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On the Amalgamation Property for Various Algebraic Logics

Tarek Sayed Ahmed

Department of Mathematics, Faculty of Science
Cairo University, Giza, Egypt
rutahmed@gmail.com

Abstract

We show that a natural class of representable algebras (of logic) has the super amalgamation property. Applications of this results are given. In particular, questions originally posed by Tarski, Henkin, Monk and Pigozzi are answered. Several techniques for failure of various forms of the amalgamation property are appropriately modified, proving new results. Answers to open questions in the recent paper [18] are summarized in tabular form at the end of the paper.

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This article is about neat reducts and the super amalgamation property for various classes studied in Algebraic Logic. Invented by Leon Henkin, the notion of neat reducts is an old venerable notion in cylindric algebra theory, that has become fashionable in recent times. Its importance stems from its intimate connection to the notion of representability. Indeed it is known that for an arbitrary ordinal α and $\mathfrak{A} \in \mathbf{CA}_\alpha$, $\mathfrak{A}$ is representable if and only if there exist $\mathfrak{B} \in \mathbf{CA}_{\alpha+\omega}$ and an embedding $e : \mathfrak{A} \to \mathfrak{Nr}_\alpha\mathfrak{B}$. Here $\mathfrak{Nr}_\alpha\mathfrak{B}$ is the neat α reduct of $\mathfrak{B}$ and $\mathfrak{A}$ is said to have the neat embedding property. Referred to as the Neat Embedding Theorem, or NET for short, the NET (which states that the class of representable algebras coincide with the class of algebras with the neat embedding property) is due to Henkin [20]. In the NET, ω cannot be truncated to any finite ordinal; for $n > 0$ and any $\alpha > 2$ algebras in $S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+n}$ are not all representable, a classical result of Monk [20]. Neat reducts have been studied intensively lately, see e.g. [4], [3], [2], [5], [7], [6]. A recent reference dealing with relation algebra reducts of $\mathbf{CA}$'s which are basically neat reducts, with conversion and composition defined using a spare dimension, is [21]. In [31] Simon gives a very interesting new proof of the representability of Tarski's quasi-projective relation algebras using a NET for **CA**'s. A variation on the NET is used by Sain [27] to prove a strong representation theorem for certain reducts of polyadic algebras. In [8] a NET is used to prove a strong interpolation theorem for certain reducts of Keisler's infinitary logics, a similar argument will be used below. A reference that unifies results on neat reducts and neat embeddings, in connection to metalogical properties, like completeness, omitting types and interpolation is [9].

The super amalgamation property, on the other hand, is a much more recent notion invented by Maksimova and it is a global algebraic property that corresponds to a strong form of interpolation in the corresponding logic. Daigneault [17] was the first to explicitly make the connection between the Craig interpolation theorem and amalgamation in the context of polyadic algebras. The first systematic use of the link to obtain results about interpolation properties from results of amalgamation, or vice versa, can be found in [25]. The principal context of [25] is the class of infinite dimensional cylindric algebras, an alternative equational formulation of first order logic. The positive results of section 2.2 in combination with the negative ones of section 2.3 of [25] answer most of the natural questions one could ask about amalgamation and interpolation properties for cylindric algebras. However, some were posed as open questions and other closely related ones appeared after Pigozzi's paper was published. [18] answers all these questions. However in [18] new questions, concerning other algebraisations of first order logic, appeared. In this paper, we relate the notion of neat embeddings to the super amalgamation property. We show that a natural class of representable algebras has the super amalgamation property. From certain consequences of this Theorem we obtain solutions to long standing open problems posed by Henkin, Monk and Tarski in [19] concerning neat embeddings, and problems in [18] concerning amalgamation and interpolation. Techniques of Pigozzi and Madárasz are adapted to prove that certain distinguished classes of algebras fail to have various forms of the amalgamation property. We start by recalling the concrete versions of the algebras we deal with:

Let α be an ordinal. Let U be a set. Then we define for $i, j < \alpha$ and $X \subseteq {}^{\alpha}U$:

$$\mathsf{C}_i X = \{s \in {}^{\alpha}U : \exists t \in X, t(j) = t(i) \text{ for all } i \neq j\},$$

$$\mathsf{S}_i^j X = \{s \in {}^{\alpha}U : s \circ [i|j] \in X\},$$

$$\mathsf{S}_{[i,j]} X = \{s \in {}^{\alpha}U : s \circ [i, j] \in X\},$$

$$\mathsf{D}_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}.$$

$[i|j]$ is the replacement on α that takes i to j and leaves every other thing fixed, while $[i,j]$ is the transposition interchanging i and j. The extra non - boolean operations we deal with are as specified above. For set X, let $\mathfrak{B}(X) = (\wp(X), \cup, \cap, \setminus, \emptyset, X)$ be the full boolean set algebra with universe $\wp(X)$. Let S be the operation of forming subalgebras, and P be that of forming products. Then

$$\mathbf{RSC}_\alpha = SP\{(\mathfrak{B}(^\alpha U), \mathsf{C}_i, \mathsf{S}_i^j)_{i,j<\alpha} : U \text{ is a set }\}.$$

$$\mathbf{RQA}_\alpha = SP\{(\mathfrak{B}(^\alpha U), \mathsf{C}_i, \mathsf{S}_i^j, \mathsf{S}_{[i,j]})_{i,j<\alpha} : U \text{ is a set }\}.$$

$$\mathbf{RCA}_\alpha = SP\{(\mathfrak{B}(^\alpha U), \mathsf{C}_i, \mathsf{D}_{ij})_{i,j<\alpha} : U \text{ is a set }\}.$$

$$\mathbf{RQEA}_\alpha = SP\{(\mathfrak{B}(^\alpha U), \mathsf{C}_i, \mathsf{D}_{ij}, \mathsf{S}_{[i,j]})_{i,j<\alpha} : U \text{ is a set }\}.$$

SC stands for the class of Pinter's substitution algebras, **QA** (**QEA**) stands for the class of quasi-polyadic (equality) algebras and **CA** stands for the class of cylindric algebras. Throughout $\mathbf{K} \in \{\mathbf{SC}, \mathbf{QA}, \mathbf{CA}, \mathbf{QEA}\}$. $\mathbf{RK}_\alpha$ as defined above stands for the class of representable algebras in $\mathbf{K}_\alpha$. From now on ordinals considered, unless otherwise explicitly specified, are infinite. It is not hard to show that $\mathbf{RK}_\alpha$ is a variety. However it is not finite schema axiomatizable [19], [26]. The question we adress in what follows is basically : Which subclasses of $\mathbf{RK}_\alpha$ enjoy (various variants) of the amalgamation property? (This problem is addressed by Pigozzi in [25] for cylindric algebras). Overall our techniques are similar to those of Pigozzi in his pioneering paper [25] but there are non-trivial significant deviations. At the end of the paper answers to open problems in the recent paper [18] are summarized in tabular form. We deal with non-representable algebras only at the appendix where we prove that the superamagamation base of $\mathbf{RCA}_\omega$ is not contained in the amalgamation base of $\mathbf{CA}_\omega$.We recall the notion of super amalgamation due to Maksimova [22]. We broaden the definition allowing the amalgam, be it strong or super to be found in a possibly bigger class.

Definition 1 .

