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Semi-Essentially Compressible Modules and Rings

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Abstract

Let R be a ring and let M be a unitary right $R-$module. A right R-module M is called (*essentially*) *compressible* if for each (essential) non-zero submodule N of M, there exists a monomorphism $\theta : M \to N$. Some authors studied (essentially) compressible modules and other use this property to characterize other rings and modules. We call an R-module M *semi-essentially compressible* if for each essential submodule N of M, there exists a monomorphism $\theta : M \to N^{(I)}$ for some set I. Semi-essentially compressible modules form a larger class than that of the essentially compressible modules. Over right FBN rings, both classes are coincide. In this paper, some results on essentially compressible modules will extend to semi-essentially compressible modules. A module M is semi-essentially compressible if and only if it is isomorphic to a direct sum of semisimple module and a semi-essentially compressible module with zero socle. It follows that every right strongly semiprime ring is a direct sum of a semisimple ring and a right strongly semiprime ring with zero socle. If M_R is a finitely generated quasi-projective module then M_R is semi-essentially compressible if and only if the $\mathrm{End}_R(M)$ is a right strongly semiprime ring.

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1 Introduction

Throughout the paper rings will be an associative ring with unit and modules will be unitary and right. Let R be a ring, then the category of right $R-$modules will be denoted by Mod$-R$. By $K \leq M$ we usually mean that K is a submodule of M, and $K \leq_e M$ indicates that K is essential submodule of M. A right R-module N is said to be M-*generated* if there exists

an epimorphism from a direct sum of copies of M_R to N_R. For any right $R-$module M, the full subcategory of Mod$-R$, whose objects are submodules of $M-$generated modules, is denoted by $\sigma[M]$. The $M-$injective hull of $N \in \sigma[M]$ is written as $\widehat{N}$.
For unexplained notions we refer to [1] and [13].
Let M be a right R-module. Consider the following properties for M:
(i) for every $N \leq_e M$, $R/An_R(M) \hookrightarrow N^I$ for some set I;
(ii) for every $N \leq_e M$, $M \in \sigma[N]$;
(iii) for every $N \leq_e M$, $M \hookrightarrow N^{(I)}$ for some set I;
(iv) for every $N \leq_e M$, $R/An_R(M) \hookrightarrow N^r$ for some $r \in \mathbf{N}$;
(v) for every $N \leq_e M$, $M \hookrightarrow N^I$ for some set I;
(vi) for every $K \leq M$, $M/\tau_K(M) \in \sigma[K]$, where $\tau_K(M) = \{U \leq M \mid \mathrm{Hom}_R(U, \widehat{KT}) = 0$, for $T = \mathrm{End}_R(\widehat{M})$ $\}$;
(vii) for every $N \leq_e M$, $M \hookrightarrow N^{(n)}$ for some $n \in \mathbf{N}$;

The conditions (i)-(v) stated for every submodule were considered in [14] transferring primeness conditions from rings to modules. In particular, modules satisfying (ii) for every submodule $N \leq M$, were called strongly prime modules and modules satisfying (vi) is called strongly semiprime in [3]. From the arguments given there it follows that all these conditions are in fact distinct.
Here we have for any module M the implications

$$\begin{array}{ccccccc} (vii) & \Rightarrow & (iii) & \Rightarrow & (v) & & \\ & & \Downarrow & & \Downarrow & & \\ (vi) & \Rightarrow & (ii) & \Rightarrow & (i) & \Leftarrow & (iv) \end{array}$$

For $M = R_R$, the conditions (ii), (iii), (iv), (vi) and (vii) are equivalent and characterize the right strongly semiprime rings introduced by Handelman [8] and further studied by Kutami-Oshiro [9] and extended to modules by Wisbauer-Beidar as strongly semiprime and properly semiprime module [3] and [2]. We shall consider some properties of modules that satisfy (iii).

Following [10] and [11], the right R-module M is called (*essentially*) *compressible* if for each (essential) non-zero submodule N of M, there exists a monomorphism $\theta : M \to N$. For example, if R is any domain then every right ideal of R is a compressible R-module. The notion of (essentially) compressible modules appeared in several papers for example see [11], [12], [15], [16] and [17]. In this paper an R-module is called semi-essentially compressible if it satisfies the (iii) condition. Clearly the class of essentially compressible is larger than the class of compressible(for example any semisimple of length > 1 is essentially compressible which is not compressible). We show that the class of semi-essentially compressible is also larger than the class of essentially compressible (see example 2.5). There are more results about essentially compressible modules in [11], we shall prove almost all results of this paper for semi-essentially compressible modules. In [17] Zhou states some characterizations of modules M such that in $\sigma[M]$ every cyclic or every finitely generated module is essentially compressible in terms of weakly injective and tight modules, using our results one can replace in some results of [17] the essentially compressible condition by semi-essentially compressible. We rewritten and repair some proofs in [11] for reader is convenient.

