International Journal of Algebra, Vol. 3, 2009, no. 17, 839 - 856

The Structure of Endomorphism Monoids in Conjugate Categories

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Abstract

We invoke structure theorems from the theory of semigroups to analyze the anatomy of categories arising from a type of conjugation by a subcategory. The equivalence classes of Green's $\mathcal{H}$-relation admit group structures when they contain idempotents, and we use this to study the endomorphism monoids of the objects in a conjugate category. We prove that in any such category, the $\mathcal{H}$-group associated to an idempotent endomorphism is always realizable as the automorphism group of its "rank object," which serves as a generalized image. As all $\mathcal{H}$-groups are automorphism groups of lower rank, we establish a resulting homogeneity of these $\mathcal{H}$-groups, giving stringent necessary conditions for the existence of conjugate categories.

Mathematics Subject Classification: 55U35,18A25

Keywords: endomorphism, monoid, semigroup, category

1 Introduction

The categories central to this article arise in certain aspects of the *representation theory* of categories. A basic problem in this area—which nowadays stretches across algebra and homotopy theory—is to give a framework for

Morita equivalence of categories of functors taking values in a structured category $\mathcal{C}$. The exact type of equivalence desired depends on the structure of $\mathcal{C}$. Strict adjoint equivalences are usually sought after when $\mathcal{C}$ is abelian, whereas Quillen equivalences are the desired sort when $\mathcal{C}$ is a model category.

We will let $[\mathcal{D}, \mathcal{C}]$ denote the category of functors $F : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is a small category. When we regard a functor $F : \mathcal{D} \to \mathcal{C}$ as a *left $\mathcal{D}$-module* in $\mathcal{C}$ the analogy with classical Morita theory becomes clear: $[\mathcal{D}, \mathcal{C}]$ is now a category of modules. The most basic question in this generalized Morita theory is as follows: under what conditions will the pair $(\mathcal{B}, \mathcal{A})$ of small categories induce an adjoint equivalence between $[\mathcal{B}, \mathcal{C}]$ and $[\mathcal{A}, \mathcal{C}]$? The idea is that while the category $\mathcal{B}$ may be the most natural parametrization for a family of constructions in $\mathcal{C}$, there may be a more manageable category $\mathcal{A}$ which will do the same job with less baggage. When this occurs, we often say that $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* for $\mathcal{C}$, or that $\mathcal{A}$ and $\mathcal{B}$ give *equivalent $\mathcal{C}$-representations.*

Hence our basic question is one of efficiency. Such questions of representations of categories have been considered since at least the 1970s (see [4] and [6] for early examples), but almost all of the progress has concerned cases where $\mathcal{C}$ is algebraic in nature. Following an example of Pirashvili [7], in 2004 Słomińska [8] established a rather general theorem on representations in the category of R-modules. In this paper Słomińska gives sufficient conditions for which a category $\mathcal{B}$ admits a subcategory $\mathcal{A}$ with equivalent representations in the category of R-modules. Not only does her theorem recover Pirashvili's example, but it also lends itself to numerous additional examples with applications to several areas of algebra.

In an effort to extend Słomińska's theorem to some topological categories, the first author has given a re-organization of Słomińska's hypotheses on the categories $\mathcal{B}$ and $\mathcal{A}$ in [1]. This leads to the notion of a *conjugate pair* of categories, the main focus of this current paper. Using model category techniques, in [1] we prove that the categories in a conjugate pair $(\mathcal{B}, \mathcal{A})$ yield equivalent $\mathcal{C}$-representations, where $\mathcal{C}$ is any model category satisfying a few additional technical assumptions. These assumptions on $\mathcal{C}$ essentially abstract behavior common to both abelian categories and stable model categories, the categories central to modern algebra and homotopy theory. The upshot is that we have generalized Słomińska's algebraic theorem in the abelian category case, while simultaneously gaining the theorem for certain topological categories in the same proof.

At this point it is evident that conjugate pairs have a rich representation theory, and hence one would like to know more about the structure of such categories. While there are numerous known examples of conjugate pairs—even some infinite families—new examples are hard to come by, indicating an inherent rarity of the phenomenon. This current work is an attempt to understand the strict structure of conjugate pairs by invoking known results on the structure of semigroups.

There is a certain equivalence relation defined on every monoid (i.e., semigroup with identity), the resulting classes being known as the $\mathcal{H}$-*classes* of the monoid. Any particular $\mathcal{H}$-class either contains exactly one idempotent or none at all; those that contain an idempotent admit a group structure. We have naturally occurring monoids in any category via its endomorphisms: the set of all self-maps on a fixed object admits a monoid structure under composition. Hence we may apply the theory of $\mathcal{H}$-classes and idempotents to analyze these various endomorphism monoids.

Our main result is that given a conjugate pair $(\mathcal{B}, \mathcal{A})$, the $\mathcal{H}$-class structure of endomorphisms in the category $\mathcal{B}$ is very rigid. In a sense, this severely limits the categories $\mathcal{B}$ which may appear as the "first coordinate" of a conjugate pair. The exact statement of our main result follows, where we are writing $\mathcal{H}_e^B$ for the $\mathcal{H}$-group associated to the idempotent $e : B \to B$.

Theorem 4.6. *Suppose that* $(\mathcal{B}, \mathcal{A})$ *is a conjugate pair of categories in which every map in* $\mathcal{A}$ *is epic in* $\mathcal{B}$*. If* $e : B \to B$ *is an idempotent in* $\mathcal{B}$ *with rank object* R*, then there exists an isomorphism* $\mathcal{H}_e^B \simeq \mathcal{H}_1^R$ *of* $\mathcal{H}$*-groups. Moreover,* $\mathcal{H}_1^R = \mathrm{Aut}_{\mathcal{B}}(R)$*, so that every* $\mathcal{H}$*-group in* $\mathcal{B}$ *is isomorphic to a group of automorphisms.*

As we will explain in the course of the article, this theorem makes a strong homogeneity statement about the $\mathcal{H}$-groups of endomorphisms in $\mathcal{B}$. For instance, this makes it relatively simple to check that a given category $\mathcal{B}$ *cannot* occur as the first coordinate in a conjugate pair of categories. With the bigger picture in mind, we hope this work highlights the potential applications of semigroups to category theory, and that it will in turn encourage more fruitful interactions between the two fields.

