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Commuting Regular Semigroup Rings[1]

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Abstract

Strongly regular semigroup rings of non-unitary commutative rings is characterized. It is proved that a commuting regular semigroup is exactly an inflation of a Clifford semigroup and that a commuting regular ring is exactly a direct sum of strongly regular ring with a zero ring. Sufficient and necessary conditions for semigroup rings of commutative rings to be commuting regular are presented.

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1 Introduction

Recently commuting regular rings and semigroups are studied in [1, 2, 3, 13]. A semigroup or a ring A is called commuting regular if for any $x, y \in A$ there exists $z \in A$ such that $xy = yxzyx$. In this note, we characterize commuting regular semigroups, commuting regular rings and commuting regular semigroup rings of commutative rings.

A semigroup S is called regular if for any $x \in S$ there exists $y \in S$ such that $x = xyx$. A regular semigroup with the central idempotents is called a Clifford semigroup. A semigroup S is Clifford semigroup if and only if S is a semilattice of groups (cf. [5]). A semigroup S is called an inflation of a semigroup T if T is a subsemigroup of S and there is a mapping ϕ of S into T such that $\phi(x) = x$ for $x \in T$ and $xy = \phi(x)\phi(y)$ for $x, y \in S$. Tamura [11, Proposition] characterizes inflations of Clifford semigroups and shows that a semigroup S is an inflation of a Clifford semigroup if and only if S^2 is a Clifford semigroup.

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Throughout this paper R is a ring not necessarily with identity. A ring R is called strongly regular if for any $x \in S$ there exists $y \in S$ such that $x = x^2y$. A regular ring is strongly regular if and only it is reduced, *i.e.*, it does not contain nonzero nilpotent elements.

In Section 2, a characterization in [10] of strongly regular semigroup rings over unitary commutative rings is generalized to non-unitary commutative rings. In Section 3, it is proved that a semigroup is commuting regular if and only if it is an inflation of a Clifford semigroup and that a ring is commuting regular if and only if it is a direct sum of strongly regular ring with a zero ring. We present sufficient and necessary conditions under which semigroup rings of commutative rings are commuting regular. From the results in this note, many results in [1, 2, 3, 13] can be deduced immediately. Because strongly regular group rings are not yet characterized, it would be difficult to describe commuting regular semigroup rings over general rings.

2 Strongly Regular Semigroup Rings

For a ring R, let $\tilde{R}$ denote the endomorphism ring of the R-R-bimodule R. Then $\tilde{R}$ has identity. If R is regular, then $\tilde{R}$ is a commutative and regular ring with identity by [4, Theorem 1], and R can be viewed as an $\tilde{R}$-algebra by defining

$$\rho x = \rho(x), \text{ for any } \rho \in \tilde{R} \text{ and } x \in R.$$

Let $R^1 = R \times \tilde{R}$ and define the addition pointwisely and the multiplication by

$$(x, \rho)(y, \tau) = (xy + \tau x + \rho y, \rho\tau).$$

Then R^1 is a regular ring with the identity $(0, 1_R)$, where 1_R denotes the identity mapping on R, and R is imbedded in R^1 as an ideal ([4, Theorem 2]). In what follows, we will consider simply R as an ideal of R^1. It is clear that R^1 is strongly regular if and only if so is R.

For a nonzero integer n, R is called n-torsion-free if $nx = 0,\ x \in R$, implies $x = 0$.

Lemma 2.1 *Let R be a regular ring and n be a nonzero integer. Then R is n-torsion-free if and only if $nR = R$.*

Proof. Suppose R is n-torsion-free. Then for any $x \in R$, there exists $y \in R$ such that $nx = (nx)y(nx)$, that is, $n(x - nxyx) = 0$. Hence $x = n(xyx)$, and so $nR = R$.

Conversely, suppose $nR = R$. Then for $x \in R$ there exists $y \in R$ such that $x = ny$. If $nx = 0$, then $n^2y = 0$. Since R is regular, there exists $z \in R$ such that $ny = (ny)z(ny) = n^2yzy = 0$, implying $x = 0$. Thus R is n-torsion-free.

Lemma 2.2 *Let R be a regular ring and n be an nonzero integer. Then $nR = R$ if and only if $n\tilde{R} = \tilde{R}$.*

Proof. Suppose $nR = R$. For any $\rho \in \tilde{R}$, if $n\rho = 0$, then for any $x \in R$, $n(\rho x) = 0$, yielding $\rho(x) = 0$ since R is n-torsion-free by Lemma 2.1. Thus $\rho = 0$. It follows that $\tilde{R}$ is n-torsion-free, and so $n\tilde{R} = \tilde{R}$ by Lemma 2.1.

Conversely, suppose $n\tilde{R} = \tilde{R}$. Let $\rho \in \tilde{R}$ such that $n\rho = 1_R$. Then for any $x \in R$ we have $x = 1_R x = n(\rho x)$, and so $nR = R$.

