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On Zero Divisors in Near-Rings

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Abstract

We study the number of zero divisors in zero symmetric near-rings and are able to give lower and upper bounds for their number for a large class of near-rings. In case of finite zero symmetric near-rings with identity we can give our best results and show that the additive group of such a near-ring must be a p-group if the number of right zero divisors is smaller than a certain bound. Amongst other results, also the size of ideals of a near-ring when given its number of zero divisors is discussed.

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1 Introduction

Throughout this paper we use right near-rings and the notation of [11]. A right near-ring $(N, +, *)$ is a $(2, 2)$-algebra, where $(N, +)$ is a (not necessarily abelian) group, $(N, *)$ is a semigroup and the right distributive law $(n_1 + n_2) * n_3 = n_1 * n_3 + n_2 * n_3$ holds. We will usually abbreviate $(N, +, *)$ by N and write ab instead of $a * b$ for a product in N. A near-ring is called zero symmetric if $n * 0 = 0$ for all $n \in N$. The most general class of examples of zero symmetric near-rings comes from the following construction: Let $(G, +)$ be a not necessarily abelian group. Then the set $M_0(G)$ of all functions $f : G \longrightarrow G$ with $f(0) = 0$ under pointwise addition $+$ and function composition $\circ$ determines a zero symmetric near-ring $(M_0(G), +, \circ)$. Evidently, also each ring is a zero symmetric (right) near-ring and so we may view near-rings as generalised rings. The reader is refered to [11] for a comprehensive discussion on near-rings.

Given a zero symmetric right near-ring N, we say that it is integral if $ab = 0$ implies $a = 0$ or $b = 0$ $(a, b \in N)$. We define the set of right zero divisors Z_r and left zero divisors Z_l of a near-ring by $Z_r := \{n \in N \mid \exists x \in N \setminus \{0\} : xn = 0\}$ and similary $Z_l := \{n \in N \mid \exists x \in N \setminus \{0\} : nx = 0\}$. Certainly, the zero 0 is a right and left zero divisor. The size of the sets Z_r and Z_l can give information about the size of N. We cite two results in this context. For a given set X, let $\mid X \mid$ be its cardinality. Furthermore, an element d in N for which the left distributive law $d * (n_1 + n_2) = d * n_1 + d * n_2$ holds for all $n_1, n_2 \in N$ is called distributive.

Theorem 1.1. *[8, Theorem 2.1] Let N be a zero symmetric non-integral near-ring. If $|Z_l|$ is finite, then $|N| \leq |Z_l|^2$. If $|Z_r|$ is finite and if there exists a non-zero distributive element in Z_r, then $|N| \leq |Z_r|^2$.*

Note that in contrast to the situation in rings we do not have a left-right dual for the zero divisors in the theorem above. Nevertheless, this result can be extended to right zero divisors for many near-rings not using any distributivity condition. What we need instead is the fact that non-zero elements n of a near-ring N give us non-zero N-groups Nn. This is certainly the case for near-rings having an identity, for example. For a near-ring N the descending chain condition on N-subgroups of N will be abbreviated by $DCCN$. There is the following variation of Theorem 1.1 not using any distributivity condition in case of near-rings fulfilling the $DCCN$ (see also Corollary 4.5).

Theorem 1.2. *[12] Let N be a zero symmetric near-ring with DCCN. Suppose $|Z_r|$ is finite and suppose there exists an element $z \in Z_r$ such that $Nz \neq 0$. Then $|N|$ is finite and $|N| \leq |Z_r|^2$.*

Certainly, these two theorems are generalisations of well known results from ring theory (see for example [7]). One of the goals of this paper is to sharpen the results of Theorem 1.2. But not only this. We will see that knowledge on the number of right zero divisors Z_r can give use knowledge about the group structure of the additive group of $(N, +)$ of the near-ring N. We will also be able to tell something about the size of ideals in near-rings in case we know something about the size of Z_r. Also, we will be concerned with the number of invertible elements in near-rings with identity and we take a look to corresponding results in ring theory. This is because any ring can be considered as a near-ring, so all of our results are also valid for rings and they may give a different point of view to the corresponding topic for rings. However, our focus lies on near-rings. In contrast to the situation in rings there have not been published many results on the topic of zero divisors in near-rings so far. For example, in ring theory so called zero divisor graphs are often studied. For near-rings we are only aware of the paper [2] dealing with this subject (see there for further references).