2 Conjugate Pairs of Categories

2.1 Dualities

We assume some knowledge of elementary category theory, but not much more than the basic definitions of *category* and (contravariant) *functor*. The books [2] and [5] provide a more than adequate background.

Roughly, a category $\mathcal{B}$ is a *conjugate category* if it factors as $\mathcal{B} = \mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^*$, where the three displayed categories on the right are all subcategories of $\mathcal{B}$ and $\mathcal{I}^*$ is isomorphic to $\mathcal{I}^{\mathrm{op}}$. Hence such a category $\mathcal{B}$ is obtained by "conjugating" $\mathcal{A}$ by $\mathcal{I}$ and its dual. In this case we call $(\mathcal{B}, \mathcal{A})$ a *conjugate pair*.

It will be helpful to see a concrete example before giving the abstraction, and the category Γ of *finite based sets* is the prime example of a conjugate category. This category has as its objects the sets $\mathbf{n}_+ = \{0, 1, \ldots, n\}$ (for $n \geq 0$). A morphism $f : \mathbf{m}_+ \to \mathbf{n}_+$ is an ordinary function between the indicated sets with the additional property that $f(0) = 0$. We call the element $0 \in \mathbf{n}_+$ the *basepoint* of $\mathbf{n}_+$ and refer to such maps as *based maps*. It is a simple matter to check that Γ forms a category under ordinary function composition.

The first step in understanding the conjugate structure of Γ is to understand the subcategories which play the role of $\mathcal{I}$ and $\mathcal{I}^*$. These are known as the subcategories of *ordered* and *collapse* maps, respectively.

Example 2.1. *The subcategory $\mathcal{O}$ of* ordered maps *in Γ is described as follows. The objects of $\mathcal{O}$ will be the same as those of Γ. A morphism $i : \mathbf{m}_+ \to \mathbf{n}_+$ will be a Γ-map with the additional property that $i(x) < i(y)$ whenever $x < y$. Note that $\mathrm{Mor}_{\mathcal{O}}(\mathbf{m}_+, \mathbf{n}_+)$ is empty when $m > n$, and $\mathrm{Mor}_{\mathcal{O}}(\mathbf{n}_+, \mathbf{n}_+)$ consists of the identity map alone. Moreover, it is a nice exercise to check that the set $\mathrm{Mor}_{\mathcal{O}}(\mathbf{m}_+, \mathbf{n}_+)$ is in one-to-one correspondence with the collection of m-element subsets of a set with n elements, the correspondence being given by taking the images of maps $i : \mathbf{m}_+ \to \mathbf{n}_+$ in $\mathcal{O}$ (and "forgetting" the basepoints).*

Example 2.2. *Here we describe the subcategory $\mathcal{O}^*$ of* collapse maps *in Γ. For every $\mathcal{O}$-map $i : \mathbf{m}_+ \to \mathbf{n}_+$ there exists a unique Γ-map $i^* : \mathbf{n}_+ \to \mathbf{m}_+$ with the property $i^* \circ i = 1_{\mathbf{m}_+}$. Recalling that the map i represents a subset of $\mathbf{n}_+$ via its image, we see that i^* must retain the elements in this image while collapsing its complement. For example, suppose that $i : \mathbf{2}_+ \to \mathbf{5}_+$ is defined by $i(1) = 3$ and $i(2) = 5$, so that i represents the subset $\{0, 3, 5\}$ of $\mathbf{5}_+$. The map $i^* : \mathbf{5}_+ \to \mathbf{2}_+$ must preserve these elements in order, so we have $i^*(3) = 1$ and $i^*(5) = 2$, while the remaining elements must map to the basepoint 0. It is easy to check now that $i^* \circ i = 1$.*

It is not hard to see that we have $(j \circ i)^* = i^* \circ j^*$, whenever the composite $j \circ i$ is defined in $\mathcal{O}$. Moreover, it is clear that this collapsing process preserves identities, so that $(1_{\mathbf{n}_+})^* = 1_{\mathbf{n}_+}$. From these observations it follows that the class of all Γ-maps of the form i^* (where i comes from $\mathcal{O}$) gives a subcategory of Γ. We call this the subcategory of collapse maps *and denote it by $\mathcal{O}^*$.*

The previous paragraph shows that $\mathcal{O}^*$ is in fact isomorphic to the opposite category of $\mathcal{O}$. To be precise, the act of dualizing the map i into i^* provides an isomorphism $(\ )^* : \mathcal{O}^{\mathrm{op}} \to \mathcal{O}^*$ which fixes the objects of $\mathcal{O}$. This is an instance of what we will call a *duality*.

Definition 2.3. *Suppose that $\mathcal{I}$ and $\mathcal{I}^*$ are subcategories of a category $\mathcal{B}$, all with the same objects. A* duality *is an object-fixing isomorphism $(\ )^* : \mathcal{I}^{\mathrm{op}} \to \mathcal{I}^*$ such that $i^* \circ i = 1_A$ for each map $i : A \to B$ in $\mathcal{I}$.*

Note that the composition $i^* \circ i$ above makes sense, as both $\mathcal{I}$ and $\mathcal{I}^*$ are subcategories of $\mathcal{B}$. Since $(\ )^*$ is a functor we have relations such as $(j \circ i)^* =$

$i^* \circ j^*$ and $1^* = 1$. Also, each map in the category $\mathcal{I}^*$ must be expressible in the form i^* for a unique $\mathcal{I}$-map i, since the duality is a categorical isomorphism. Moreover, simple categorical arguments show that each map in $\mathcal{I}$ must be monic, and hence every map in $\mathcal{I}^*$ is epic (with respect to maps in $\mathcal{B}$). Two additional examples of dualities follow.

