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Gassner Representation of the Pure Braid Group P_4

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Abstract. The reduced Gassner representation is a multi-parameter representation of P_n, the pure braid group on n strings. Specializing the parameters $t_1, t_2, \ldots, t_n$ to nonzero complex numbers $z_1, z_2, \ldots, z_n$ gives a representation $G_n(z_1, \ldots, z_n) : P_n \rightarrow GL(\mathbb{C}^{n-1}) = GL_{n-1}(\mathbb{C})$. It was proved that the representation is irreducible if and only if $z_1 \ldots z_n \neq 1$. In our work, we consider the case $n = 4$ and we determine the composition factors of $G_4(z_1, \ldots, z_4)$ when it is reducible. Our main theorem shows that the reduced Gassner representation $G_4(z_1, \ldots, z_4)$ of degree three is either a direct sum of one-dimensional representations or it has a composition factor of degree 2, namely, $\widehat{G_4}(z_1, \ldots, z_4)$, which is an extension of the irreducible representation $G_3(z_1, z_2, z_3)$ to P_4.

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1. Introduction.

The pure braid group, P_n , is a normal subgroup of the braid group, B_n , on n strings. It has a lot of linear representations. One of them is the Gassner representation which comes from the embedding $P_n \rightarrow \text{Aut}(F_n)$, by means of Magnus representation [2, p.119]. According to Artin, the automorphism corresponding to the braid generator σ_i takes x_i to $x_i x_{i+1} {x_i}^{-1}$, x_{i+1} to x_i , and fixes all other free generators. Applying this standard Artin representation to the generator of the pure braid group, we get a representation of the pure braid group by automorphisms. Such a representation has a composition factor, the reduced Gassner representation $G_n(t_1, \ldots, t_n)$: $P_n \rightarrow GL_{n-1}(\mathbb{Z}[{t_1}^{\pm 1}, \ldots, {t_n}^{\pm 1}])$, where $t_1, \ldots, t_n$ are indeterminates. Unlike the Burau representation of the braid group, B_n, that also arises as a subgroup of the automorphism group of a free group with n generators, there is no explicit representation which is agreed to be "the" reduced Gassner representation of the pure braid group. In our work, we define the reduced Gassner representation up to equivalence. Specializing the indeterminates $t_1, \dots, t_n$ to nonzero complex numbers $z_1, \dots, z_n$, we get a representation $G_n(z_1, \dots, z_n) : P_n \to GL_{n-1}(\mathbb{C}) = GL(\mathbb{C}^{n-1})$ which is irreducible if and only if $z_1 \dots z_n \neq 1$ (see [1]). It was shown that the reduced Burau representation $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$ is either irreducible or it has a composition factor of degree $n-2$ that is irreducible. The composition factor is $\widehat{\beta}_n(z) : B_n \to GL_{n-2}(\mathbb{C}) = GL(\mathbb{C}^{n-2})$ where $\widehat{\beta}_n(z)|B_{n-1} = \beta_{n-1}(z)$. In our work, we get a similar result concerning the pure braid group on 4 strings. We consider the representation $G_4(z_1, \dots, z_4) : P_4 \to GL_3(\mathbb{C}) = GL(\mathbb{C}^3)$ and we describe the composition factors of $G_4(z_1, \dots, z_4)$ when it is reducible.

In section two, we present the reduced Gassner representation of P_4 by choosing a free generating set with n generators to be our basis and by applying the Magnus representation as was suggested by J. Birman in [2].

In section three, we diagonalize the matrix corresponding to some element in P_4 by an invertible matrix, say T, and conjugate the reduced Gassner representation of P_4 by the same matrix T. Diagonalizing one of the matrices means that the eigen vectors of this diagonal matrix are the standard unit vectors in $\mathbb{C}^3$ or a linear combination of some of them in case of equal eigenvalues. Our objective is to look for invariant subspaces, if exist. Then we present our main theorem that shows that if $G_4(z_1, \dots, z_4) : P_4 \to GL_3(\mathbb{C}) = GL(\mathbb{C}^3)$ is reducible then either the representation is a direct sum of one-dimensional representations or its composition factor $\widehat{G_4}(z_1, \dots, z_4) : P_4 \to GL_2(\mathbb{C}) = GL(\mathbb{C}^2)$ is an extension of the (irreducible) reduced Gasssner representation $G_3(z_1, \dots, z_3)$ to P_4.