2 Preliminary Notes

We start with a discussion of some basic properties of the sets Z_r and Z_l.

Lemma 2.1. *Let N be a zero symmetric near-ring with DCCN. Then $NZ_r \subseteq Z_r$ and if $m \notin Z_r$, then $Nm = N$.*

Proof. Let $m \in N \setminus Z_r$. Then any m^k, k a natural number, must be in $N \setminus Z_r$. The DCCN guarantees that the chain of N-groups $Nm \supseteq Nm^2 \supseteq Nm^3$.... terminates. So there is some natural number l such that $Nm^l = Nm^{l+1} = Nm(m^l)$. Consequently, for any $n \in N$ there exists $k \in N$ such that $nm^l = (km)m^l$. Since m^l is not a right zero divisor, we get $n = km$, so $N \subseteq Nm$. Clearly, $Nm \subseteq N$ and hence, $N = Nm$.

Let $z \in Z_r$ and $n \in N$. If $n \in Z_r$, then clearly $nz \in Z_r$. If $n \in N \setminus Z_r$, then $Nn = N$ and since $z \in Z_r$, there is an element $j \in N \setminus \{0\}$ such that $jz = 0$. Now $j = mn$ for some non-zero $m \in N$ and consequently, $jz = m(nz) = 0$. This shows that $nz \in Z_r$. $\square$

Given a zero symmetric near-ring N with identity and $DCCN$ it is now easy to see that the set $N \setminus Z_r$ is the set of all invertible elements with respect to near-ring multiplication $*$. This is because let $m \in N \setminus Z_r$. Then $N = Nm$ and there is $k \in N \setminus Z_r$ such that $k * m = 1$. Similary we find $k_1 \in N \setminus Z_r$ such that $k_1 * k = 1$. Therefore, $m = 1 * m = (k_1 * k) * m = k_1 * 1 = k_1$ and consequently $m * k = 1$. In fact, $(N \setminus Z_r, *)$ is a group which is called the group of units U. A lot of group theoretical information on U can be found in [3], for example. In near-rings having no identity the situation is different. $N \setminus Z_r$ turns out to be a union of groups (see [13]).

As another immediate consequence of Lemma 2.1 we have that any non-trivial N-subgroup of N of a zero symmetric near-ring N with $DCCN$ is contained in Z_r and so is any ideal or left ideal of the near-ring. This may not be true if we replace Z_r by Z_l as we will see later (Example 6.1).

Definition 2.2. *Let N be a near-ring and $a, b \in N$. $a \equiv b \Leftrightarrow \forall n \in N : na = nb$. The number of equivalence classes w.r.t. this equivalence relation will be denoted as $\mid N/\equiv\mid$.*

If N has an identity, then clearly $a \equiv b \Leftrightarrow a = b$ and consequently, for each non-zero element $n \in N$ we have $n \not\equiv 0$ that is, $Nn \neq \{0\}$.

As usual, $(0 : z) := \{x \in N \mid xz = 0\}$ is the (left) annihilator of an element $z \in N$. We now have the following lemma on left zero divisors.

Lemma 2.3. *Let N be a zero symmetric near-ring with DCCN. Suppose $\forall z \in Z_r \setminus \{0\} : z \not\equiv 0$. Then $Z_l \subseteq Z_r$. In particular, the annihilator $(0 : z)$ is contained in Z_r for any non-zero $z \in N$.*

Proof. Let z be a left zero divisor. So, there is some non-zero a such that $za = 0$. If z is not a right zero divisor, then $Nz = N$ by Lemma 2.1. Consequently, $Na = (Nz)a = \{0\}$ which cannot happen by assumption. Finally, note that each element of an annihilator of the form $(0 : z)$ is a left zero divisor. □

The containment $Z_l \subseteq Z_r$ may be proper (see Example 6.1) in contrast to (finite) rings with identity.