Example 2.4. *Let P be any poset with greatest lower bounds. We may define a category $\mathcal{B}$, whose objects are the elements of P, as follows. A morphism $a : x \to y$ in $\mathcal{B}$ is just the statement that $a \leq x$ and $a \leq y$. The composition of $a : x \to y$ and $b : y \to z$ is given by the greatest lower bound of a and b. We have the subcategory $\mathcal{P}$ consisting of the maps $x : x \to y$ and the corresponding dual subcategory $\mathcal{P}^*$ with maps $x^* = x : y \to x$ (when $x \leq y$). It follows from $x^* \circ x = 1_x$ that the obvious functor $(\ )^* : \mathcal{P}^{\mathrm{op}} \to \mathcal{P}^*$ is a duality.*

Example 2.5. *Let G be a group and define the category $\mathcal{C}(G)$ as follows. There will be only one object, namely G itself. The elements of G will represent the morphisms of $\mathcal{C}(G)$, with composition given by the group operation. Every morphism $g : G \to G$ gives a dual morphism $g^* = g^{-1} : G \to G$, and clearly $g^* \circ g = 1$. Hence inversion gives a duality between $\mathcal{C}(G)^* = \mathcal{C}(G)$ and its opposite.*

2.2 Conjugate pairs of categories

We are finally in a position to describe the notion of conjugate categories. The exact definition requires several sophisticated concepts of category theory, including pullbacks and the structure of EI-categories. To expound on all of this would take us too far afield, but fortunately we can describe the structural aspects relevant to our current work without taking this detour. For the curious reader, the precise definition may be found in [1].

As previously mentioned, the most instructive example arises from the category Γ of finite based sets. Each Γ-map $\gamma : \mathbf{m}_+ \to \mathbf{n}_+$ carries two important pieces of data: its image, and the elements of the domain that are *not* sent to 0 (the complement of the "kernel" $\gamma^{-1}(0)$ of γ). The former is a subset of $\mathbf{n}_+$ and the latter is a subset of $\mathbf{m}_+$, and hence each may be represented by maps in the category $\mathcal{O}$. Suppose that we represent the image of γ by $i : \mathbf{r}_+ \to \mathbf{n}_+$ and the complement of the kernel of γ by $j : \mathbf{q}_+ \to \mathbf{m}_+$.

By collapsing the kernel, we may define a map $\gamma' : \mathbf{q}_+ \to \mathbf{n}_+$ for which the basepoint of $\mathbf{q}_+$ is the only element sent to the basepoint of the codomain. Explicitly, this map satisfies $\gamma = \gamma' \circ j^*$ as one may check. Note that the image of γ' is equal to the image of γ, hence we may restrict the codomain of γ' to this image and obtain a surjective map $\alpha : \mathbf{q}_+ \to \mathbf{r}_+$ with the property that $\alpha^{-1}(0) = \{0\}$. Precisely, α is defined by $\gamma' = i \circ \alpha$. In summary, each Γ-map

$\gamma : \mathbf{m}_+ \to \mathbf{n}_+$ has a uniquely determined three-fold factorization of the form

$$\begin{array}{ccc} \mathbf{m}_+ & \xrightarrow{\gamma} & \mathbf{n}_+ \\ {\scriptstyle j^*}\big\downarrow & & \big\uparrow{\scriptstyle i} \\ \mathbf{q}_+ & \xrightarrow{\alpha} & \mathbf{r}_+ \end{array}$$

where

- j^* is the $\mathcal{O}^*$-map that collapses the kernel $\gamma^{-1}(0)$ of γ,
- i is the $\mathcal{O}$-map representing the inclusion of the image of γ into its codomain, and
- α is the unique surjection making the diagram commute.

Note that maps "like α" in fact give a subcategory of Γ. We let $\mathcal{E}$ denote the category with the same objects as Γ, where the morphisms are the based surjections $\alpha : \mathbf{q}_+ \to \mathbf{r}_+$ with $\alpha^{-1}(0) = \{0\}$.

The fact that all maps in Γ factor this way may be expressed by writing $\Gamma = \mathcal{O} \circ \mathcal{E} \circ \mathcal{O}^*$, so that Γ is created by "conjugating" $\mathcal{E}$ by $\mathcal{O}$ and its dual $\mathcal{O}^*$. We say that $(\Gamma, \mathcal{E})$ is a *conjugate pair* and that Γ itself is a *conjugate category*. This is Pirashvili's original example in [7], where it is proven that Γ and $\mathcal{E}$ give equivalent representations in the category of abelian groups.

Let us now describe the relevant structure of a generic conjugate pair $(\mathcal{B}, \mathcal{A})$ of categories. First, we assume that $\mathcal{A}$ is a subcategory of $\mathcal{B}$. We also have subcategories $\mathcal{I}$ and $\mathcal{I}^*$ of $\mathcal{B}$ with a fixed underlying duality $(\ )^* : \mathcal{I}^{\mathrm{op}} \to \mathcal{I}^*$. Each of these subcategories should have all the objects of $\mathcal{B}$. The fundamental property of a conjugate pair $(\mathcal{B}, \mathcal{A})$ is that each map $\beta : A \to B$ in $\mathcal{B}$ admits a three-fold factorization

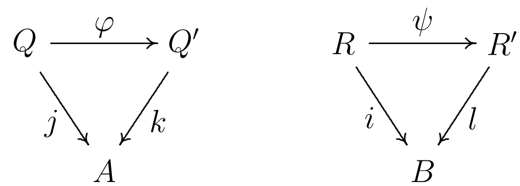
$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle j^*}\big\downarrow & & \big\uparrow{\scriptstyle i} \\ Q & \xrightarrow{\alpha} & R \end{array} \tag{1}$$

with $i, j \in \mathcal{I}$ and $\alpha \in \mathcal{A}$. We may write $\mathcal{B} = \mathcal{I} \circ \mathcal{A} \circ \mathcal{I}^*$ and refer to $\mathcal{B}$ itself as a *conjugate category*.

We do not assume that the components of such a factorization are unique, only that they are unique up to compatible isomorphisms in $\mathcal{I}$. To be precise, suppose that

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle k^*}\big\downarrow & & \big\uparrow{\scriptstyle l} \\ Q' & \xrightarrow{\delta} & R' \end{array}$$

is a second factorization of β. In a conjugate pair, there will exist isomorphisms $\varphi : Q \to Q'$ and $\psi : R \to R'$ in $\mathcal{I}$ such that the diagrams



commute. Hence the object R in diagram (1) need not be unique, but it is unique up to isomorphism. This will be useful to us later, so we give this object a special name.