(1) *Let V be a class of algebras (usually but not always assumed to be a variety) and $L_1, L_2 \subseteq V$. L_2 is said to have the amalgamation property, or AP for short over L_1, with respect to V, if for all $\mathfrak{A}_0 \in L_1$ all $\mathfrak{A}_1$ and $\mathfrak{A}_2 \in L_2$, and all monomorphisms i_1 and i_2 of $\mathfrak{A}_0$ into $\mathfrak{A}_1$, $\mathfrak{A}_2$, respectively, there exists $\mathfrak{A} \in V$, a monomorphism m_1 from $\mathfrak{A}_1$ into $\mathfrak{A}$ and a monomorphism m_2 from $\mathfrak{A}_2$ into $\mathfrak{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$. In this case we say that $\mathfrak{A}$ is an amalgam of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$ via m_1 and m_2 or even simply an amalgam.*

(2) *We say that L_2 has the strong AP, or SAP for short over L_1 with respect to V, if in addition to (1), we have $m_1 \circ i_1(A_0) = m_1(A_1) \cap m_2(A_2)$. In*

this case we say that $\mathfrak{A}$ is a strong amalgam of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$, via m_1 and m_2, or even simply a strong amalgam.

(3) *Assume that V is a class of boolean algebras with extra operations. We say that L_2 has $SUPAP$ over L_1 with respect to V, if in addition to (1) we have*

$$(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y))$$

where $\{j,k\} = \{1,2\}$. Here $\leq$ is the boolean order. In this case we say that $\mathfrak{A}$ is a super amalgam of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$, via m_1 and m_2, or even simply a super amalgam.

(4) *When $L_1 = L_2 = L$ in (1) above we say that L has AP with respect to V. If furthermore $L = V$, we say that V simply has AP. A similar observation holds for SAP and $SUPAP$.*

Note that the more conventional definition of AP, SAP and $SUPAP$ is when $L_1 = L_2 = V$. We note that $SUPAP$ is stronger than SAP for boolean algebras with extra operations, and for (varieties of) cylindric algebras, it is strictly stronger, a result of Sagi and Shelah [32]. From now on, all algebras considered have an **SC** reduct. For **CA**'s and **QEA**'s and $i,j < \alpha$ $i \neq j$ $\mathsf{s}_i^j x = \mathsf{c}_i(x.\mathsf{d}_{ij})$. For most of the following proofs, we shall need the following (abstract) axiomatization Σ of $\mathbf{SC}_\alpha$. If $\mathfrak{A} \in \mathbf{K}_\alpha$ then its **SC** reduct satisfies Σ. The proof of this can be recovered from [19] the section on substitutions (section 1.5) starting p. 189, accompanied by [26].

Definition 2 . *An algebra in $\mathbf{SC}_\alpha$ is of the form*

$$\mathfrak{A} = (A, +, \cdot, , -, 0, 1, \mathsf{c}_i, \mathsf{s}_i^j)_{i,j\in\alpha}$$

where $(A, +, \cdot, -, 0, 1)$ is a boolean algebra $\mathsf{c}_i, \mathsf{s}_i^j$ are unary operations on $\mathfrak{A}$ $(i, j < \alpha)$ satisfying the following equations for all $i, j, k, l \in \alpha$

1. $\mathsf{c}_j 0 = 0$, $x \leq \mathsf{c}_i x$, $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$, *and* $\mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$,
2. $\mathsf{s}_i^i x = x$,
3. s_j^i *are boolean endomorphisms,*
4. $\mathsf{s}_j^i \mathsf{c}_i x = \mathsf{c}_i x$,
5. $\mathsf{c}_i \mathsf{s}_j^i x = \mathsf{s}_j^i x$ *whenever* $i \neq j$,
6. $\mathsf{s}_j^i \mathsf{c}_k x = \mathsf{c}_k \mathsf{s}_j^i x$, *whenever* $k \notin \{i, j\}$,

7. $\mathsf{c}_i\mathsf{s}_i^j x = \mathsf{c}_j\mathsf{s}_j^i x$,

8. $\mathsf{s}_i^j\mathsf{s}_k^l x = \mathsf{s}_k^l\mathsf{s}_i^j x$, *whenever* $|\{i,j,k,l\}| = 4$

9. $\mathsf{s}_i^l\mathsf{s}_l^j x = \mathsf{s}_i^l\mathsf{s}_i^j x$

Our axiomatization is slightly different than that adopted by Pinter [23], [24]. The axioms are of course sound (i.e true in set algebras) but severely incomplete. We follow the notation and terminology of [19] with the obvious changes for algebras other than **CA**'s. Reducts cf. [19] Definition 2.6.1, neat reducts, cf. [19] Definition 2.6.28, and dimension restricted free algebras, cf. [19] Definition 2.5.31, are defined like the **CA** case. For $\mathfrak{B} \in \mathbf{K}_\beta$, α an ordinal and ρ a sequence of length α with $Rg\rho \subseteq \beta$, the ρ reduct of $\mathfrak{A}$ is denoted by $\mathfrak{Rd}_\alpha^\rho\mathfrak{A}$ or even simply by $\mathfrak{Rd}^\rho\mathfrak{A}$. Note that $\mathfrak{Rd}_\alpha^\rho\mathfrak{A} \in \mathbf{K}_\alpha$. If $\alpha < \beta$ and ρ is the inclusion map, then $\mathfrak{Rd}_\alpha^\rho\mathfrak{A}$, denoted in this special case by $\mathfrak{Rd}_\alpha\mathfrak{A}$, is the α reduct of $\mathfrak{A}$ obtained by discarding the operations indexed by ordinals in $\beta \smallsetminus \alpha$. $\mathfrak{Nr}_\alpha\mathfrak{A}$ denotes the *neat* α reduct of $\mathfrak{A}$, which is the subalgebra of $\mathfrak{Rd}_\alpha\mathfrak{A}$ consisting of the α dimensional elements of $\mathfrak{A}$. $\mathfrak{Nr}_\alpha L$ denotes the class of all neat α reducts of algebras in L. For a cardinal $\beta > 0$, $L \subseteq \mathbf{K}_\alpha$ and $\rho : \beta \to \wp(\alpha)$, $\mathfrak{Fr}_\beta^\rho L$ stands for the dimension restricted L free algebra on β generators. The sequence $\langle \eta/Cr_\beta^\rho L : \eta < \beta\rangle$ L-freely generates $\mathfrak{Fr}_\beta^\rho L$, cf. [19] Theorem 2.5.35. $\mathfrak{Fr}_\beta^\rho L$ is treated in [25] under the name of free algebras over L subject to certain defining relations, cf. [25] Definition 1.1.5. The next definition is taken from [19] Definition 1.11.9. The definition is redundant (gives nothing new) in the case of **QA** and **QEA**. Finite substitutions are already there and they coincide with the newly added ones. This needs proof though.[1]

Definition 3 .

(i) *For a transformation* $\tau \in {}^\alpha\alpha$, *the support of* τ, *or sup*τ *for short, is the set* $\{i \in \alpha : \tau(i) \neq i\}$. τ *is finite if sup*τ *is finite. For a finite transformation* τ *we write* $[u_0|v_0, u_1|v_1 \ldots u_{k-1}|v_{k-1}]$ *if sup*$\tau = \{u_0 \ldots u_{k-1}\}$, $u_0 < u_1 \ldots < u_{k-1}$ *and* $\tau(u_i) = v_i$ *for* $i < k$.