3 Near-rings where each right zero divisor is nilpotent

Before being in the position to prove our main theorems in the next sections, we have to collect some knowledge on near-rings where each zero divisor is nilpotent. Some of the results included in this section may be available in the literature but the author is not aware of a thorough discussion which we could use for citation. Therefore, we include our own proofs and do citation when we are aware of references. Before we can start with our first lemma to be proved, we have to fix some more notation.

An element $m \in N$ is nilpotent if there is a natural number n, such that $m^n = 0$. A left ideal L of a near-ring is nil if every element in L is nilpotent. In this case $L \subseteq J_2(N)$, $J_2(N)$ being the Jacobson radical of type 2 (see [11]).

Lemma 3.1. *Let N be a zero symmetric near-ring with DCCN such that $\forall z \in Z_r \setminus \{0\}$: $z \not\equiv 0$. Suppose each element in Z_r is nilpotent. Then $J_2(N) = Z_r$ or $J_2(N) = N$.*

Proof. In case of an integral near-ring, $Z_r = \{0\}$ and the statement is clear since N is simple by Lemma 2.1. In case every element in N is a right zero divisor the statement is also clear, because then N is nil and by [11, Corollary 5.45] $J_2(N) = N$. Suppose N is not integral and let $z \in Z_r \setminus \{0\}$ be a non-zero zero divisor. Then there is a natural number n such that $z^n = 0$ but $z^{n-1} \neq 0$. Then, $z \in (0 : z^{n-1})$ and $(0 : z^{n-1}) \subseteq Z_r$ by Lemma 2.3. So, $(0 : z^{n-1})$ is a nil left ideal and therefore contained in $J_2(N)$. Hence, $Z_r \subseteq J_2(N)$. On the other hand, if $J_2(N) \neq N$, $J_2(N) \subseteq Z_r$ by Lemma 2.1. □

Note that if N has an identity, then we do not have non-zero elements $z \equiv 0$. By [11, Proposition 5.43], $J_2(N) \neq N$ in case N has an identity and a minimal N-subgroup (N has to be taken zero symmetric). So we get immediately:

Corollary 3.2. *Let N be a zero symmetric near-ring with identity and DCCN. If each right zero divisor is nilpotent, then $J_2(N) = Z_r$.*

We now introduce local near-rings.

Definition 3.3. *[9] Let $L := \{z \in N \mid Nz \neq N\}$. A near-ring N with identity is said to be local if L is an N-subgroup.*

The next corollary can be found in [5, Proposition 1.13.9]. We include our own proof.

Corollary 3.4. *Let N be a zero symmetric near-ring with identity and DCCN. Then the following holds: N is local $\Leftrightarrow$ each right zero divisor is nilpotent.*

Proof. $\Leftarrow$: By Corollary 3.2, Z_r is an N-subgroup. By Lemma 2.1 and according to Definition 3.3, $L = Z_r$.

$\Rightarrow$: If N is local, then $L = Z_r$. By [9, Theorem 4.2] there are no other idempotents than 0 and 1 in N. Let z in Z_r. Then, for any natural number n, $z^n \in Z_r$. Therefore, z must be nilpotent for otherwise, the semigroup generated by z contains a non-zero idempotent. $\square$

Local near-rings are studied in [9], for example. From there we take the following result:

Proposition 3.5. *[9, Corollary 7.5] The additive group of a finite local near-ring is a p-group.*

4 On the number of zero divisors

We now prove the main theorem of our paper and show that the number of right zero divisors Z_r can give us knowledge on the additive group structure of a near-ring N. We formulate our theorem for the case of near-rings with identity. Note that in this case the set Z_r is just the set of non-invertible elements w.r.t. multiplication. The case of near-rings not having an identity is discussed in Corollary 4.5, as far as possible.

