n} \phi_F(k)\Big).$$

Theorem 3.4 ([8])**.** *Let $\mathcal{C}$ be a regular Auslander-Reiten component of the finite dimensional algebra Λ. Assume the quiver Q of the sections $\mathcal{S}_j$ of $E(S(\mathcal{C}$ $))$ is infinite and is not of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$.*

1) *Then any finitely presented functor $F \in \mathrm{gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is either of finite length or it has infinite Gelfand Kirillov dimension.*
2) *The category of finitely presented functors $\mathrm{gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is not noetherian.*
3) *If $E(S(\mathcal{C}))$ has sections of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$, then $\mathrm{gr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is noetherian of Gelfand-Kirillov dimension 1 or 2.*

We obtain the following application of the results for Section 2.

Theorem 3.5 ([8])**.** *Let $\mathcal{C}$ be a regular Auslander-Reiten component of the finite dimensional algebra Λ. Assume $E(S(\mathcal{C}))$ has sections of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$. Then the quotient category of the finitely presented functors modulo the functors of finite length, $\mathrm{Qgr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ is noetherian, of dimension one, with Serre duality.*

If the sections of $E(S(\mathcal{C}))$ are infinite not of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$ or $\mathbb{A}_\infty^\infty$, then $\mathrm{Qgr}(\mathcal{A}_{\mathrm{gr}}(\mathcal{C}))$ *is not noetherian, but the category* $\mathrm{Qgr}(\mathrm{Fin}(C))$ *satisfies Serre duality.*

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