(ii) *Let* $\mathfrak{A} \in \mathbf{SC}_\alpha$ *be such that* $\alpha \setminus \Delta x$ *is infinite for every* $x \in A$. *If* $\tau = [u_0|v_0, u_1|v_1 \ldots u_{k-1}|v_{k-1}]$ *is a finite transformation, if* $x \in A$ *and if* $\pi_0 \ldots \pi_{k-1}$ *are in this order the first* k *ordinals in* $\alpha \setminus (\Delta x \cup Rgu \cup Rgv)$, *then*

$$\mathsf{s}_\tau x = \mathsf{s}_{v_0}^{\pi_0} \ldots \mathsf{s}_{v_{k-1}}^{\pi_{k-1}}\mathsf{s}_{\pi_0}^{u_0} \ldots \mathsf{s}_{\pi_{k-1}}^{u_{k-1}} x.$$

[1] Indeed let $\mathfrak{A} \in \mathbf{QA}_\alpha$ and $x \in A$. It suffices to show that for any μ, $k, l < \alpha$ if $\mu \notin \Delta x$, then ${}_\mu\mathsf{s}(k,l)x = \mathsf{s}_{[k,l]}x$. We proceed as follows using the axiomatization in [26]. ${}_\mu\mathsf{s}(k,l)x = \mathsf{s}_k^\mu\mathsf{s}_l^k\mathsf{s}_\mu^l x = \mathsf{s}_k^\mu\mathsf{s}_{[k,l]}\mathsf{s}_k^l\mathsf{s}_\mu^l\mathsf{c}_\mu x = \mathsf{s}_k^\mu\mathsf{s}_{[k,l]}\mathsf{s}_\mu^l\mathsf{c}_\mu x = \mathsf{s}_{[k,l]}\mathsf{s}_\mu^l\mathsf{s}_l^\mu\mathsf{c}_\mu x = \mathsf{s}_{[k,l]}\mathsf{s}_\mu^l\mathsf{c}_\mu x = \mathsf{s}_{[k,l]}\mathsf{c}_\mu x$ $= \mathsf{s}_{[k,l]}x$

Notation . *We represent the function* $\mathsf{s}^{\mu_{\pi_0}}_{\nu_{\pi_0}} \circ \mathsf{s}^{\mu_{\pi_1}}_{\nu_{\pi_1}} \circ ...... \circ \mathsf{s}^{\mu_{\pi_{k-1}}}_{\nu_{\pi_{k-1}}}$ *simply as* s^{μ}_{ν}, *whenever* μ, ν *are functions from the same finite subset of* ω *into* α *and* π *is the unique strictly increasing sequence such that* $Rg\pi = Do\mu = Do\nu$.

Lemma 4 . *Let* $\mathfrak{A} \in \mathbf{SC}_\alpha$. *The following hold for all* $x \in A$ *and all* $i, j, l, k < \alpha$:

(1) $\mathsf{s}^i_j \mathsf{s}^k_i \mathsf{c}_i x = \mathsf{s}^k_j \mathsf{c}_i x$.

(2) $\mathsf{s}^i_j \mathsf{s}^l_i \mathsf{c}_i \mathsf{c}_k x = \mathsf{s}^k_j \mathsf{s}^l_k \mathsf{c}_i \mathsf{c}_k x$.

(3) $\mathsf{s}^i_j \mathsf{s}^i_l x = \mathsf{s}^i_l x$, *if* $i \neq l$.

(4) $\mathsf{s}^i_j \mathsf{s}^j_i x = \mathsf{s}^i_j x$.

(5) *For all* $i, j < \alpha$, $\mathsf{s}_{[i|j]} x = \mathsf{s}^i_j x$

Assume that $\alpha \smallsetminus \Delta x$ *is infinite for every* $x \in A$ *then,*

(6) *If* $k < \omega$, *if* $\mu, \nu, \pi, \rho \in {}^k\alpha$, *if* μ, π, ρ *are one-one, and if* $(Rg\pi \cup Rg\rho) \cap (\Delta x \cup Rg\mu \cup Rg\nu) = 0$, *then* $\mathsf{s}^{\pi}_{\nu} \mathsf{s}^{\mu}_{\pi} x = \mathsf{s}^{\rho}_{\nu} \mathsf{s}^{\mu}_{\rho} x$.

(7) *let* τ *be a finite transformation of* α. *If* $k < \omega$, *if* $\mu, \pi \in {}^k\alpha$, *if* μ *and* π *are one-one, if* $\{\lambda : \tau(\lambda) \neq \lambda\} \subseteq Rg\mu$, $Rg\pi \cap (\Delta x \cup Rg\mu \cup \tau(Rg\mu)) = 0$, *then* $\mathsf{s}_\tau x = \mathsf{s}^{\pi}_{\tau\circ\mu} \mathsf{s}^{\mu}_{\pi} x$.

Proof.

(1)
$$\begin{aligned} \mathsf{s}^i_j \mathsf{s}^k_i \mathsf{c}_i x &= \mathsf{s}^i_j \mathsf{s}^k_j \mathsf{c}_i x && \text{by} \quad (2.9) \\ &= \mathsf{s}^i_j \mathsf{c}_i \mathsf{s}^k_j x && \text{by} \quad (2.6) \\ &= \mathsf{c}_i \mathsf{s}^k_j x && \text{by} \quad (2.4) \\ &= \mathsf{s}^k_j \mathsf{c}_i x && \text{by} \quad (2.6). \end{aligned}$$

(2)
$$\begin{aligned} \mathsf{s}^i_j \mathsf{s}^l_i \mathsf{c}_i \mathsf{c}_k x &= \mathsf{s}^l_i \mathsf{c}_i \mathsf{c}_k x && \text{by} \quad (4.1) \\ &= \mathsf{s}^l_i \mathsf{c}_k \mathsf{c}_i x && \text{by} \quad (2.1) \\ &= \mathsf{s}^k_j \mathsf{s}^l_k \mathsf{c}_k \mathsf{c}_i x && \text{by} \quad (4.1) \\ &= \mathsf{s}^k_j \mathsf{s}^l_k \mathsf{c}_i \mathsf{c}_k x && \text{by} \quad (2.1). \end{aligned}$$

(3)
$$\begin{aligned} \mathsf{s}^i_j \mathsf{s}^i_l x &= \mathsf{s}^i_j \mathsf{c}_i \mathsf{s}^i_l x && \text{by} \quad (2.5) \\ &= \mathsf{c}_i \mathsf{s}^i_l x && \text{by} \quad (2.4) \\ &= \mathsf{s}^i_l x && \text{by} \quad (2.5) \end{aligned}$$

$$(4) \qquad \mathsf{s}_j^i \mathsf{s}_i^j x = \mathsf{s}_j^i \mathsf{s}_j^j x \quad \text{by} \quad (2.9)$$
$$= \mathsf{s}_j^i x.$$

$$(5) \qquad \mathsf{s}_{[i|j]} x = \mathsf{s}_j^{\pi_0} \mathsf{s}_{\pi_0}^i x \quad \text{for} \quad \pi_0 \notin \Delta x$$
$$= \mathsf{s}_j^{\pi_0} \mathsf{s}_{\pi_0}^i \mathsf{c}_{\pi_0} x$$
$$= \mathsf{s}_j^i \mathsf{c}_{\pi_0} x \quad \text{by} \quad (4.1)$$
$$= \mathsf{s}_j^i x.$$