2. Definitions

Definition 1. *The braid group on n strings, B_n, is an abstract group which has a presentation with generators:*

$$\sigma_1, \dots, \sigma_{n-1}$$

and defining relations :

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad for \quad i = 1, 2, ..., n-2$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad if \quad |i-j| \geq 2$$

The generators $\sigma_1, \dots, \sigma_{n-1}$ are called the standard generators of B_n. Let t be an indeterminate and $\mathbb{C}[t^{\pm 1}]$ represent the Laurent polynomial ring over complex numbers.

Definition 2. *The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \to S_n$, defined by $\sigma_i \to (i, i+1)$, $1 \leq i \leq n-1$. It has the* *following generators:*

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}{\sigma_i}^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1},\quad 1\le i<j\le n$$

We will construct for each $j = 1, \ldots, n$ a free normal subgroup of rank $n-1$, namely, U_j. Let U_j be the subgroup generated by the elements

$$A_{1,j}, A_{2,j}, \ldots, A_{j-1,j}, A_{j,j+1}, \ldots, A_{j,n},$$

where $A_{i,j}$ are those generators of P_n. In other words, the generators of U_j are $A_{i,j}$ where $A_{i,j} = A_{j,i}$ whenever $i > j$ and $j = 1, 2, \ldots, n$. It is known that U_j generates a free subgroup of P_n which is isomorphic to the subgroup U_n freely generated by $\{A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}\}$. This is intuitively clear because it is quite arbitrary how we assign indices to the braid "strings".

We will follow the idea suggested by J. Birman to define our reduced Gassner representation of the pure braid group. Let F_4 be the free group of rank 4, with free basis $x_1, \ldots, x_4$. In F_4 , we introduce the free generating set $g_1, \ldots, g_4$ with $g_i = x_1 \ldots x_i$. The Jacobian matrix is defined as follows:

$$J(A_{i,j}) = \begin{pmatrix} D_1(A_{i,j}(g_1)) & \ldots & D_4(A_{i,j}(g_1)) \\ \vdots & & \vdots \\ D_1(A_{i,j}(g_4)) & \ldots & D_4(A_{i,j}(g_4)) \end{pmatrix},$$

where $D_j = \phi d_j$. Here d_j is the Fox derivative defined in [2] and $\phi(g_i) = t_1 \ldots t_i$.

The action of the braid generator σ_i on the basis $\{g_1, \ldots, g_4\}$ is defined as follows:

$$\sigma_i : \begin{cases} g_k \to g_k, & k \neq i \\ g_i \to g_{i+1}{g_i}^{-1}g_{i-1}, & i \neq 1 \\ g_1 \to g_2{g_1}^{-1}, & i = 1 \end{cases}$$

Under the action of the pure braid generators on the basis $\{g_1, g_2, g_3, g_4\}$, we determine the Gassner representation of P_4. Having done this, we observe that the last row of these 4×4 matrices is $(0, 0, 0, 1)$. Hence the last row and column may be deleted to obtain a 3×3 representation of P_4 and is denoted by $G_4(t_1, \ldots, t_4)$.