Definition 2.6. *Suppose that β is a map in a conjugate category, factoring as in diagram* (1)*. We call the object R a* rank object *of β.*

The remaining structural aspects of conjugate pairs concern further assumptions on the category $\mathcal{I}$ and a means of computing the factorization of a composite $\beta \circ \gamma$ from the factorizations of β and γ. For the most part this structure will not be needed for our main result. Examples of conjugate pairs other than $(\Gamma, \mathcal{E})$ follow; other examples may be found in [1] and [8].

Example 2.7. *Let $\mathcal{I}$ denote the category on the two objects 0 and 1 with a single non-identity morphism as displayed below:*

$$0 \xrightarrow{\;i\;} 1.$$

Let $\mathcal{B}$ denote the category on the same objects with two non-identity maps

$$0 \underset{i^*}{\overset{i}{\rightleftarrows}} 1$$

satisfying $i^* \circ i = 1$. Taking $\mathcal{A}$ to be the discrete category on the same objects, it is immediate that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair. This is, in a sense, the smallest possible example.

This example provides a nice illustration of how $\mathcal{C}$-representations and conjugate pairs interact. Suppose, for instance, that $\mathcal{C}$ is the category of vector spaces over a field (in this case, we are discussing *linear representations*). To give a functor $F : \mathcal{B} \to \mathcal{C}$ is to give a diagram of vector spaces and linear transformations

$$F_0 \underset{T^*}{\overset{T}{\rightleftarrows}} F_1$$

with $T^* \circ T = 1$. Note then that $T \circ T^*$ is an idempotent on F_1, which in linear algebra is frequently called a *projection operator*. It is a standard result in linear algebra that such an operator admits a decomposition

$$F_1 \simeq F_0 \oplus \ker(T \circ T^*).$$

Hence to give a functor $F : \mathcal{B} \to \mathcal{C}$ is simply to specify a pair $(F_0, \ker(T \circ T^*))$ of vector spaces. Since $\mathcal{A}$ is discrete on two objects, functors $G : \mathcal{A} \to \mathcal{C}$ correspond to pairs of vector spaces. Thus $\mathcal{B}$ and $\mathcal{A}$ do indeed give equivalent representations in the category of vector spaces.

Example 2.8. *Let P, $\mathcal{B}$, $\mathcal{P}$ and $\mathcal{P}^*$ be as in Example 2.4. We note that each map $a : x \to y$ in $\mathcal{B}$ factors as*

$$\begin{array}{ccc} x & \xrightarrow{\ a\ } & y \\ {\scriptstyle a^*}\big\downarrow & & \big\uparrow{\scriptstyle a} \\ a & \xrightarrow{\ 1\ } & a. \end{array}$$

With $\mathcal{A}$ the discrete category on the elements of P, we have that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair.

Example 2.9. *Let $\mathcal{B}$ denote the category of Γ-maps β such that the inverse image $\beta^{-1}(x)$ of each nonzero point x is either empty or a singleton. Said differently, a map $\beta \in \mathcal{B}$ may send lots of elements to the basepoint, but modulo this, it is injective. Let Σ denote the subcategory of based permutations on the objects of $\mathcal{B}$. With $\mathcal{O}$ and $\mathcal{O}^*$ as before, it is not hard to see that $\mathcal{B} = \mathcal{O} \circ \Sigma \circ \mathcal{O}^*$, so that $(\mathcal{B}, \Sigma)$ is a conjugate pair.*

Example 2.10. *(After example 2.5 of [8]) Given a finite subset A of $\mathbb{N}$ (without 0), we will write A_+ for the based set $A \cup \{0\}$, where 0 will always act as a basepoint. Let $\mathcal{B}$ denote the category with objects the finite sets A_+ and morphisms the based maps $f : A_+ \to B_+$ such that $f(a) = a$ whenever $f(a) \neq 0$. There is the subcategory $\mathcal{I}$ of subset inclusions, with dual $\mathcal{I}^*$ consisting of the obvious analogues of the collapse maps of Example 2.2. It is clear that any map f in $\mathcal{B}$ factors as $i \circ j^*$, where j^* collapses the "kernel" $f^{-1}(0)$ and i is the inclusion of the image. Taking $\mathcal{A}$ to be discrete, $(\mathcal{B}, \mathcal{A})$ is a conjugate pair.*

Example 2.11. *The $(\Gamma, \mathcal{E})$ example can be enlarged. Let $\mathcal{B}$ denote the category with objects the finite based sets A_+ as above, but with morphisms all based maps. We can mimic the behavior of the $(\Gamma, \mathcal{E})$ example by letting $\mathcal{I}$ be the subcategory of injective maps (not necessarily ordered) and $\mathcal{A}$ the category of based surjections with trivial kernel. The same arguments as before would show that $(\mathcal{B}, \mathcal{A})$ is a conjugate pair.*

3 Monoids and Semigroup Theory

3.1 Motivation

A *semigroup* is a set endowed with an associative binary operation, and a *monoid* is a semigroup with an identity element. Hence a monoid satisfies the axioms for a group with the exception of the existence of inverses. Monoids arise naturally in categories, and we will make use of the well-established theory of semigroups to analyze these monoids in conjugate categories.

To see how monoids enter our story, suppose that $\mathcal{C}$ is any category. Given an object X of $\mathcal{C}$, an *endomorphism* is simply a map $f : X \to X$. We will let $\mathrm{End}_{\mathcal{C}}(X)$ denote the set of all endomorphisms of X. Note that any two endomorphisms on X may be composed, and the axioms for a category show that composition is an associative operation with 1_X serving as the identity. Hence each object X in a category $\mathcal{C}$ gives us the *endomorphism monoid* $\mathrm{End}_{\mathcal{C}}(X)$. The subset of self-isomorphisms of X forms a group known as the *automorphism group* of X, denoted $\mathrm{Aut}_{\mathcal{C}}(X)$.