(6) can be established if $Rg\pi \cap Rg\rho = 0$ as follows

$$\mathsf{s}_\nu^\pi \mathsf{s}_\pi^\mu x = \mathsf{s}_{\nu_0}^{\pi_0} \dots \mathsf{s}_{\nu_{k-1}}^{\pi_{k-1}} \mathsf{s}_{\pi_0}^{\mu_0} \dots \mathsf{s}_{\pi_{k-1}}^{\mu_{k-1}} x$$
$$= \mathsf{s}_{\nu_0}^{\pi_0} \mathsf{s}_{\pi_0}^{\mu_0} \dots \mathsf{s}_{\nu_{k-1}}^{\pi_{k-1}} \mathsf{s}_{\pi_{k-1}}^{\mu_{k-1}} x \quad \text{by} \quad (2.8)$$
$$= \mathsf{s}_{\nu_0}^{\rho_0} \mathsf{s}_{\rho_0}^{\mu_0} \mathsf{c}_{\pi_0} \mathsf{c}_{\rho_0} \dots \mathsf{s}_{\nu_{k-1}}^{\rho_{k-1}} \mathsf{s}_{\rho_{k-1}}^{\mu_{k-1}} \mathsf{c}_{\pi_{k-1}} \mathsf{c}_{\rho_{k-1}} x \quad \text{by} \quad (4.2)$$
$$= \mathsf{s}_{\nu_0}^{\rho_0} \mathsf{s}_{\rho_0}^{\mu_0} \dots \mathsf{s}_{\nu_{k-1}}^{\rho_{k-1}} \mathsf{s}_{\rho_{k-1}}^{\mu_{k-1}} x$$
$$= \mathsf{s}_{\nu_0}^{\rho_0} \dots \mathsf{s}_{\nu_{k-1}}^{\rho_{k-1}} \mathsf{s}_{\rho_0}^{\mu_0} \dots \mathsf{s}_{\rho_{k-1}}^{\mu_{k-1}} x \quad \text{by} \quad (2.8)$$
$$= \mathsf{s}_\nu^\rho \mathsf{s}_\rho^\mu x.$$

To prove (6) in the general case, note that there is a $\xi \in {}^k\alpha$ with ξ one-one, $Rg\xi \cap (\Delta x \cup Rg\mu \cup Rg\nu) = 0$, and $Rg\xi \cap (Rg\pi \cup Rg\rho) = 0$, and then use the special case just established.

(7) follows from (6), (2.4), (2.8) and (4). ■

Theorem 5 . *Let $\mathfrak{A} \in \mathbf{K}_\alpha$ such that $\alpha \smallsetminus \Delta x$ is infinite for every x. For finite $\Gamma \subseteq \alpha$, $\mathsf{c}_{(\Gamma)} x$ abbreviates $\mathsf{c}_{i_0} \dots \mathsf{c}_{i_n} x$, where $\Gamma = \{i_0 \dots, i_n\}$. Let σ and τ be finite transformations on α, Γ, Δ finite subsets of α and $x \in A$. Then*

(i) s_τ *is a boolean endomorphism of* $\mathfrak{A}$.

(ii) $\mathsf{s}_{\sigma \circ \tau} = \mathsf{s}_\sigma \circ \mathsf{s}_\tau$.

(iii) If $(\alpha \smallsetminus \Gamma) \upharpoonright \sigma = (\alpha \smallsetminus \Gamma) \upharpoonright \tau$, then $\mathsf{s}_\sigma \mathsf{c}_{(\Gamma)} x = \mathsf{s}_\tau \mathsf{c}_{(\Gamma)} x$.

(iv) if $\sigma \upharpoonright \Delta x = \tau \upharpoonright \Delta x$, then $\mathsf{s}_\sigma x = \mathsf{s}_\tau x$.

(v) If $\tau^{-1}\Gamma = \Delta$ and $\tau \upharpoonright \Delta$ is one to one, then $\mathsf{c}_{(\Gamma)} \mathsf{s}_\tau x = \mathsf{s}_\tau \mathsf{c}_{(\Delta)} x$.

Proof. We follow closely [19] Thms 1.11.11, 1.11.12. It is tedious but not hard to show that the axiomatization given above is adequate for the existence of such substitution operations with such properties. We now give details. They are merely manipulations of substitutions. We proceed like [19] p. 238.

(i) Let x, y be given; τ be a finite transformation and let $k<\omega$. Let μ be a one-one function in ${}^k\alpha$ such that $\{\lambda : \tau(\lambda)\neq\lambda\}\subseteq Rg\mu$. Choose $\pi\in {}^k\alpha$ such that π is one-one and $Rg\pi\cap(\Delta x\cup\Delta y\cup Rg\mu\cup\tau(Rg\mu))=0$. Then

$$\begin{aligned} \mathsf{s}_\tau(x+y) &= \mathsf{s}^\pi_{\tau\circ\mu}\mathsf{s}^\mu_\pi(x+y) && \text{by (4.7)}\\ &= \mathsf{s}^\pi_{\tau\circ\mu}\mathsf{s}^\mu_\pi x+\mathsf{s}^\pi_{\tau\circ\mu}\mathsf{s}^\mu_\pi y && \text{by (2.3)}\\ &= \mathsf{s}_\tau x+\mathsf{s}_\tau y && \text{by (4.7)} \end{aligned}$$

Furthermore, it is clear from (2.3) that $\mathsf{s}_\tau - x = -\mathsf{s}_\tau x$. Thus s_τ is a boolean endomorphism of $\mathfrak{A}$.

For (ii), let σ,τ be finite transformations of α. Then there is a $k<\omega$ and $\mu\in {}^k\alpha$ such that μ is one-one, $\{\lambda:\tau(\lambda)\neq\lambda\}\cup\{\lambda:\sigma(\lambda)\neq\lambda\}\subseteq Rg\mu$, and $\sigma(Rg\mu)\cup\tau(Rg\mu)\subseteq Rg\mu$. There exist $\tau,\rho\in {}^k\alpha$ such that π,ρ are one-one, $Rg\pi\cap(\Delta x\cup Rg\mu)=0$, and $Rg\rho\cap(\Delta\mathsf{s}_\tau x\cup Rg\mu\cup Rg\pi)=0$. Thus, by (4.7)

$$\mathsf{s}_\sigma\mathsf{s}_\tau x=\mathsf{s}^\rho_{\sigma\circ\mu}\mathsf{s}^\mu_\rho\mathsf{s}^\pi_{\tau\circ\mu}\mathsf{s}^\mu_\pi x \tag{1}$$

We now claim that, for any $\lambda\leq k$,

$$\mathsf{s}_\sigma\mathsf{s}_\tau x=\mathsf{s}^{\pi\restriction\lambda}_{\lambda\restriction\sigma\circ\tau\circ\mu}\mathsf{s}^\rho_{\sigma\circ\mu}\mathsf{s}^\mu_\rho\mathsf{s}^{\pi\restriction(k\smallsetminus\lambda)}_{\tau\circ\mu\restriction(k\smallsetminus\lambda)}\mathsf{s}^\mu_\pi x \tag{2}$$