Definition 3. *The reduced Gassner representation* $G_4(t_1, \ldots, t_4)$: $P_4 \to GL_3(\mathbb{Z}[{t_1}^{\pm 1}, \ldots, {t_n}^{\pm 1}])$ *is given by*

$$A_{1,2} = \begin{pmatrix} t_1t_2 & 1-t_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{1,3} = \begin{pmatrix} t_3 & t_3(t_1-1) & 1-t_1 \\ t_3-1 & -t_3+t_1t_3+1 & 1-t_1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ t_2(1-t_3) & t_2t_3 & 1-t_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_{1,4} = \begin{pmatrix} t_4 & 0 & t_4(t_1-1) \\ t_4-1 & 1 & t_4(t_1-1) \\ t_4-1 & 0 & -t_4+t_1t_4+1 \end{pmatrix},$$

$$A_{2,4} = \begin{pmatrix} 1 & 0 & 0 \\ t_2(1-t_4) & t_4 & t_4(t_2-1) \\ t_2(1-t_4) & t_4-1 & -t_4+t_2t_4+1 \end{pmatrix}, \; A_{3,4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_3(1-t_4) & t_3t_4 \end{pmatrix}.$$

Here $A_{i,j}$ means $G_4(t_1,\dots,t_4)(A_{i,j})$. For simplicity, we still denote the matrix for $A_{i,j}$ by $A_{i,j}$.

3. Composition Factors of $G_4(z_1,\dots,z_4)$

Specialize the indeterminates t_1, t_2, t_3 and t_4 to nonzero complex numbers a, b, c and d respectively. For $z = (a,b,c,d) \in (\mathbb{C}^*)^4$, it was proved in [1] that the complex specialization of the reduced Gassner representation of P_4, namely, $G_4(z) : P_4 \to GL_3(\mathbb{C})$ is irreducible if and only if $abcd \neq 1$. Now, we suppose that $G_4(z)$ is reducible and $abcd = 1$. Our main work is to determine the composition factors of the reducible representation.

Theorem 1. *Let $G_4(z) : P_4 \to GL_3(\mathbb{C}) = GL(\mathbb{C}^3)$ be the complex specialization of the reduced Gassner representation. Let $z = (a,b,c,d) \in (\mathbb{C}^*)^4$ and $abcd = 1$.*

(i) If $a = 1$ or $d = 1$ then $G_4(a,\dots,d)$ is the direct sum of one-dimensional representations.

(ii) If $a \neq 1$ and $d \neq 1$ then the irreducible representation $G_3(a,b,c)$: $P_3 \to GL_2(\mathbb{C}) = GL(\mathbb{C}^2)$ extends to a (necessarily irreducible) representation $\widehat{G_4}(a,\dots,d) : P_4 \to GL_2(\mathbb{C}) = GL(\mathbb{C}^2)$.

Proof. (i) Let $a = 1$. Under the assumption that $abcd = 1$, we get that $bcd = 1$. Making this substitution in Definition 3, we obtain a 3×3 representation of P_4 where the $(1,2)$ and $(1,3)$ entries of each of the matrices are zeros. Therefore, we may delete the first row and the first column to obtain a 2×2 representation. This is true because the subspace generated by e_2 and e_3 is invariant. Here, we regard $M_3(\mathbb{C})$ as acting from the left on column vectors and acting from right on row vectors. The 2×2 representation is reduced further to one-dimensional representations since the subspace generated by the column vector $(1-b, 1-bc)$ is invariant and the subspace generated by the row vector $(1-bc, b-1)$ is also invariant. Thus, by Theorem 4 of [1], $\widehat{G_4}(a,\dots,d) : P_4 \to GL_2(\mathbb{C}) = GL(\mathbb{C}^2)$ is the direct sum of one-dimensional representations.

Let $d = 1$. As above, we have that $abc = 1$. After substitution in Definition 3, we get 3×3 matrices whose $(3,1)$ and $(3,2)$ entries are zeros. The subspace generated by e_1 and e_2 is invariant and we may delete the last row and the last column to obtain a 2×2 representation. Again, the subspace generated by the column vector $(1-a, 1-ab)$ and the subspace generated by the row vector $(1-ab, a-1)$ are invariant under the 2×2 representation. Therefore, $\widehat{G_4}(a, \ldots, d) : P_4 \to GL_2(\mathbb{C}) = GL(\mathbb{C}^2)$ is the direct sum of one-dimensional representations.