Given an arbitrary category, not much can be said about the structure of its various endomorphism monoids. The point of our work is that the endomorphism monoids in conjugate categories carry extra structure, distinguishing them from general categories. In order to carry out this analysis, we must first review some basic structural results of semigroup theory.

Remark. *We will be making use of the material on Green's relations from [3, Chapter 1]. In fact, all of the results given in the remainder of this section are taken directly from this text. Since we are concerned only with monoids and not general semigroups, we will be translating those results into our setting. This will have the effect of simplifying a bit of the development.*

3.2 Green's relations

Throughout this discussion M will denote a fixed monoid with identity 1. Given $a \in M$, the *right ideal generated by* a is the subset $aM = \{am \mid m \in M\}$. One easily checks that this subset deserves its name, since $xy \in aM$ whenever $x \in aM$ and $y \in M$. Similarly, we have the subset $Ma = \{ma \mid m \in M\}$, the *left ideal generated by* a. Note also that each of the ideals aM and Ma contains a since M has an identity; this is one point where our development is slightly simpler than that of general semigroup theory.

Just as in the case of groups, upon fixing an element $a \in M$ we have the *right and left translation maps* $\rho_a : M \to M$ and $\lambda_a : M \to M$ defined by

$$\rho_a(x) = xa$$
$$\lambda_a(x) = ax$$

respectively. Unfortunately, we diverge from group theory at this point since these maps need not be bijections: after all, the inverse of ρ_a "should" be right translation by a^{-1}, which need not exist in M. For this reason one introduces several equivalence relations on M—known as *Green's relations*—to study this defect.

Green's relations are defined in terms of the ideals in M. We define an equivalence relation $\overset{\mathcal{L}}{\sim}$ on M by $a \overset{\mathcal{L}}{\sim} b$ if a and b generate the same left ideal, that is, if $Ma = Mb$. A separate relation $\overset{\mathcal{R}}{\sim}$ is similarly defined using right ideals. Finally, we define the $\mathcal{H}$-relation by $a \overset{\mathcal{H}}{\sim} b$ if both $a \overset{\mathcal{L}}{\sim} b$ and $a \overset{\mathcal{R}}{\sim} b$ hold. We will denote the equivalence class of $a \in M$ under the $\mathcal{L}$-relation by $\mathcal{L}_a$. The analogous notation will be used for the other relations. The most basic properties of these relations are listed in the following proposition. The proofs of these assertions are all straightforward, so we omit them.

Proposition 3.1. *The $\mathcal{L}$-relation on a monoid M has the following properties:*

- *Elements $a, b \in M$ are $\mathcal{L}$-related if and only if there exist elements $m_1, m_2 \in M$ with $a = m_1 b$ and $b = m_2 a$.*
- *The $\mathcal{L}$-relation is preserved by right multiplication.*
- *For any $a \in M$ we have $\mathcal{L}_a \subseteq Ma$.*
- *For elements $a, b \in M$, the right translation ρ_b restricts to a map $\rho_b : \mathcal{L}_a \to \mathcal{L}_{ab}$.*

Analogous results hold for the $\mathcal{R}$-relation in M, obtained by exchanging all instances of "left" and "right" in the above. In particular, the left translation λ_b restricts to a map $\lambda_b : \mathcal{R}_a \to \mathcal{R}_{ba}$.

Of course, every statement we make about left ideals or the left relation has a corresponding right-sided dual and vice versa. Hence we will only prove (or even state) roughly half of our results.

The usefulness of these relations in studying bijections is apparent in the next proposition. This result will be significant, so we will provide a careful proof.

Proposition 3.2. *Let M be a monoid and let $a, b \in M$. Suppose that $a \overset{\mathcal{R}}{\sim} b$, so that there are elements $s, t \in M$ with $as = b$ and $bt = a$. Then*

$$\mathcal{L}_a \underset{\rho_t}{\overset{\rho_s}{\rightleftarrows}} \mathcal{L}_b$$

is an inverse pair of maps (i.e., ρ_s is a bijection with inverse ρ_t). Moreover, the maps ρ_s and ρ_t are $\mathcal{R}$-class preserving.

Proof. The domains and codomains are correct by Proposition 3.1. An arbitrary element of $\mathcal{L}_a$ has the form ma for some $m \in M$. We compute:

$$\begin{aligned} \rho_t \circ \rho_s(ma) &= mast \\ &= mbt \\ &= ma. \end{aligned}$$

Hence $\rho_t \circ \rho_s = 1_{\mathcal{L}_a}$. Composition in the other order is the identity map on $\mathcal{L}_b$ by the same argument. This establishes the first claim.

To show that ρ_s is $\mathcal{R}$-class preserving we must prove that $x \stackrel{\mathcal{R}}{\sim} xs$ for all $x \in \mathcal{L}_a$. Note that it is true in general that $(xs)M \subseteq xM$. Since $\rho_t \circ \rho_s(x) = x$ we have $xst = x$, and this implies that $xM \subseteq (xs)M$. Hence $xM = (xs)M$ and $x \stackrel{\mathcal{R}}{\sim} xs$, as desired. □

3.3 $\mathcal{H}$-groups and idempotents

Our interest in Green's relations will primarily focus on the $\mathcal{H}$-relation. We will see that certain $\mathcal{H}$-classes in a monoid form groups (under the induced operation), and pinpointing when this occurs is fairly straightforward.

Proposition 3.3. *Let M be a monoid and let $a, s \in M$.*

- *If $a \stackrel{\mathcal{H}}{\sim} as$ then the map ρ_s restricts to a bijection $\rho_s : \mathcal{H}_a \to \mathcal{H}_a$.*
- *Dually, if $s \stackrel{\mathcal{H}}{\sim} as$ then the map λ_a restricts to a bijection $\lambda_a : \mathcal{H}_s \to \mathcal{H}_s$.*

Proof. We prove only the first statement. Recall that our assumption is that a and as are both $\mathcal{L}$- and $\mathcal{R}$-related. Hence, applying Proposition 3.2 (with $b = as$) implies that $\rho_s : \mathcal{L}_a \to \mathcal{L}_{as}$ is a bijection. But $\mathcal{L}_{as} = \mathcal{L}_a$ since $a \stackrel{\mathcal{L}}{\sim} as$, so we actually have a bijection $\rho_s : \mathcal{L}_a \to \mathcal{L}_a$. As ρ_s is $\mathcal{R}$-class preserving this further restricts to $\rho_s : \mathcal{H}_a \to \mathcal{H}_a$, yet we do not know that this restricts to a bijection itself.