For $\lambda=0$, (2) reduces to (1). Assuming that (2) holds for $\lambda<k$ we prove it for $\lambda+1$. Let

$$\tau(\mu_\lambda)=\mu_\eta; \tag{3}$$

we then have

$$\begin{aligned} \mathsf{s}_\sigma\mathsf{s}_\tau x &= \mathsf{s}^{\pi\restriction\lambda}_{\sigma\circ\tau\circ\mu\restriction\lambda}\mathsf{s}^\rho_{\sigma\circ\mu}\mathsf{s}^\mu_\rho\mathsf{s}^{\mu_\eta}_{\rho_\eta}\mathsf{s}^{\pi\restriction(k\smallsetminus\lambda)}_{\tau\circ\mu\restriction(k\smallsetminus\lambda)}\mathsf{s}^\mu_\pi x && \text{by (2) and (2.8)}\\ &= \mathsf{s}^{\pi\restriction\lambda}_{\sigma\circ\tau\circ\mu\restriction\lambda}\mathsf{s}^\rho_{\sigma\circ\mu}\mathsf{s}^\mu_\rho\mathsf{s}^{\mu_\eta}_{\rho_\eta}\mathsf{s}^{\pi_\lambda}_{\rho_\eta}\mathsf{s}^{\pi\restriction(k\smallsetminus\lambda)}_{\tau\circ\mu\restriction(k\smallsetminus\lambda)}\mathsf{s}^\mu_\pi x && \text{by (3) and (2.9)}\\ &= \mathsf{s}^{\pi\restriction\lambda}_{\sigma\circ\tau\circ\mu\restriction\lambda}\mathsf{s}^\rho_{\sigma\circ\mu}\mathsf{s}^{\rho_\eta}_{\sigma\mu_\eta}\mathsf{s}^{\pi_\lambda}_{\rho_\eta}\mathsf{s}^\mu_\rho\mathsf{s}^{\pi\restriction(k\smallsetminus\lambda)}_{\tau\circ\mu\restriction(k\smallsetminus\lambda)}\mathsf{s}^\mu_\pi x && \text{by (2.8)}\\ &= \mathsf{s}^{\pi\restriction\lambda}_{\sigma\circ\tau\circ\mu\restriction\lambda}\mathsf{s}^\rho_{\sigma\circ\mu}\mathsf{s}^{\rho_\eta}_{\sigma\mu_\eta}\mathsf{s}^\mu_\rho\mathsf{s}^{\pi\restriction(k\smallsetminus\lambda)}_{\tau\circ\mu\restriction(k\smallsetminus\lambda)}\mathsf{s}^\mu_\pi x && \text{by (2.9)}\\ &= \mathsf{s}^{\pi\restriction(\lambda+1)}_{\sigma\circ\tau\circ\mu\restriction(\lambda+1)}\mathsf{s}^\rho_{\sigma\circ\mu}\mathsf{s}^\mu_\rho\mathsf{s}^{\pi\restriction(k\smallsetminus(\lambda+1))}_{\tau\circ\mu\restriction(k\smallsetminus(\lambda+1))}\mathsf{s}^\mu_\pi x && \text{by (3) and (2.8)} \end{aligned}$$

By taking $\lambda=k$ in (2) , and by (2.4) and (4.3) and using (4.7) we get

$$\mathsf{s}_\sigma\mathsf{s}_\tau x=\mathsf{s}^\pi_{\sigma\circ\tau\circ\mu}\mathsf{s}^\mu_\pi x=\mathsf{s}_{\sigma\circ\tau}x.$$

(iii) We may restrict ourselves to the case $|\Gamma|=1$, say $\Gamma=\{k\}$. There is a $\mu\in\alpha\smallsetminus(\Delta x\cup\{k\}))$. Then

$$\begin{aligned} \mathsf{s}_\sigma\mathsf{c}_k x &= \mathsf{s}_\sigma\mathsf{s}^k_\mu\mathsf{c}_k x && \text{by (2.4)}\\ &= \mathsf{s}_{\sigma\circ[k|\mu]}\mathsf{c}_k x && \text{by (ii), (4.5)}\\ &= \mathsf{s}_{\tau\circ[k|\mu]}\mathsf{c}_k x\\ &= \mathsf{s}_\tau\mathsf{s}^k_\mu\mathsf{c}_k x && \text{by (ii), (4.5)}\\ &= \mathsf{s}_\tau\mathsf{c}_k x && \text{by (2.4)} \end{aligned}$$

(iv) Let $\Gamma = \{i : \sigma(i) \neq \tau(i)\}$. Then $|\Gamma| < \omega$, $\Gamma \subseteq \alpha \smallsetminus \Delta x$ and hence $\mathsf{c}_{(\Gamma)}x = x$ and $\sigma \restriction (\alpha \smallsetminus \Gamma) = \tau \restriction (\alpha \smallsetminus \Gamma)$. Thus by (iii) we get $\mathsf{s}_\sigma x = \mathsf{s}_\tau x$.

(v) It is sufficient to treat the case in which $|\Gamma| = 1$, say $\Gamma = \{k\}$. So, we have two cases. First one, we may have $\Delta = 0$. Then, for some $\mu \in \alpha \smallsetminus \{k\}$ we have

$$\begin{aligned}
\mathsf{c}_k\mathsf{s}_\tau x &= \mathsf{c}_k\mathsf{s}_{[k|\mu]\circ\tau}x \\
&= \mathsf{c}_k\mathsf{s}^k_\mu\mathsf{s}_\tau x \qquad \text{by} \quad (\text{ii}), (4.5) \\
&= \mathsf{s}^k_\mu\mathsf{s}_\tau x \qquad \text{by} \quad (2.5) \\
&= \mathsf{s}_\tau x.
\end{aligned}$$

Second, we have $\Delta = \{\lambda\}$ for some λ. Then

$$(([k|\lambda, \lambda|k] \circ \tau)^{-1})\lambda = \{\lambda\},$$

and hence $[k|\lambda, \lambda|k] \circ \tau$ can be written as a product of transpositions and replacements ρ such that $(\rho^{-1})\lambda = \{\lambda\}$. It is easily seen using (2.6) that for any such transformation ρ we have $\mathsf{s}_\rho\mathsf{c}_\lambda x = \mathsf{c}_\lambda\mathsf{s}_\rho x$ for every $x \in A$, and hence

$$(\star) \quad \mathsf{s}_{[k|\lambda,\lambda|k]\circ\tau}\mathsf{c}_\lambda x = \mathsf{c}_\lambda\mathsf{s}_{[k|\lambda,\lambda|k]\circ\tau}x.$$

We need some equations before going on:

(I) If $\{k, \lambda\} \cap \{\mu, \nu\} = 0$, then $\mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu\mathsf{c}_\mu\mathsf{c}_\nu x = \mathsf{s}^\nu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\nu\mathsf{c}_\mu\mathsf{c}_\nu x$.

Indeed,

$$\begin{aligned}
\mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu\mathsf{c}_\mu\mathsf{c}_\nu x &= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu\mathsf{c}_\nu\mathsf{c}_\mu x && \text{by}(2.1) \\
&= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\nu_\mu\mathsf{s}^\lambda_\nu\mathsf{c}_\nu\mathsf{c}_\mu x && \text{by}(4.1) \\
&= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\nu_\mu\mathsf{s}^\lambda_\nu\mathsf{c}_\mu\mathsf{c}_\nu x && \text{by}(2.1) \\
&= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\nu_\mu\mathsf{c}_\mu\mathsf{s}^\lambda_\nu\mathsf{c}_\nu x && \text{by}(2.6) \\
&= \mathsf{s}^\mu_k\mathsf{s}^\nu_\mu\mathsf{s}^k_\lambda\mathsf{c}_\mu\mathsf{s}^\lambda_\nu\mathsf{c}_\nu x && \text{by}(2.8) \\
&= \mathsf{s}^\mu_k\mathsf{s}^\nu_\mu\mathsf{c}_\mu\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\nu\mathsf{c}_\nu x && \text{by}(2.6) \\
&= \mathsf{s}^\nu_k\mathsf{c}_\mu\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\nu\mathsf{c}_\nu x && \text{by}(4.1) \\
&= \mathsf{s}^\nu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\nu\mathsf{c}_\mu\mathsf{c}_\nu x && \text{by}(2.6).
\end{aligned}$$