Lemma 2.3 *The semigroup ring RS is strongly regular if and only if R is strongly regular and S is a semilattice of groups $G_\alpha,\ \alpha \in \Lambda$, such that every RG_α is strongly regular.*

Proof. If RS is strongly regular, then R as a homomorphic image of RS is strongly regular and S is regular by [12, Lemma 2]. Since the idempotents of RS are central, so are those of S, and hence S is a semilattice of groups $G_\alpha,\ \alpha \in \Lambda$. If S is a semilattice of groups $G_\alpha,\ \alpha \in \Lambda$, then by [8, Remark (ii)], RS is strongly regular if and only if every RG_α is strongly regular, from which the lemma follows.

Lemma 2.4 *If RS is strongly regular, then so is $\tilde{R}S$.*

Proof. If RS is strongly regular, then by Lemma 2.3 we can assume that S is a group. Thus by [7, Theorem 4.7], S is a locally finite group such that $nR = R$ for an order n of any finite subgroup of S. Since R is regular, $\tilde{R}$ is regular by [4, Theorem 1]. By Lemma 2.2 and [7, Theorem 4.7], $\tilde{R}S$ is regular.

It remains to prove that $\tilde{R}S$ is reduced. Suppose $a^2 = 0$ for some $a \in \tilde{R}S$. Let $a = \sum \rho_s s \in \tilde{R}S,\ \rho_s \in \tilde{R},\ s \in S$. Then $\sum_{st=u} \rho_s\rho_t = 0$ for any $u \in S$. For any $r \in R$, let $x = \sum(\rho_s r)s \in RS$. Then

$$x^2 = \sum_{u\in S}\Big(\sum_{st=u}(\rho_s r)(\rho_t r)\Big)u = \sum_{u\in S}\Big(\Big(\sum_{st=u}\rho_s\rho_t\Big)r^2\Big)u = 0.$$

Since RS is reduced, we have $x = 0$, yielding $\rho_s r = 0$. Therefore, $\rho_s = 0$, and so $a = 0$, as desired.

Lemma 2.5 *The semigroup ring RS is strongly regular if and only if so is R^1S.*

Proof. Without loss of generality, we can assume that R is strongly regular. Since RS is an ideal of R^1S such that $R^1S/RS \cong \tilde{R}S$, the lemma follows from Lemma 2.4.

A ring R and a semigroup S are said to be torsion-disjoint if $s^n = t^n$ and $mr = 0$ imply $s = t$ or $r = 0$ for any $s, t \in S,\ r \in R$ and positive integers m, n such that m divides n.

A periodic Abelian group H is said to be primitive relative to a unitary ring R if each element $h \in H$ has an order $n = n(h)$ such that the equation $x_1^2 + x_2^2 + x_3^2 = 0$ has only the trivial solution in the ring $K[t]/(F_n(t))$, where $(F_n(t))$ is the principal ideal generated by the cyclotomic polynomial $F_n(t)$ in the polynomial ring $K[t]$ (cf. [10]).

Theorem 2.6 *For a commutative ring R and a semigroup S, the semigroup ring RS is strongly regular if and only if R is strongly regular and S is a semilattice of periodic groups G_α, $\alpha \in \Lambda$, such that one of the following conditions holds:*

1. *R and G_α are torsion-disjoint, and G_α is Abelian;*
2. *R is torsion-free, and G_α is a direct product of the quaternion group with an Abelian group primitive relative to R^1.*

Proof. By Lemma 2.3, it is sufficient to prove that RG is strongly regular if and only if (1) or (2) hold for a group $G = G_\alpha$. If G is Abelian, then RG is strongly regular if and only if (1) holds by [6, Lemma 7]. Suppose G is not Abelian. By Lemma 2.5, RG is strongly regular if and only if so is R^1G. Since R is torsion-free if and only if R^1 contains the field of rational numbers by Lemma 2.1 and Lemma 2.2, it follows that R^1G is strongly regular if and only if (2) holds from [10].

3 Commuting Regular Semigroup Rings

Theorem 3.1 *For a semigroup S the following statements are equivalent.*

1. *S is commuting regular.*
2. *S^2 is a Clifford semigroup.*
3. *S is an inflation of a Clifford semigroup.*

Proof. (1) $\Rightarrow$ (2). For $x \in S$ and an idempotent $e \in S$ there exists $z \in S$ such that $xe = exzex$, whence $exe = xe$. Symmetrically, we have $exe = ex$. Thus $ex = xe$. It follows that the idempotents in S are central. For any $x, y \in S$ there exists $z \in S$ such that $xy = yxzyx$, and there exists $u \in S$ such that $zyx = yxzuyxz$, whence $xy = (yx)^2zuyxz$. Noting that the square of each element in a commuting regular semigroup is regular, we see that $(yx)^2 = (yx)^2v(yx)^2$ for some $v \in S$. Let $e = (yx)^2v$. Then e is an idempotent such that $(yx)^2 = e(yx)^2$ and so $xy = exy$. There exists $w \in S$ such that $xy = e(xy) = xyewxye = xyewexy$, which implies xy is regular, and so S^2 is regular. Hence S^2 is Clifford.