Theorem 4.1. *Let N be a finite zero-symmetric near-ring with identity. Then N belongs to one of the following three classes:*

1. *N is a near-field, hence $(N, +)$ is an (elementary abelian) p-group.*
2. *N is a local near-ring, hence $(N, +)$ is a p-group.*
3. *N is a near-ring with $\mid Z_r \mid \geq 2 \cdot \sqrt{\mid N \mid} - 1$.*

Proof. Suppose N is integral. If $\mid N/\equiv \mid = 2$ (see Definition 2.2), then, due to the existence of an identity, $N = \{0, 1\}$, so N is the field $\mathbb{Z}_2$. If $\mid N/\equiv \mid \geq 3$, N is a planar near-ring and therefore a near-field due to the required identity in N by [11, Theorem 9.50 and Corollary 8.91]. The additive groups of finite near-fields are known to be elementary abelian (see [11]).

Suppose N is not integral, so we have non-zero right zero divisors. If each of these right zero divisors is nilpotent, then N is a local near-ring by Corollary 3.4, so $(N,+)$ is a p-group by Proposition 3.5.

Suppose N is not a near-field and not a local near-ring, so we must have a right zero divisor z being not nilpotent. Then, the semigroup generated by z contains a non-zero idempotent right zero divisor e. By the Peirce decomposition we have $N = (0:e) + Ne$ with $Ne \cap (0:e) = \{0\}$. Since e is a zero divisor, $Ne \subseteq Z_r$ by Lemma 2.1. Since $N \neq (0:e)$, we again have $(0:e) \subseteq Z_r$ by Lemma 2.3. By the fact that $Ne \cap (0:e) = \{0\}$ we have $\mid Z_r \mid \geq \mid Ne \mid + \mid (0:e) \mid -1$ (we have to count 0 only once). Moreover, $\mid N \mid = \mid Ne \mid \cdot \mid (0:e) \mid$.

Now let $x := \mid Ne \mid$, $y := \mid (0:e) \mid$, $a := \mid Z_r \mid +1$, $b := \mid N \mid$. So we get the two inequalities $x \cdot y = b$ and $x + y \leq a$. Since $(x-y)^2 \geq 0$, we have $x^2 + y^2 \geq 2xy$ and therefore $a^2 \geq (x+y)^2 \geq 4xy$. So, $a^2 \geq 4b$ and consequently, $a \geq 2 \cdot \sqrt{b}$. This proves the last statement of the theorem.

□

Remark 4.2. *Near-fields can be considered as integral local near-rings. So item (1) of Theorem 4.1 is a sub-case of item (2).*

In particular, if the additive group of a zero symmetric near-ring with identity is not a p-group, then we can be assured that there are a lot of elements not being invertible w.r.t. near-ring multiplication. The proof follows immediately from Theorem 4.1.

Corollary 4.3. *Let N be a finite zero-symmetric near-ring with identity. If $(N,+)$ is not a p-group, then $\mid Z_r \mid \geq 2 \cdot \sqrt{\mid N \mid} - 1$.*

In the following we will see that the bound in Theorem 4.1 is indeed reached by certain near-rings.

Example 4.4. *Take some finite near-field N and consider the direct product $N \times N$. Each element of the type $(n,0)$ or $(0,n)$ $(n \neq 0)$ is a right zero divisor. Since the set of right zero divisors is not additively closed, $N \times N$ is not a local near-ring and certainly not a near-field. We have exactly $\mid Z_r \mid = 2 \cdot \sqrt{\mid N \mid} - 1$.*

Also, for rings we cannot do better. We might take a field N in example 4.4 which gives us a ring $N \times N$ having exactly $\mid Z_r \mid = 2 \cdot \sqrt{\mid N \mid} - 1$ zero divisors.

Certainly, in general there are more than $2 \cdot \sqrt{\mid N \mid} - 1$ zero divisors in a near-ring N. (A good source for examples is the *GAP* package *SONATA* [1]).

Note that finiteness and the existence of an identity in the near-rings in Theorem 4.1 were used for getting a near-field in item (1) instead of only an integral near-ring and to get local near-rings with additive p-groups in item (2).

We can give a more general version of this theorem in our next corollary. What we need are near-rings with $DCCN$ such that given $n \neq 0$ then $Nn \neq 0$, so $n \not\equiv 0$, in order to apply Lemma 3.1. Near-rings without identity where $Nn \neq 0$ for each $n \neq 0$ exist. For example, regular near-rings have this property but do not necessarily have an identity. (A near-ring is regular if for all $n \in N$ there is $x \in N$ such that $n = nxn$.) Furthermore, we require that the near-rings under consideration have only a finite number of right zero divisors. What is more, we are only considering near-rings which are not integral.