(II) For all $k, \lambda, \mu < \alpha, k \neq \lambda$ and $\mu \notin \Delta x$ we have $\mathsf{s}_{[k/\lambda,\lambda/k]}x = \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu x$.
We have,

$$\begin{aligned}
\mathsf{s}_{[k/\lambda,\lambda/k]}x &= \mathsf{s}^{\mu_0}_\lambda\mathsf{s}^\mu_k\mathsf{s}^k_{\mu_0}\mathsf{s}^\lambda_\mu x \qquad \text{for} \quad \mu, \mu_0 \notin \Delta x \\
&= \mathsf{s}^\mu_k\mathsf{s}^{\mu_0}_\lambda\mathsf{s}^k_{\mu_0}\mathsf{s}^\lambda_\mu x \qquad \text{by} \quad (2.8) \\
&= \mathsf{s}^\mu_k\mathsf{s}^{\mu_0}_\lambda\mathsf{s}^k_{\mu_0}\mathsf{c}_{\mu_0}\mathsf{s}^\lambda_\mu x \\
&= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{c}_{\mu_0}\mathsf{s}^\lambda_\mu x \qquad \text{by} \quad (4.1) \\
&= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu x.
\end{aligned}$$

(III) $\mathsf{c}_k\mathsf{s}_{[k|\lambda,\lambda|k]}\mathsf{c}_\mu = \mathsf{s}_{[k|\lambda,\lambda|k]}\mathsf{c}_\lambda\mathsf{c}_\mu x$ where k, λ, μ are distinct.
Indeed,

$$\begin{aligned}
\mathsf{c}_k\mathsf{s}_{[k|\lambda,\lambda|k]}\mathsf{c}_\mu x &= \mathsf{c}_k\mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu\mathsf{c}_\mu x && \text{by (II)}\\
&= \mathsf{c}_\mu\mathsf{s}^k_\mu\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu\mathsf{c}_\mu x && \text{by(2.7)}\\
&= \mathsf{c}_\mu\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu\mathsf{c}_\mu x && \text{by(4.3)}\\
&= \mathsf{s}^k_\lambda\mathsf{c}_\mu\mathsf{s}^\lambda_\mu\mathsf{c}_\mu x && \text{by(2.6)}\\
&= \mathsf{s}^k_\lambda\mathsf{c}_\lambda\mathsf{s}^\mu_\lambda\mathsf{c}_\mu x && \text{by(2.7)}\\
&= \mathsf{s}^k_\lambda\mathsf{c}_\lambda\mathsf{c}_\mu x && \text{by(2.4)}\\
&= \mathsf{s}^k_\lambda\mathsf{c}_\mu\mathsf{c}_\lambda x && \text{by(2.1)}\\
&= \mathsf{c}_\mu\mathsf{s}^k_\lambda\mathsf{c}_\lambda x && \text{by(2.6)}\\
&= \mathsf{s}^\mu_k\mathsf{c}_\mu\mathsf{s}^k_\lambda\mathsf{c}_\lambda x && \text{by(2.4)}\\
&= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{c}_\mu\mathsf{c}_\lambda x && \text{by(2.6)}\\
&= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{c}_\lambda\mathsf{c}_\mu x && \text{by(2.1)}\\
&= \mathsf{s}^\mu_k\mathsf{s}^k_\lambda\mathsf{s}^\lambda_\mu\mathsf{c}_\lambda\mathsf{c}_\mu x && \text{by(2.4)}\\
&= \mathsf{s}_{[k|\lambda,\lambda|k]}\mathsf{c}_\lambda\mathsf{c}_\mu x && \text{by (II).}
\end{aligned}$$

Now applying $\mathsf{s}_{[k|\lambda,\lambda|k]}$ to both sides of $(\star)$ we get

$$\begin{aligned}
\mathsf{s}_\tau\mathsf{c}_\lambda x &= \mathsf{s}_{[k|\lambda,\lambda|k]}\mathsf{c}_\lambda\mathsf{s}_{[k|\lambda,\lambda|k]\circ\tau}x\\
&= \mathsf{c}_k\mathsf{s}_{[k|\lambda,\lambda|k]}\mathsf{s}_{[k|\lambda,\lambda|k]\circ\tau}x && \text{by (III)}\\
&= \mathsf{c}_k\mathsf{s}_\tau x.
\end{aligned}$$

■

$\mathbf{DKc}_\alpha$ denotes all $\mathfrak{A} \in \mathbf{K}_\alpha$ such that $\alpha \smallsetminus \Delta x$ is infinite for all $x \in A$.

Lemma 6 . *Let $\alpha \geq \omega$. Then for every $\beta \geq \alpha$, $\mathbf{DKc}_\alpha \subseteq \mathfrak{Nr}_\alpha\mathbf{K}_\beta$*

Sketch of Proof.
The proof is analogous to that of Theorem 2.6.49(i) in [19].

- Let $R = Id \restriction (\alpha \times A) \cup \{((k, x), (\lambda, y)) : k, \lambda < \alpha, x, y \in A, \lambda \notin \Delta x, y = \mathsf{s}^k_\lambda x\}$. It is easy to see that R is an equivalence relation on $(\alpha \times A)$. Define the quotient algebra $\mathfrak{C}$ as in [19] p.412, with substitutions for $k, l < \alpha$ defined as follows:

$$\mathsf{s}^{l\,\prime}_k(u, x)/R = (u, \mathsf{s}^l_k x/R), x \in A, u \in \alpha \smallsetminus \{k, l\}$$

$$\mathsf{s}'_{[k,l]}(u, x)/R = (u, \mathsf{s}_{[k,l]}x/R), x \in A, u \in \alpha \smallsetminus \{k, l\}.$$

Then $\mathfrak{C} \in \mathbf{K}_\alpha$.

- Let $h(x) = (u, x)/R$ for $x \in A$ and $u \in \alpha \smallsetminus \Delta x$, then h is a well defined isomorphism from $\mathfrak{A}$ into $\mathfrak{C}$.

- Define extra operations with the new index α on $\mathfrak{C}$ like the **CA** case defining (taking as an example substitutions corresponding to transpositions) $\mathsf{s}_{[k,\alpha]}(u, x)/R = (u, \mathsf{s}_{[k,\alpha]}x/R)$ for any $u \in \alpha \smallsetminus \{k\}$. Then $\mathfrak{A} \subseteq Nr_\alpha\mathfrak{B}$ where $\mathfrak{B} \in \mathbf{K}_{\alpha+1}$ is the expanded algebra.

- Neatly embedding $\mathfrak{A}$ into k extra dimensions, for $k \in \omega$, is done via a straightforward induction and for the transfinite case using ultraproducts.

■

The following is a generalization of Pigozzi's Theorem 2.2.10 in [25]. It addresses other algebras and it proves $SUPAP$ which is stronger than SAP. The proof is similar to Pigozzi's proof, but it has the advantage that it is self contained. Our proof is more streamlined side stepping non trivial lemmas not used in their full generality. (Pigozzi's proof is somewhat terse, cryptic and circuitous, it is done in several stages and it is scattered over a series of non-trivial lemmas.) Furthermore, in our proof, ultrafilters are used in place of maximal ideals. Such ultrafilters represent rich complete theories and are termed as Henkin ultrafilters in [8]. (Using ultrafilters makes the proof more natural and the analogy with the methods of Henkin of constructing models out of constants becomes clearer.) The proof is also a variation on a theme occuring in [13] and [8].