Since $a \stackrel{\mathcal{R}}{\sim} as$, there is an element $t \in M$ with $a = ast$. The same argument as above shows that ρ_t restricts to a map $\rho_t : \mathcal{H}_a \to \mathcal{H}_a$. These maps are defined functionally in the same way as in Proposition 3.2, hence checking that they form an inverse pair is automatic. Thus $\rho_s : \mathcal{H}_a \to \mathcal{H}_a$ is a bijection with inverse ρ_t. The second assertion follows from the dual of Proposition 3.2. □

We finally arrive at the focal point of our work. Recall that an element e of a monoid M is an *idempotent* if $e^2 = e$.

Proposition 3.4. *Let e be an idempotent in the monoid M. Then the $\mathcal{H}$-class $\mathcal{H}_e$ is a group under the operation induced by M. Moreover, no $\mathcal{H}$-class in M can contain more than one idempotent.*

Proof. Since $e^2 = e$, the hypotheses of both statements in Proposition 3.3 hold (with $a = s = e$). Thus the maps ρ_e and λ_e provide bijections $\mathcal{H}_e \to \mathcal{H}_e$. In particular, for all $h \in \mathcal{H}_e$ we have $he, eh \in \mathcal{H}_e$. Rephrased, this says $e \stackrel{\mathcal{H}}{\sim} he$ and $e \stackrel{\mathcal{H}}{\sim} eh$. Applying Proposition 3.3 again, these relations imply that $\rho_h : \mathcal{H}_e \to \mathcal{H}_e$ and $\lambda_h : \mathcal{H}_e \to \mathcal{H}_e$ are bijections. Hence $\mathcal{H}_e h = \mathcal{H}_e = h\mathcal{H}_e$ for all $h \in \mathcal{H}_e$. Thus we see that the multiplication on M passes down to a closed associative operation on $\mathcal{H}_e$.

To show that e is a right-sided identity for $\mathcal{H}_e$ we use that $\rho_e : \mathcal{H}_e \to \mathcal{H}_e$ is a bijection. Given $h \in \mathcal{H}_e$ we have $\rho_e(h) = he = \rho_e(he)$. Since ρ_e is injective this gives $he = h$. A similar argument on λ_e shows that e is a left-sided identity for $\mathcal{H}_e$. Hence e is in fact a two-sided identity for $\mathcal{H}_e$.

To see the existence of inverses, fix $h \in \mathcal{H}_e$. As $\rho_h : \mathcal{H}_e \to \mathcal{H}_e$ is surjective, there must be an element $x \in \mathcal{H}_e$ with $\rho_h(x) = xh = e$. Hence x is a left inverse for h. Using λ_h we similarly produce an element y with $hy = e$. But

$$y = ey = xhy = xe = x,$$

so x is a two-sided inverse for h. Hence $\mathcal{H}_e$ is a group, as claimed. The last assertion in the proposition is now immediate. □

Since we are mainly concerned with endomorphism monoids in categories, we should establish some standard notation for this setting. We will eventually compare the $\mathcal{H}$-groups associated to various objects, so it will be convenient to record the relevant objects as well as the idempotents.

Notation 3.5. *Fix an object X of a category $\mathcal{C}$ and consider the endomorphism monoid* $\mathrm{End}_{\mathcal{C}}(X)$*. Given an idempotent $e : X \to X$ (so that $e \circ e = e$), we will denote by $\mathcal{H}_e^X$ the $\mathcal{H}$-group associated to e in the monoid* $\mathrm{End}_{\mathcal{C}}(X)$*.*

Of course, in this categorical setting the identity maps are idempotents, and in this case the $\mathcal{H}$-groups are well-known: they are the automorphism groups of the objects.

Proposition 3.6. *Let $\mathcal{C}$ be any category and fix an object X of $\mathcal{C}$. In the endomorphism monoid* $\mathrm{End}_{\mathcal{C}}(X)$ *we have $f \overset{\mathcal{H}}{\sim} 1_X$ if and only if $f : X \to X$ is an isomorphism. Hence the subgroup $\mathcal{H}_1^X$ of* $\mathrm{End}_{\mathcal{C}}(X)$ *is exactly the group* $\mathrm{Aut}_{\mathcal{C}}(X)$ *of automorphisms of X.*

Proof. Decoding the first assertion in Proposition 3.1, we see that $f \overset{\mathcal{L}}{\sim} 1_X$ if and only if there is a map l with $l \circ f = 1_X$. Similarly, $f \overset{\mathcal{R}}{\sim} 1_X$ if and only if there is a map r with $f \circ r = 1_X$. We claim that $l = r$, so that $f : X \to X$ is in fact an isomorphism. Note that we have

$$l \circ f \circ r = l \circ (f \circ r) = l \circ 1_X = l.$$

Likewise, if we instead associate on the first two terms we get

$$l \circ f \circ r = (l \circ f) \circ r = 1_X \circ r = r,$$

giving $l = r$, as claimed. □

4 The Main Result

4.1 The proof

Throughout this section $(\mathcal{B}, \mathcal{A})$ will denote a fixed conjugate pair with duality $(\ )^* : \mathcal{I}^{\text{op}} \to \mathcal{I}^*$. Fix once and for all an object B of $\mathcal{B}$ and an idempotent endomorphism $e : B \to B$. We know that e admits a three-fold factorization of the form

$$\begin{array}{ccc} B & \xrightarrow{e} & B \\ {\scriptstyle j^*}\downarrow & & \uparrow{\scriptstyle i} \\ Q & \xrightarrow{e'} & R \end{array} \tag{2}$$

with $i, j \in \mathcal{I}$ and $e' \in \mathcal{A}$. Recall that we refer to the object R as a *rank object* of e.