2 Semi-essentially compressible modules

Lemma 2.1. *Let $M \leq \bigoplus_I L_i$ for R-modules M and L_i where $i \in I$ for some set I. Then there exist a subset J of I such that $\pi_J(M) \cap L_j \neq 0$ for all $j \in J$ and $Ker(\pi_J) \cap M = 0$.*

Proof. Let $\Lambda = \{A \subseteq I \mid \bigoplus_A \cap M = 0\}$ then by zorn's Lemma Λ has maximal element, say A_0. Put $J = I \setminus A_0$, if $\pi_J(M) \cap L_{j_0} = 0$ for some $j_0 \in J$ then we claim that $A_1 \in \Lambda$ where $A_1 = A_0 \cup \{j_0\}$ because if $0 \neq x \in \bigoplus_{A_1} L_i \cap M$ then $x = l_A + l_{j_0}$ where $l_A \in \bigoplus_A L_i$ and $l_{j_0} \in L_{j_0}$ so that $\pi_J(x) = \pi_J(l_A) + \pi(l_{j_0}) = l_{j_0}$ and hence $l_{j_0} \in \pi_J(M) \cap L_{j_0} = 0$. Thus $0 \neq x = l_A \in \bigoplus_A \cap M = 0$ which is a contradiction.
Now if $\pi_J(m) = 0$ then $m \in \bigoplus_{A_0} L_i \cap M = 0$ then $\mathrm{Ker}(\pi_J) \cap M = 0$.

A right $R-$module M is *retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for every non-zero submodule N of M.

Theorem 2.2. *Every semi-essentially compressible module is retractable.*

Proof. Let $\Lambda = \{L \leq M \mid$ there exist a monomorphism from L to $M\}$ then we can chose a maximal independent elements of Λ such as $\bigoplus L_i$ so there exist a submodule W of M such that $(\bigoplus L_i \bigoplus W)$is an essential submodule of M and $W \tilde{\cap} N = 0$. By our assumption there exist a monomorphism $\theta : M \to (\bigoplus L_i)^{(I)} \bigoplus (W)^{(I)}$ if we get $\alpha = \pi\theta$ where $\pi : (\bigoplus L_i)^{(I)} \bigoplus (W)^{(I)} \to (\bigoplus L_i)^{(I)}$ is natural projection then we claim that $L := N \cap \mathrm{Ker}\alpha = 0$ since $\alpha(L) = 0 \Leftrightarrow \pi\theta(L) = 0 \Leftrightarrow \theta(L) \subseteq W^{(I)}$ then by above lemma there exist a subset B of I such that $\theta(L) \hookrightarrow W^{(B)}$ and $\theta(L) \cap W_i \neq 0$. We know that $L \cong \theta(L)$ so $N \tilde{\cap} W \neq 0$ which is contradiction.

Almost all results in [11] about essentially compressible modules can be extended to semi-essentially compressible modules, we shall collect these results in Theorem 2.3 Some of the results are proved similarly, hance we omit their proofs.
Let M be an R-module. M is *mono-equivalent* to an R-module M' if there

exist R-monomorphisms $\alpha : M \to M'$ and $\beta : M' \to M$, and in this case M and M' are called mono-equivalent.