Corollary 4.5. *Let N be a zero symmetric near-ring with $DCCN$ which is not integral. Suppose $\mid Z_r \mid$ is finite and $Nn \neq \{0\}$ for all $n \in N \setminus \{0\}$. Then N is finite and belongs to one of the following two classes.*

1. *Each right zero divisor is nilpotent and $J_2(N) = Z_r$ in case $J_2(N) \neq N$.*
2. *N is a near-ring with $\mid Z_r \mid \geq 2 \cdot \sqrt{\mid N \mid} - 1$.*

Proof. Let $z \in Z_r$ and consider $\psi : N \longrightarrow Nz, n \mapsto nz$. ψ is an N-homomorphism with kernel $(0 : z)$. By Lemma 2.1 and Lemma 2.3 we have $Nz \subseteq Z_r$ and $(0 : z) \subseteq Z_r$. By the homomorphism theorem, $N/(0 : z) \cong Nz$. Thus $\mid N \mid = \mid (0 : z) \mid \cdot \mid Nz \mid$, which has to be finite (also following from this is $\mid N \mid \leq \mid Z_r \mid^2$). If each right zero divisor is nilpotent we apply Lemma 3.1. Suppose we have one right zero divisor which is not nilpotent. Then we can proceed as in the proof of item (3) of Theorem 4.1 and the proof is finished. □

5 On the number of invertible elements

Theorem 4.1 and Corollary 4.5 give us a lower bound for the number of right zero divisors in a near-ring. Now we will give also an upper bound but we must require an identity for the near-ring. Let U be the set of all invertible elements w.r.t. near-ring multiplication in a finite zero symmetric near-ring with identity, so $U = N \setminus Z_r$. Our task now will be in finding a lower bound for $\mid U \mid$ and as a consequence obtaining an upper bound for $\mid Z_r \mid$.

Let $(N, +)$ be a group. As usual, the exponent of the group is the least common multiple of all the orders of the group elements. For $k \in N$ let $ord(k)$ be the order of k w.r.t. the group operation $+$. Then we have the following:

Theorem 5.1. *[11, Proposition 9.111] Let N be a finite near-ring with identity 1. Let n be the exponent of $(N, +)$. Then $ord(1) = n$.*

Let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $\mid U \mid = \phi(n)$, ϕ being the Euler-Phi function. We now give a generalisation of this result to arbitrary zero symmetric near-rings with identity. To fix our notation, let n and m be two integers. Then $n \cdot m$ is the usual product in $\mathbb{Z}$.

The proof of the following lemma is based on the observation that the identity 1 in a near-ring N generates a subring of N. This result can be found in [10]. We want to work out some more details.

Lemma 5.2. *Let N be a finite zero symmetric near-ring with identity and U its group of units. Let n be the exponent of $(N, +)$ and ϕ be the Euler-Phi function. Then there is a subgroup of order $\phi(n)$ contained in U.*

Proof. Let 1 be the identity of N. By $+$ and $*$ we denote the group operation and the multiplication in N. Consider the cyclic group $< 1 >$ (additively) generated by 1. Let $m \cdot 1 := 1 + \ldots + 1$ (m times 1) and $k \cdot 1 := 1 + \ldots + 1$ (k times 1) be two elements of $< 1 >$. By the right distributive law in N and by using the fact that 1 is the identity we get $(m \cdot 1) * (n \cdot 1) = (m \cdot k) \cdot 1$. This expression is certainly contained in $< 1 >$. Consequently, $S := (< 1 >, +, *)$ is a subnear-ring of N containing the identity. Actually, S is easily seen to be a ring isomorphic to $\mathbb{Z}_n$ since the function $\psi :< 1 > \longrightarrow \mathbb{Z}_n, m \cdot 1 \mapsto [m]_n$ turns out to be a near-ring isomorphism.