Lemma 4.1. *Suppose that* $(\mathcal{B}, \mathcal{A})$ *is a conjugate pair in which all maps in* $\mathcal{A}$ *are epic in* $\mathcal{B}$. *Then in the situation of diagram* (2) *above, the following identities hold:*

- $i^* \circ e = e' \circ j^*$
- $e' \circ j^* \circ i = 1_R$
- $e \circ i = i$.

Proof. The assumption is that $e = i \circ e' \circ j^*$ and composing this with i^* on the left gives the first identity. For the second identity, take the relation $e \circ e = e$ and replace each occurrence of e by $i \circ e' \circ j^*$. Composing on the left with i^* and on the right with j leaves

$$e' \circ j^* \circ i \circ e' = e'.$$

But $e' \in \mathcal{A}$ and all maps in $\mathcal{A}$ are epic in $\mathcal{B}$, so we may cancel the e' terms on the right to obtain $e' \circ j^* \circ i = 1_R$. Composing this relation with i on the left gives the third identity. □

It is worth noting that in the case of ordinary functions on sets, each of the above identities is simply a re-statement of the fact that an idempotent fixes its own image. For instance, consider the third equality. Recalling that i serves as the inclusion of the image of e into its codomain, the equation $e \circ i = i$ simply states that $e(e(x)) = e(x)$ for each $x \in B$.

Remark. *We will need this lemma from here until the end, so we will assume from now on that the maps in* $\mathcal{A}$ *are epic in* $\mathcal{B}$. *It should be noted that this extra assumption is in fact true for all known examples of conjugate pairs, although it is not known if this is purely a consequence of the definition.*

Given the factorization in diagram (2) above, we may define the "conjugation" map

$$c_i : \mathcal{H}_e^B \longrightarrow \mathrm{End}_{\mathcal{B}}(R)$$

which sends the endomorphism $\beta : B \to B$ to $c_i(\beta) = i^* \circ \beta \circ i$. Note that from Lemma 4.1 we have

$$\begin{aligned} c_i(e) &= i^* \circ e \circ i \\ &= i^* \circ i \\ &= 1_R \end{aligned}$$

so that c_i preserves identities. In fact, much more is true. (Because of the large number of factors in the compositions that follow, we will drop the composition symbol $\circ$ in favor of juxtaposition when convenient.)

Proposition 4.2. *The map* $c_i : \mathcal{H}_e^B \longrightarrow \mathrm{End}_{\mathcal{B}}(R)$ *preserves composition.*

Proof. Take endomorphisms $\alpha, \beta \in \mathcal{H}_e^B$. Since $\mathcal{H}_e^B$ is a group under composition with identity e, we have $e \circ \alpha = \alpha = \alpha \circ e$ and similarly for β. From this we have

$$\begin{aligned} c_i(\alpha) \circ c_i(\beta) &= (i^* \alpha i)(i^* \beta i) \\ &= i^*(e\alpha) i i^*(e\beta) i. \end{aligned}$$

Now replace each occurrence of e by $e = ie'j^*$ and use $i^* i = 1_R$ to obtain

$$\begin{aligned} c_i(\alpha) \circ c_i(\beta) &= e'j^* \alpha (ie'j^*) \beta i \\ &= e'j^*(\alpha e)\beta i \\ &= e'j^*(e\alpha)\beta i. \end{aligned}$$

Making the replacement $e = ie'j^*$ once again and invoking Lemma 4.1 gives

$$\begin{aligned} c_i(\alpha) \circ c_i(\beta) &= e'j^*(ie'j^*)\alpha\beta i \\ &= (e'j^* i) e'j^* \alpha\beta i \\ &= 1_R (e'j^*) \alpha\beta i \\ &= (i^* e)\alpha\beta i \end{aligned}$$

From $e \circ \alpha = \alpha$ the last line becomes

$$\begin{aligned} c_i(\alpha) \circ c_i(\beta) &= i^*(\alpha \circ \beta) i \\ &= c_i(\alpha \circ \beta). \end{aligned}$$

Hence c_i preserves composition, as claimed. ∎

Proposition 4.3. *We have $c_i(\beta) \in \mathcal{H}_1^R$ for each $\beta \in \mathcal{H}_e^B$. Hence we may restrict the codomain of c_i to obtain a group homomorphism*

$$c_i : \mathcal{H}_e^B \longrightarrow \mathcal{H}_1^R.$$

Proof. By Proposition 3.6, to show that $c_i(\beta) \in \mathcal{H}_1^R$ we need only show that $c_i(\beta)$ is an isomorphism. Since $\mathcal{H}_e^B$ is a group with identity e, there is an endomorphism $\alpha \in \mathcal{H}_e^B$ with $\alpha \circ \beta = e = \beta \circ \alpha$. We claim that $c_i(\alpha)$ is inverse to $c_i(\beta)$. To see this, note that Proposition 4.2 implies that

$$\begin{aligned} c_i(\alpha) \circ c_i(\beta) &= c_i(\alpha \circ \beta) \\ &= c_i(e) \\ &= 1_R. \end{aligned}$$

Hence $c_i(\alpha)$ is a left inverse for $c_i(\beta)$, and the analogous argument shows that it is a right inverse as well. Therefore $c_i(\beta)$ is an isomorphism, as claimed. □

Our main result asserts that $c_i : \mathcal{H}_e^B \longrightarrow \mathcal{H}_1^R$ is in fact an isomorphism. In this sense, the $\mathcal{H}$-group structure of each endomorphism monoid $\mathrm{End}_{\mathcal{B}}(B)$ is completely determined by the automorphism groups in the category. All that remains is the verification that c_i is injective and surjective.