In the following, we extensively use the fact that ideals in $\mathbf{K}_\alpha$'s function like the **CA** case. For $\mathfrak{B} \in \mathbf{K}_\alpha$ and $X \subseteq \mathfrak{B}$, $\mathfrak{Ig}^{\mathfrak{B}}X$ denotes the ideal generated by X.

Theorem 7 . *Let κ be any ordinal > 1. Let $L = \{\mathfrak{A} \in \mathbf{K}_{\kappa+\omega} : \mathfrak{A} = \mathfrak{Sg}^{\mathfrak{A}}\mathfrak{Nr}_\kappa\mathfrak{A}\}$. Then L has $SUPAP$.*

Proof. Assume that α is infinite. Let μ be a cardinal > 0, $\rho : \mu \to \mathcal{P}(\alpha)$ be a function such that $|\alpha \setminus \rho(i)| \geq \omega$ for all $i \in \mu$. Let $\mathfrak{A} = \mathfrak{Fr}_\mu^\rho(\mathbf{K}_\alpha)$. We first show the following interpolation result. For all $X_1, X_2 \subseteq \mathfrak{A}$, if $a \in \mathfrak{Sg}^{\mathfrak{A}}X_1$ and $b \in \mathfrak{Sg}^{\mathfrak{A}}X_2$, such that $a \leq b$, then there exists $c \in \mathfrak{Sg}^{\mathfrak{A}}(X_1 \cap X_2)$ such that $a \leq c \leq b$. We refer to c as an interpolant of a and b. This is proved in [13] for cylindric algebras when $|\alpha| = \mu = \omega$. Here we generalize the result to arbitrary ordinals and cardinals and to $\mathbf{K}_\alpha$.

(1) First for any ordinal $\beta > \alpha$, the sequence $\langle \eta/Cr_\mu^\rho(\mathbf{K}_\beta) : \eta < \mu\rangle$ $\mathbf{K}_\alpha$ -freely generates $\mathfrak{Nr}_\alpha\mathfrak{Fr}_\mu^\rho(\mathbf{K}_\beta)$. Let $\mathfrak{B} \in \mathbf{K}_\alpha$ and $a = \langle a_\eta : \eta < \mu\rangle \in {}^\mu B$ be such that $\Delta a_\eta \subseteq \rho(\eta)$ for all $\eta < \mu$. Since $\mathfrak{Fr}_\mu^\rho(\mathbf{K}_\alpha)$ is in $\mathbf{DKc}_\alpha$, we have $|\alpha \setminus \bigcup_{\xi\in\Gamma} \rho\xi| \geq \omega$ for each finite $\Gamma \subseteq \mu$. Assuming that

Rga generates $\mathfrak{B}$, we have $\mathfrak{B} \in \mathbf{DKc}_\alpha$. Therefore $\mathfrak{B} \in S\mathfrak{Nr}_\alpha\mathbf{K}_\beta$. Let $\mathfrak{D} = \mathfrak{Fr}_\mu^\rho(\mathbf{K}_\beta)$. Then $x = \langle \eta/Cr_\mu^\rho\mathbf{K}_\beta : \eta < \mu\rangle \in {}^\mu D\ S\mathfrak{Nr}_\alpha\mathbf{K}_\beta$ - freely generates $\mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{D}}Rgx$. Indeed consider $\mathfrak{C} \in \mathfrak{Nr}_\alpha\mathbf{K}_\beta$ and $y \in {}^\mu C$ such that $\Delta y_\eta \subseteq \rho\eta$ for all $\eta < \mu$. Let $\mathfrak{C}' \in \mathbf{K}_\beta$ be such that $\mathfrak{C} = \mathfrak{Nr}_\alpha\mathfrak{C}'$. Then clearly $y \in {}^\mu C'$ and $\Delta y_\eta \subseteq \alpha$ for all $\eta < \mu$. Then there exists $h \in Hom(\mathfrak{D}, \mathfrak{C}')$ such that $h \circ x = y$. Hence $h \in Hom(\mathfrak{Rd}_\alpha\mathfrak{D}, \mathfrak{Rd}_\alpha\mathfrak{C}')$, thus $h \in Hom(\mathfrak{Sg}^{\mathfrak{Rd}_\alpha\mathfrak{D}}Rgx, \mathfrak{Sg}^{\mathfrak{Rd}_\alpha\mathfrak{C}'}h(Rgx))$. Since $Rgx \subseteq Nr_\alpha D$, we have $h \in Hom(\mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{D}}Rgx, \mathfrak{C})$. The conclusion follows. Therefore there exists $h : \mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{D}}Rgx \to \mathfrak{B}$ such that $h(\eta/Cr_\mu^\rho\mathbf{K}_\beta) = a_\eta$. But $\mathfrak{Nr}_\alpha\mathfrak{Fr}_\mu^\rho(\mathbf{K}_\beta) = \mathfrak{Nr}_\alpha(\mathfrak{Sg}^{\mathfrak{D}}Rgx) = \mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{D}}Rgx$. Therefore, as claimed, $\langle \eta/Cr_\mu^\rho(\mathbf{SC}_\beta) : \eta < \mu\rangle\ \mathbf{K}_\alpha$ freely generates $\mathfrak{Nr}_\alpha\mathfrak{Fr}_\mu^\rho(\mathbf{K}_\beta)$.

(2) Identifying $\eta/Cr_\mu^\rho\mathbf{K}_\alpha$ with $\eta/Cr_\mu^\rho\mathbf{K}_\beta$ for each $\eta < \mu$, we can and will assume that $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{D}$. We next show that for all $X \subseteq \mathfrak{A}$, we have (*) $\mathfrak{Sg}^{\mathfrak{A}}X = \mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{D}}X$. Let $\mathfrak{D}' = \mathfrak{Sg}^{\mathfrak{D}}X$ and let $\mathfrak{A}' = \mathfrak{Sg}^{\mathfrak{A}}X$. Both $\mathfrak{D}' \in \mathbf{DKc}_\beta$ and $\mathfrak{A}' \in \mathbf{DKc}_\alpha$. For $\mathfrak{F} \in \mathbf{DKc}_\lambda$ ($\lambda \geq \omega$), and τ is a finite transformation on λ, let $\mathsf{s}_\tau^{\mathfrak{F}}$ be the unary operation of substitution as defined above. In this part we follow [19] 2.6.65. Let $\mathfrak{D}'' = \{\mathsf{s}_\sigma^{\mathfrak{D}'}x : x \in \mathfrak{A}'$ and σ is a finite transformation of β such that $\sigma \upharpoonright \alpha$ is one to one $\}$. Since $\mathsf{s}_{Id}x = x$ for all $x \in A'$ it is clear that $A' \subseteq D''$. It is not hard to show that D'' is a subuniverse of D', hence $D' = D''$. Now suppose $y \in D'$ and $\Delta x \subseteq \alpha$. There exist $y \in A'$ and a finite transformation σ of β such that $\sigma \upharpoonright \alpha$ is one to one and $x = \mathsf{s}_\sigma^{\mathfrak{D}''}y$. Let τ be a finite transformation of β such that $\tau \upharpoonright \alpha = Id$ and $(\tau \circ \sigma)\alpha \subseteq \alpha$. Then $x = \mathsf{s}_\tau^{\mathfrak{D}''}x = \mathsf{s}_\tau^{\mathfrak{D}''}\mathsf{s}_\sigma y = \mathsf{s}_{\tau\circ\sigma}^{\mathfrak{D}''}y = \mathsf{s}_{\tau\circ\sigma\upharpoonright\alpha}^{\mathfrak{A}'}y$. We have proved (*).