(2) $\Rightarrow$ (1). Note that the idempotents of S are central, and for any $x, y \in S$ there exists an idempotent e such that $xy = exy$. Set $u = ex,\ v = ey$. Then $u, v \in S^2$. Thus there exist $u', v' \in S$ such that $u = uu'u,\ v = vv'v$. Since uu' and $v'v$ are central idempotents, we have

$$\begin{aligned} yxu'v'exyu'v'yx &= v(uu')v'uvu'(v'v)u \\ &= v(v'v)v'uvu'(uu')u \\ &= (vv')uv(u'u) \\ &= u(uu')(vv')v \\ &= uv \\ &= xy. \end{aligned}$$

Thus, S is commuting regular.

(2) $\Leftrightarrow$ (3) follows from [11, Proposition].

Theorem 3.2 *For a ring R the following statements are equivalent.*

1. *R is commuting regular.*
2. *R^2 is strongly regular.*
3. *R is a direct sum of a strongly regular ring with a zero ring.*

Proof. (1) $\Rightarrow$ (3). Suppose R is commuting regular. Then by Theorem 3.1 the idempotents of R are central and products of any two elements of R are regular. Let I be the ideal generated by the idempotents in R. Then for any $x \in I$ there exists an idempotent e such that $x = ex$. Thus I is strongly regular. Let $N = \{x \in R | xI = Ix = 0\}$. Then N is an ideal of R and $I \cap N = 0$. Note that $R^2 \subset I$. Then $N^2 \subset R^2 \cap N \subset I \cap N = 0$.

Now for any $x \in R$, x^2 is regular, and so there exists an idempotent e such that $x^2 = ex^2$, whence $(x - ex)^2 = 0$. For any $a \in I$, there exists an idempotent f such that $a = fa$. Noting that $((x - ex)f)^2 = (x - ex)^2 f = 0$ and $(x - ex)f \in I$, we see that $(x - ex)f = 0$ since I is reduced. Thus $(x - ex)a = (x - ex)fa = 0$ and $a(x - ex) = af(x - ex) = 0$. It follows that $x - ex \in N$. Since $x = ex + (x - ex)$, we have $R = I + N$. Thus $R = I \oplus N$.

(3) $\Rightarrow$ (2). Suppose $R = I \oplus N$, where I is strongly regular and $N^2 = 0$. Then $R^2 = I^2 = I$ is strongly regular.

(2) $\Rightarrow$ (1). Let S be the multiplicative semigroup of R. Then $S^2 \subset R^2$. Hence the idempotents of S^2 are central and for any $x, y \in S$ there exists $r \in R$ such that $xy = xyrxy,\ r = rxyr$, whence $r \in S^2$. It follows that S^2 is a Clifford semigroup. By Theorem 3.1 S and so R are commuting regular.

We remark that a ring which is a direct sum of a strongly regular ring with a zero ring is characterized in [9]

Lemma 3.3 *For a ring R and a semigroup S, the following conditions are equivalent.*

1. *RS is commuting regular.*
2. *R^2S^2 is strongly regular.*
3. *R is commuting regular and S is an inflation of a semilattice of periodic groups G_α, $\alpha \in \Lambda$, such that every R^2G_α is strongly regular.*

Proof. (1) $\Leftrightarrow$ (2) follows from Theorem 3.2.

(2) $\Leftrightarrow$ (3) follows from Theorem 3.2 and Lemma 2.3.

Notice that a band is a semilattice of groups if and only if it is a semilattice, if and only if it is a semilattice of trivial groups. Hence we have

Corollary 3.4 *For a band S, the semigroup ring RS is commuting regular if and only if R is commuting regular and S is a semilattice.*

Theorem 3.5 *For a commutative ring R and a semigroup S, the following conditions are equivalent.*

1. *RS is commuting regular.*
2. *R^2S^2 is strongly regular.*
3. *R is commuting regular and S is an inflation of a semilattice of periodic groups G_α, $\alpha \in \Lambda$, such that one of the following conditions holds.*
   (a) *R^2 and G_α are torsion-disjoint and G_α is Abelian.*
   (b) *R^2 is torsion-free, and G_α is a direct product of the quaternion group with an Abelian group primitive relative to $(R^2)^1$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 3.3.

(1) $\Leftrightarrow$ (3) follows from Lemma 3.3 and Theorem 2.6.

Corollary 3.6 *For a finite ring R such that $R^2 \neq 0$, the semigroup ring RS is commuting regular if and only if R is a commutative commuting regular ring, S is a commutative, commuting regular and periodic semigroup, and R and S are torsion-disjoint.*

Proof. We first observe that a strongly regular finite ring is commutative and that a semigroup S is a commutative, commuting regular and periodic semigroup if and only if S is an inflation of a semilattice of periodic Abelian groups by Theorem 3.1. Then the corollary follows from Theorem 3.5.

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