Let U_S be the group of units of S. It follows that U_S has $\phi(n)$ invertible elements. Since S and N have the same identity, an element a being invertible in S is invertible in N. Therefore $U_S \subseteq U$ and U_S is a subgroup of order $\phi(n)$ of U. □

We immediately have the following corollary by observing that $N = N \setminus Z_r \cup Z_r$.

Corollary 5.3. *Let N be a finite zero symmetric near-ring with identity. Let n be the exponent of $(N, +)$. Then $\mid Z_r \mid \leq \mid N \mid - \phi(n)$.*

The rings of integers $\mathbb{Z}_n$ provide us with examples that the bound given in Corollary 5.3 may be reached.

Now we can add an upper bound to the number of zero divisors in item (3) in Theorem 4.1. The proof is now obvious.

Theorem 5.4. *Let N be a finite zero symmetric near-ring with identity. Suppose N is not a near-field and N is not a local near-ring. Let n be the exponent of $(N, +)$. Then we have*

$$2 \cdot \sqrt{\mid N \mid} - 1 \leq \mid Z_r \mid \leq \mid N \mid - \phi(n)$$

Certainly, this result also holds for finite rings with identity. However, the groups of units of rings with identity are well studied (see for example [4]). So, the contribution may be minor here but still may add some different point of view. In the next section we will also compare our results with some ring theoretic results.

6 Left zero divisors and the situation in rings

So far we were only considering right zero divisors. In this section we will put some observations on left zero divisors. Lemma 2.3 shows that in a zero symmetric near-ring with $DCCN$ and identity any left zero divisor is a right zero divisor. The converse is not true, not any right zero divisor must be a left zero divisor as we will see in Example 6.1. Note that this is in contrast to the situation in finite rings with identity where the set of right zero divisors and left zero divisors coincide. Let R be a finite ring with identity and let d be a right zero divisor, so $rd = 0$ for some non-zero r in R. Suppose d is not a left zero divisor. Then $\psi_d : R \longrightarrow R, r \mapsto dr$ is injective, hence bijective, because d is distributive. Therefore $r \in rR = r(dR) = (rd)R = 0$, a contradiction.

In general it is not the case that the sets of left and right zero divisors are the same. There are near-rings N where the N-subgroups of N are not contained in Z_l. In this case Z_l does not coincide with the set of non-invertible elements in near-rings with identity, whereas Z_r does, in case N has the $DCCN$. Also Theorem 4.1 does not hold by replacing Z_r with Z_l and it is not known to the author if it is possible to use left zero divisors for a similar version of Theorem 4.1.

By finding a surprisingly small near-ring we can illustrate all these facts in the next example. Thus, it seems that the set of right zero divisors plays a more vital role in structure theory of right near-rings than the set of left zero divisors does.

Example 6.1. *We give the operation table of a zero symmetric near-ring with identity on the additive group* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *which is labelled as library near-ring number* 12 *in the* GAP *package* SONATA *(note that* SONATA *produces left near-rings, so we have to transpose the multiplication table). We see that we have* 3 *right zero divisors* 0*,* a*,* b*. Note that this must be the case as a consequence of Theorem 5.4 (the near-ring is neither local nor a near-field). On the other hand, we have only* 2 *left zero divisors* 0 *and* a*, so* $Z_l \not\geq 2 \cdot \sqrt{4} - 1$ *and Theorem 4.1 does not hold for left zero divisors. Also,* $Z_l \neq Z_r$ *because* b *is a right zero divisor but not a left zero divisor. Therefore, the* N*-group* $Nb = \{0, b\}$ *is not contained in* Z_l*.*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$*$	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

Before we turn over to the next section we want to point out that for rings there is a paper by K. Koh [7] which gives a result very similar to our results in the near-ring case. There is the following theorem which we formulate in our language and for finite rings (the structures in [7] are not necessarily finite).

Theorem 6.2. *[7, Theorem 2] Let R be a finite ring with identity. If there is a minimal right ideal I such that $I^2 \neq \{0\}$, then $\mid Z_r \mid \geq 2 \cdot \sqrt{\mid R \mid} - 1$.*

So, the bound for the zero divisors is the same as found in our Theorem 4.1. Actually, Theorem 6.2 is an immediate corollary of Theorem 4.1 by observing that in case there is a minimal right ideal I such that $I^2 \neq \{0\}$ in a ring, we have an idempotent right zero divisor e. Therefore, R is not a field and not a local ring. Now we can apply Theorem 4.1 to get the result. At this point it seems necessary to point out that we do not have a simple carry over of ring results to near-ring results, however. The proof in [7] makes use of the distributivity of the ring elements.