Theorem 2.3.
(a) *Every direct sum of semi-essentially compressible modules is semi-essentially compressible.*
(b) *Being semi-essentially compressible is a Morita invariant property of modules.*
(c) *Every semi-essentially compressible module is retractable.*
(d) *The following conditions are equivalent.*
 (i) *M_R is semi-essentially compressible*
 (ii) *$M \simeq M_1 \oplus M_2$ where M_1 is a semisimple R-module and M_2 is an semi-essentially compressible R-module with zero socle.*
 (iii) *M_R is semi-mono equivalent to $M_1 \oplus M_2$ where M_1, M_2 are semi-essentially compressible R-modules such that M_1 is nonsingular and M_2 is singular.*
 If further R is a right hereditary ring, then (i)-(iii) are equivalent to :
 (iv) *$M \cong M_1 \oplus M_2$ where M_1, M_2 are semi-essentially compressible R-modules such that M_1 is singular and M_2 is projective.*
(e) *Let M be a right semi-essentially compressible R-module then:*
 e-(1) *If N is either an essential submodule of M_R or a fully invariant submodule of M_R, then N_R is also a semi-essentially compressible module.*
 e-(2) *Let $N \leq M_R$ such that $\theta(N) \subseteq N^{(I)}, (\pi\theta)^{-1}(N) \subseteq N$ where I is a non-empty set, $\theta : M \to M^{(I)}$ is an injective homomorphism and $\pi : M^{(I)} \to M$ is canonical projection. Then M/N is a semi-essentially compressible module.*
 e-(3) *The $ann_R(M)$ is a semiprime ideal of R.*
 e-(4) *$\widehat{M}_R$ has no fully invariant essential submodule.*
 e-(5) *$M = Z(M) \oplus L$ for some semi-essential compressible submodule L, if M_R is quasi-injective.*
 e-(6) *If M_R is nonsingular then it embeds in a free R-module.*
 e-(7) *If further M_R is finitely generated with $S = End_R(M)$ then we have:*
  (I) *M_R does not contain an infinite direct sum of non-zero fully invariant submodules. Consequently, u.dim $({}_SM) < \infty$ if R is a commutative ring.*
  (II) *If ${}_SM$ is a prime module, then for each nonzero submodule U of M_R, there exists a positive integer n and $f_i \in Hom_R(U, M)$ $(1 \leq i \leq n)$ such that M embeds in $\sum_i^n f_i(U)$. Consequently if ${}_SM$ is a prime and $Z(M_R) = 0$, then u.dim $({}_SM) < \infty$ if and only if M_R has a uniform submodule.*

Proof. e-(4) Let N be a fully invariant essential submodule of $\widehat{M}_R$. By hypothesis, there exist a submodule L of $N^{(I)}$ for some set I and a isomorphism $\theta : L \to M$. Now θ can be extended to $\bar{\theta} \in \mathrm{Hom}_R(N^{(I)}, (\widehat{M}))$. If $D : \widehat{M} \to \widehat{M}^I$ denotes the diagonal map, given by $D(m) = (m, m, ...)$ if we suppose $h = D\bar{\theta}$ we can extend h to $\bar{h} : \widehat{M}^I \to \widehat{M}^I$ Thus we have $\bar{h}(N^I) \leq N^I$ and hence $\bar{\theta}(N^{(I)}) \leq N$ so that $\bar{\theta}(L) \leq N$ and $f(L) \leq N$ thus $M \leq N$. It follows that $\widehat{M} = \mathrm{End}_R(\widehat{M})M \subseteq \mathrm{End}_R(\widehat{M})N = N$.

e-7-(I) Let $N = N_1 \oplus N_2 \oplus \cdots$ be any direct sum of fully invariant submodules of M_R. It is well known that there exists a submodule N_0 of M_R such that $N \cap N_0 = 0$ and $N \oplus N_0$ is essential submodule of M_R. By our assumption, there is a monomorphism $\varphi : M \to (N \oplus N_0)^{(t)}$ for some positive integer t. Since M_R is finitely generated, we can assume that $\varphi(M) \subseteq N_1^{(t)} \oplus \cdots \oplus N_n^{(t)} \oplus N_0^{(t)}$ for some positive integer n. It follows, by hypothesis, that $\pi_{ij}\varphi(N_{n+1} \oplus N_{n+2} \cdots) \subseteq \pi_{ij}\varphi(M) \cap (N_{n+1} \oplus N_{n+2} \oplus \cdots) = 0$ where π_{ij} are the natural projections from $N_1^{(t)} \oplus \cdots \oplus N_n^{(t)} \oplus N_0^{(t)}$ on j'th component of $N_i^{(t)}$ for $0 \leq i \leq n$ and $1 \leq j \leq t$. Thus $N_{n+1} \oplus N_{n+2} \cdots \subseteq \cap_{i,j}\mathrm{Ker}\pi_{ij}\varphi \subseteq \mathrm{Ker}\varphi = 0$.

Theorem 2.4. *If R is a right FBN ring or a semiprime right Goldie ring, then an $R-$module M is essentially compressible module if and only if it is semi-essentially compressible.*

Proof. Let M_R be semi-essentially compressible. If R is a right FBN ring then M_R embeds in a direct sum of critical compressible R-modules and hence M_R is essentially compressible, see [11, Theorem 2.6] and its proof. Also if R is semiprime right Goldie, then M_R embeds in a free R-module and so it is essentially compressible, see [11, Theorem 2.3].

Example 2.5. [6] provides example of quasi-injective simple directly finite rings which are not Artinian. Let R be a such ring R being a simple ring is semi-essentially compressible R-module, but if R_R is essentially compressible then by [11, Theorem 5.9], R should be Artinian that is not. Therefore R_R is semi-essentially compressible which is not essentially compressible.