Proposition 4.4. *The homomorphism $c_i : \mathcal{H}_e^B \longrightarrow \mathcal{H}_1^R$ is injective.*

Proof. We will show that the kernel of c_i consists of e alone. To that end, suppose that $c_i(\beta) = i^*\beta i = 1$. Replacing β by $e\beta$ we have $i^*e\beta i = e'j^*\beta i = 1$, the first equality following from Lemma 4.1. Composing this with $e'j^*$ on the right yields $e'j^*\beta e = e'j^*$, hence $e'j^*\beta = e'j^*$. Composing this with i on the left gives $ie'j^*\beta = ie'j^*$, which is simply $e\beta = e$, or $\beta = e$. Hence the conjugation map c_i is injective. □

Proposition 4.5. *The homomorphism $c_i : \mathcal{H}_e^B \longrightarrow \mathcal{H}_1^R$ is surjective.*

Proof. Let $\varphi \in \mathcal{H}_1^R$ and let $\theta = i\varphi e'j^* \in \mathrm{End}_{\mathcal{B}}(B)$. If we can show that $\theta \in \mathcal{H}_e^B$ then we may apply c_i to it, and Lemma 4.1 then gives

$$\begin{aligned} c_i(\theta) &= i^*\theta i \\ &= (i^*i)\varphi(e'j^*i) \\ &= \varphi. \end{aligned}$$

Hence we need only prove that $\theta \in \mathcal{H}_e^B$ to see that c_i is surjective.

For this, we have no choice but to show that θ is $\mathcal{H}$-related to e, that is, that θ is both $\mathcal{L}$- and $\mathcal{R}$-related to e. The key fact is that φ is an isomorphism in $\mathcal{B}$ by Proposition 3.6, hence we have access to φ^{-1}. We claim that the following identities hold:

- $\theta \circ w = e$ when $w = i\varphi^{-1}e'j^*$
- $e \circ x = \theta$ when $x = i\varphi e'j^*$
- $y \circ \theta = e$ when $y = i\varphi^{-1}i^*$
- $z \circ e = \theta$ when $z = i\varphi i^*$.

Verifying these is straightforward and simply makes repeated use of the relations in Lemma 4.1. The first two relations show that $\theta \overset{\mathcal{R}}{\sim} e$, while the latter two show that $\theta \overset{\mathcal{L}}{\sim} e$. Hence $\theta \in \mathcal{H}_e^B$, thereby completing the proof. □

This constitutes the proof of our main result:

Theorem 4.6. *Suppose that* $(\mathcal{B}, \mathcal{A})$ *is a conjugate pair of categories in which every map in* $\mathcal{A}$ *is epic in* $\mathcal{B}$. *If* $e : B \to B$ *is an idempotent in* $\mathcal{B}$ *with rank object* R, *then there exists an isomorphism* $\mathcal{H}_e^B \simeq \mathcal{H}_1^R$ *of* $\mathcal{H}$*-groups. Moreover,* $\mathcal{H}_1^R = \mathrm{Aut}_{\mathcal{B}}(R)$, *so that every* $\mathcal{H}$*-group in* $\mathcal{B}$ *is isomorphic to a group of automorphisms.*

We can now explain this homogeneity exhibited by conjugate categories. The structure of $\mathcal{I}$ as an EI-category gives a partial ordering on the isomorphism classes of objects, hence the rank objects are so ordered. An endomorphism monoid $\mathrm{End}_{\mathcal{B}}(B)$ may have many $\mathcal{H}$-groups, but with the exception of $\mathrm{Aut}_{\mathcal{B}}(B)$ they all have occurred "previously" at a lower level. Obviously the identity map on B will have B itself as a rank object, but otherwise the idempotents on B will have lower rank ("lower" to be interpreted in this partial ordering). By our theorem, in this latter case the corresponding $\mathcal{H}$-group is itself the automorphism group of an object at a lower level. Said differently, as one moves up the poset of objects, the only new $\mathcal{H}$-group is the $\mathcal{H}$-group of the identity map, the others being replicates of previous automorphism groups.

4.2 Calculations and problems

Let us illustrate the theorem for a small example inside the category Γ.

Example 4.7. *We know that* $\mathrm{Aut}_\Gamma(\mathbf{4}_+)$ *is isomorphic to the symmetric group* S_4 *(remember, 0 must remain fixed). What of the other idempotents? Borrowing notation typically reserved for permutations, the map* $e : \mathbf{4}_+ \to \mathbf{4}_+$ *given by*

$$e = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 & 4 \end{pmatrix}$$

is an idempotent. This has rank object $\mathbf{2}_+$ *since its image consists of two non-zero points. Hence* $\mathcal{H}_e$ *should be isomorphic to* $\mathrm{Aut}_\Gamma(\mathbf{2}_+)$, *which is in turn* *isomorphic to* $S_2 \simeq \mathbb{Z}_2$*. Using Proposition 3.1, it is not terribly hard to check that*

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 4 & 0 & 2 \end{pmatrix}$$

is the only endomorphism $\mathcal{H}$*-related to* e*. Note that* $e \circ f = f = f \circ e$ *and* $f^2 = e$*, so that* $\mathcal{H}_e$ *is in fact isomorphic to* $\mathbb{Z}_2$*.*

It is instructive to compute the automorphism groups in known examples of conjugate categories. In Γ, we see that $\mathrm{Aut}_\Gamma(\mathbf{n}_+)$ is isomorphic to S_n, and hence every $\mathcal{H}$-group is isomorphic to a symmetric group. The same holds true in the conjugate categories of Examples 2.9 and 2.11. In the latter, we would have $\mathrm{Aut}_\mathcal{B}(A_+) \simeq S_{|A|}$. Each automorphism group is trivial in all the remaining examples discussed here. These observations beg the following question:

Problem 4.8. *Does there exist a conjugate category admitting an automorphism group which is neither symmetric nor trivial?*

None of the known examples of conjugate pairs (here or elsewhere) provide evidence of an affirmative answer to this question. If the answer to this is indeed positive, a deeper and more interesting question would be of the following variety:

Problem 4.9. *Classify all groups which are realizable as an automorphism group in a conjugate category.*

One can easily "cheat" and realize *any* group G as an automorphism group in a conjugate category. Specifically, the category $\mathcal{C}(G)$ of Example 2.5 can be shown to be a conjugate category. It is a simple verification that $\mathrm{Aut}_{\mathcal{C}(G)}(G)$ is isomorphic to G itself, so G is realizable. Hence one should exclude this trivial construction in order to make these problems meaningful.

ACKNOWLEDGEMENTS. We are indebted to the Jepson Summer Science Institute at the University of Mary Washington for its support of this project. We would also like to thank Prof. Janusz Konieczny for helpful conversations on semigroup theory.

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Received: January, 2009