(ii) **Let** $d \neq 1$, **and** $a \neq 1$. We diagonalize the matrix that corresponds to the pure braid $A_{1,4}A_{1,3}A_{1,2}$ by a matrix T. The diagonal form becomes

$$A_{1,4}A_{1,3}A_{1,2} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & abcd \end{pmatrix}$$

Since $abcd = 1$ and $a \neq 1$, it follows that any non-zero invariant subspace of all the matrices of P_4 should contain at least one of e_3 or $ue_1 + ve_2$, where $(u, v) \neq (0, 0)$. The conjugating matrix T is

$$T = \begin{pmatrix} 0 & 0 & \dfrac{bcd-1}{bc(d-1)} \\ 0 & 1 & \dfrac{cd-1}{c(d-1)} \\ 1 & 0 & 1 \end{pmatrix}.$$

We observe that $det(T) = \dfrac{1-bcd}{bc(d-1)} = \dfrac{a-1}{abc(d-1)} \neq 0$. Now we conjugate the reduced Gassner representation, $G_4(a, \ldots, d)$, by the invertible matrix T to get an equivalent representation of degree 3. The matrices are

$$T^{-1}A_{1,2}T = \begin{pmatrix} 1 & abc-1 & 0 \\ 0 & ab & 0 \\ 0 & \dfrac{-bc(a-1)(d-1)}{bcd-1} & \dfrac{bcd(1-a)}{bcd-1} \end{pmatrix},$$

$$T^{-1}A_{1,3}T = \begin{pmatrix} abc & c(1-abc) & 0 \\ a(b-1) & 1+ac-abc & 0 \\ \dfrac{-bc(a-1)(d-1)}{bcd-1} & \dfrac{bc^2(a-1)(d-1)}{bcd-1} & \dfrac{bcd(1-a)}{bcd-1} \end{pmatrix},$$

$$T^{-1}A_{2,3}T = \begin{pmatrix} 1 & 0 & 0 \\ 1-b & bc & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$T^{-1}A_{1,4}T = \begin{pmatrix} ad & 0 & 0 \\ ad(1-b) & 1 & 0 \\ \dfrac{bcd(a-1)(d-1)}{bcd-1} & 0 & \dfrac{bcd(1-a)}{bcd-1} \end{pmatrix},$$

$$T^{-1}A_{2,4}T = \begin{pmatrix} 1-d+bd & d-1 & 0 \\ (b-1)d & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$T^{-1}A_{3,4}T = \begin{pmatrix} cd & -c(d-1) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We observe that the subspace generated by the eigenvector $e_3 = (0, 0, 1)$ is invariant under this representation. This is due to the fact that the $(1, 3)$ and $(2, 3)$ entries in all of the matrices are zeros. Here, we used the hypothesis that $abcd = 1$. Therefore, we may delete the last row and the last column to obtain a 2×2 representation of P_4.

On the other hand, we consider the complex specialization of the Gassner representation $P_3 \to GL_3(\mathbb{C}) = GL(\mathbb{C}^3)$, which is obtained by having the generators of P_3 act on the basis g_1, g_2, g_3 and applying the Magnus representation. Then we conjugate the representation by the invertible matrix T, defined above. We observe that the $(1, 3)$ and $(2, 3)$ entries of all the matrices are zeros. Therefore, we may delete the last row and column to obtain a 2×2 representation which is exactly the same representation as $\widehat{G_4}(a, \dots, d)$ restricted to the subgroup P_3. Since $abcd = 1$ and $d \neq 1$, it follows that $abc \neq 1$ which implies the irreducibility of the 2×2 matrix representation of P_3 (see [1]). As a result, the irreducibility on P_4 follows. ∎

References

[1] M. Abdulrahim, *Complex Specializations of the reduced Gassner Representation of the Pure Braid Group*, Proc of AMS 125 (no.6) (1997), 1617-1624.

[2] J. S. Birman, *Braids, Links and Mapping Class Groups*, Annals of Mathematical Studies 82, Princeton University Press, New Jersey (1975).

[3] E. Formanek, *Braid group representations of low degree*, Proc. London Math. Soc. 73 (1996), 279-322.

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