3 Right strongly semiprime endomorphism rings

Proposition 3.1. *A ring R is right strongly semiprime if and only if R_R is semi-essentially compressible $R-$module.*

Proof. This is evident.

Corollary 3.2. *Let R be a right strongly semiprime ring, then:*

(a)$R \cong R_1 \oplus R_2$ *where* R_1 *is a semisimple ring and* R_2 *is a right semi-essentially compressible ring with zero right socle.*
(b)*If* I *is an ideal of* R *such that* $I = ann_R U$ *for some* R*-module* U*. Then* R/I *is strongly semiprime ring.*
(c)*If* S *is a commutative right denominator subset of* R*, then* RS^{-1} *is right strongly semiprime ring.*

Proof. (a) By Proposition 3.1 R_R is semi-essentially compressible and hence $R = \mathrm{Soc}(R_R) \oplus J$ by Theorem 2.3(d) where $J \leq R_R$. Now since R is a semiprime ring, $\mathrm{Soc}(R_R)$ is generated by a central idempotent of R. It follows that J is an ideal of R. The result is now clear by Theorem 2.3(e).
(b) Let J/I be an essential right ideal of R/I then J is an essential right ideal of R so by our assumption there exist elements $a_1, \cdots, a_t$ in J such that $\bigcap \mathrm{r.ann} a_i = 0$. If $a_i r \in I$ then $a_i r U = 0$ for $(1 \leq i \leq t)$ and $rU = 0$ thus $r \in I$ and $\bigcap \mathrm{r.ann}(\bar{a}_i) = 0$ as elements of R/I. Thus R/I is a strongly semiprime ring.
(c) Suppose R is a right strongly semiprime ring. For $\varphi : R \to RS^{-1}$ we have $\varphi(R) \cong R/K$ where $K = \mathrm{Ker}\varphi = \{r | rs = 0 \text{ for some } s \in S\}$. We claim R/K is right strongly semiprime ring. Let J/K be an essential ideal of R/K then J is an essential right ideal of R so there exist $x_1, ..., x_n \in J$ such that $\cap_1^n \mathrm{r.ann}_R(x_i) = 0$. Now if $x_i r \in K$ $(1 \leq i \leq n)$ then $x_i r s_i = 0$ for some $s_i \in S$ $(1 \leq i \leq n)$. Let $s = s_1 \cdots s_n$ then $rs \in \bigcap \mathrm{r.ann}(x_i) = 0$ so that $r \in K$ and $\cap_1^n \mathrm{r.ann}_{R/K}(\bar{x_i}) = 0$ thus R/K and hence $\varphi(R)$ are strongly semiprime ring and $\varphi(R) \preceq_e RS^{-1}$ and RS^{-1} is strongly semiprime ring.

Theorem 3.3. *Suppose that* M_R *is quasi-projective and let* $S = End_R(M)$*.*

(a) *If* M_R *is finitely generated semi-essentially compressible, then* S *is a strongly semiprime ring.*
(b) *If* S *is a strongly semiprime ring, then* M_R *is semi-essentially compressible.*

Proof. (a) Let I be an essential right ideal of S. Then by [7, Proposition 2.1(c)], IM is an essential submodule of M_R and hence there exists a monomorphism $f \in \mathrm{Hom}_R(M, (IM)^n) \cong I^n$ for some positive integer n. Clearly $f = (f_1, \cdots, f_n)$ with $\bigcap_1^n \mathrm{r.ann}(f_i) = 0$. Thus S is a strongly semiprime ring.
(b) Let N be an essential submodule of M_R. Then by [7, Proposition 2.1(b)] , $I = \mathrm{Hom}_R(M, N)$ is an essential right ideal of S and hence by the semi-essentially compressible condition on S, I^n contains a right regular element $(f_1, \cdots, f_n)$ for some positive integer n. Now $0 = \bigcap \mathrm{r.ann}_S(f_i) = \mathrm{Hom}_R(M, \bigcap \ker f_i)$ and so by our assumption, $\bigcap \ker f_i = 0$. Consequently, M_R is embedded

in N^n, proving that M_R is semi-essentially compressible.

Corollary 3.4. *Let R be a right strongly semiprime ring and $e^2 = e \in R$ such that $eR(1-e) = 0$ then eRe is a right strongly semiprime ring.*

Proof. Apply Theorem 3.3 for $M = eR$ and note that eR is a fully invariant submodule of R_R because $eR(1-e) = 0$. Hence eR is semi-essentially compressible by Proposition 3.1 and Theorem 2.3 (e).

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