7 On the size of ideals in near-rings

Following from Lemma 2.1 we have that any ideal I of a zero symmetric near-ring N with $DCCN$ is contained in Z_r. Therefore, it is clear that the size of Z_r restricts the possible size of ideals of a near-ring. We will investigate this more closely for near-rings with identity.

Before, we need a preliminary result which we will prove in the more general setting of near-rings not necessarily having an identity. It tells us something about the number of right zero divisors in a factor near-ring.

Lemma 7.1. *Let N be a finite zero symmetric near-ring and let I be an ideal of N. Let $Z_r(N/I)$ be the number of right zero divisors in the factor near-ring N/I and $Z_r(N)$ be the number of right zero divisors in N. Then, $Z_r(N) \geq Z_r(N/I) \cdot \mid I \mid$.*

Proof. Suppose $z + I$ is a right zero divisor in N/I, so there is an element $n + I$ ($n \notin I$) such that $nz \in I$. Suppose z is not a right zero divisor in N. Then, $Nz = N$ by Lemma 2.1 and there is an element e such that $ez = z$. Consequently, for all $m \in N$ we have $(me - m)z = 0$. But z is not a right zero divisor, so $me = m$ and e is a right identity in N. Furthermore, there is an element $\overline{z}$ such that $\overline{z}z = e$ and since e is a right identity, $\overline{z}$ is also not a right zero divisor. Then we have $z\overline{z} = ze\overline{z} = z\overline{z}z\overline{z}$. On the other hand, $ez\overline{z} = z\overline{z}$ and therefore $(z\overline{z} - e)z\overline{z} = 0$. Since z and $\overline{z}$ both are not right zero divisors, $z\overline{z}$ is not a right zero divisor and we have $z\overline{z} - e = 0$ and hence $e = z\overline{z}$.

So, $nz \in I$ gives $n = ne = nz\overline{z} \in I$ which is a contradiction to $n + I \notin I$. Therefore, z must be a right zero divisor in N. Since $z + I = (z + i) + I$ for any $i \in I$, we have that each $z + i$ must be a right zero divisor in N by the same arguments as above. Therefore, each right zero divisor in N/I gives us

$\mid I \mid$ right zero divisors in N. So we must have at least $Z_r(N/I)\cdot \mid I \mid$ right zero divisors in N. $\square$

The proof of Lemma 7.1 required to show that an element which is not a right zero divisor is kind of invertible (to a right identity of the near-ring) even if N does not have an identity. Much more on this subject and on the multiplicative semigroup of near-rings can be found in [13]. For the remainder of this section we keep the notation $Z_r(N)$ and $Z_r(N/I)$ for the number of right zero divisors of N and N/I, respectively. We formulate and prove the main theorem of this section again for near-rings with identity. Using Corollary 4.5 one could easily adapt it to suitable near-rings without identity.

Theorem 7.2. *Let N be a finite zero symmetric near-ring with identity. Let I be an ideal of N. If $\mid N/I \mid$ is not a power of some prime, then:*

$$Z_r(N) \geq (2 \cdot \sqrt{\mid N/I \mid} - 1)\cdot \mid I \mid$$

Proof. By Theorem 4.1 we must have $Z_r(N/I) \geq 2 \cdot \sqrt{\mid N/I \mid} - 1$ and by Lemma 7.1, $Z_r(N) \geq Z_r(N/I)\cdot \mid I \mid \geq (2 \cdot \sqrt{\mid N/I \mid} - 1)\cdot \mid I \mid$. $\square$

The result of Theorem 7.2 clearly also applies to rings. A theorem in the same spirit for rings R can be found in [6, Theorem 2.7] where the authors use logarithms of $\mid Z_r \mid$ and $\mid R \mid$ to study the interplay between the size of ideals and the number of zero divisors.

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