(3) Now suppose that there exists no interpolant in $\mathfrak{Sg}^{\mathfrak{A}}(X_1 \cap X_2)$. Then there exists no interpolant in $\mathfrak{Sg}^{\mathfrak{D}}(X_1 \cap X_2)$. Indeed let c be an interpolant in $\mathfrak{Sg}^{\mathfrak{D}}(X_1 \cap X_2)$. Let $\Gamma = (\beta \setminus \alpha) \cap \Delta c$. Then Γ is finite. Let $c' = \mathsf{c}_{(\Gamma)}c$. Then $a \leq c'$. Also $b = \mathsf{c}_{(\Gamma)}b$, so that $c' \leq b$. hence $a \leq c' \leq b$. But $c' \in \mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{D}}(X_1 \cap X_2) = \mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{D}}(X_1 \cap X_2) = \mathfrak{Sg}^{\mathfrak{A}}(X_1 \cap X_2)$.

(4) Now assume that β is a regular cardinal such that $\beta > |\Delta x|$ for all $x \in \mathfrak{D}$, so that $|\mathfrak{D}| = \beta$. We now have $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{D}$, $a, b \in \mathfrak{A}$, $a \leq b$ and there is no interpolant in $\mathfrak{D}$. We will arrive at a contradiction. We follow closely [8], which in turn follows [25]. Arrange $\beta \times \mathfrak{Sg}^{\mathfrak{D}}X_1$ and $\beta \times \mathfrak{Sg}^{\mathfrak{D}}X_2$ as β - termed sequences

$$\langle (k_i, x_i) : i \in \beta\rangle \text{ and } \langle (l_i, y_i) : i \in \beta\rangle,$$

respectively. Because β is regular and $\beta > |\Delta x|$ for all $x \in \mathfrak{D}$ we can define simultaneously by transfinite recursion β-termed sequences (of witnesses)

$$\langle u_i : i \in \beta\rangle \text{ and } \langle v_i : i \in \beta\rangle$$

such that for all $i \in \beta$ we have:

$$u_i \in \beta \smallsetminus (\Delta a \cup \Delta b) \cup \bigcup_{j \le i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j < i\} \cup \{v_j : j < i\}$$

and

$$v_i \in \beta \smallsetminus (\Delta a \cup \Delta b) \cup \bigcup_{j \le i} (\Delta x_j \cup \Delta y_j) \cup \{u_j : j \le i\} \cup \{v_j : j < i\}.$$

For a cylindric algebra $\mathfrak{F}$ we write $Bl\mathfrak{F}$ to denote its boolean reduct. For $i, j < \mu$, $i \neq j$, recall that for **CA**'s and **QEA**'s, we have $\mathsf{s}_i^j x = \mathsf{c}_i(\mathsf{d}_{ij} \cdot x)$ and $\mathsf{s}_i^i x = x$. (s_i^j is a unary operation that abstracts the operation of substituting the variable v_i for the variable v_j such that the substitution is free.) For a boolean algebra $\mathfrak{B}$ and $Y \subseteq \mathfrak{B}$, we write $fl^{\mathfrak{B}} Y$ to denote the boolean filter generated by Y in $\mathfrak{B}$. Now let

$$Y_1 = \{a\} \cup \{-\mathsf{c}_{k_i} x_i + \mathsf{s}_{u_i}^{k_i} x_i : i \in \beta\},$$

$$Y_2 = \{-c\} \cup \{-\mathsf{c}_{l_i} y_i + \mathsf{s}_{v_i}^{l_i} y_i : i \in \beta\},$$

$$H_1 = fl^{Bl\mathfrak{Sg}^{\mathfrak{D}}(X_1)} Y_1, \ H_2 = fl^{Bl\mathfrak{Sg}^{\mathfrak{D}}(X_2)} Y_2,$$

and

$$H = fl^{Bl\mathfrak{Sg}^{\mathfrak{D}}(X_1 \cap X_2)}[(H_1 \cap \mathfrak{Sg}^{\mathfrak{D}}(X_1 \cap X_2) \cup (H_2 \cap \mathfrak{Sg}^{\mathfrak{D}}(X_1 \cap X_2)].$$

Then H is a proper filter of $\mathfrak{Sg}^{\mathfrak{D}}(X_1 \cap X_2)$. To prove this it is sufficient to consider any pair of finite, strictly increasing sequences of natural numbers

$$\eta(0) < \eta(1) \cdots < \eta(n-1) < \omega \text{ and } \xi(0) < \xi(1) < \cdots < \xi(m-1) < \omega,$$

and to prove that the following condition holds: This is proved by induction on $n + m$, cf [8], [13].

$$\begin{array}{l} \text{For any } \ b_0, b_1 \in \mathfrak{Sg}^{B}(X_1 \cap X_2) \ \text{ such that} \\ \qquad a.\prod_{i<n}(-\mathsf{c}_{k_{\eta(i)}} x_{\eta(i)} + \mathsf{s}_{u_{\eta(i)}}^{k_{\eta(i)}} x_{\eta(i)}) \le b_0 \\ \quad \text{and} \\ \qquad (-c).\prod_{i<m}(-\mathsf{c}_{l_{\xi(i)}} y_{\xi(i)} + \mathsf{s}_{v_{\xi(i)}}^{l_{\xi(i)}} y_{\xi(i)}) \le b_1 \\ \text{we have} \\ \qquad b_0.b_1 \neq 0. \end{array} \tag{4}$$

The base of the induction (i.e when $n + m = 0$) follows from the fact that no interpolant exists in $\mathfrak{Sg}^{\mathfrak{D}}(X_1 \cap X_2)$).

(5) Proving that H is a proper filter of $\mathfrak{Sg}^{\mathfrak{D}}(X_1\cap X_2)$, let H^* be a (proper boolean) ultrafilter of $\mathfrak{Sg}^{\mathfrak{D}}(X_1\cap X_2)$ containing H. We obtain ultrafilters F_1 and F_2 of $\mathfrak{Sg}^{\mathfrak{D}}(X_1)$ and $\mathfrak{Sg}^{\mathfrak{D}}(X_2)$, respectively, such that

$$H^*\subseteq F_1,\quad H^*\subseteq F_2$$

and (**)

$$F_1\cap\mathfrak{Sg}^{\mathfrak{D}}(X_1\cap X_2)=H^*=F_2\cap\mathfrak{Sg}^{\mathfrak{D}}(X_1\cap X_2).$$

Now for all $x\in\mathfrak{Sg}^{\mathfrak{D}}(X_1\cap X_2)$ we have

$$x\in F_1 \text{ if and only if } x\in F_2.$$

Also from how we defined our ultrafilters, F_i for $i\in\{1,2\}$ satisfy the following condition:

(+) For all $k<\mu$, for all $x\in\mathfrak{Sg}^{\mathfrak{D}}X_i$ if $\mathsf{c}_kx\in F_i$ then s_l^kx is in F_i for some $l\notin\Delta x$. Let V be the set of all finite transformations on β, i.e

$$V=\{\sigma\in{}^{\beta}\beta:|\{\sigma(i)\neq i\}|<\omega\}.$$

Let $i\in\{1,2\}$. Let $\mathfrak{D}_i=\mathfrak{Sg}^{\mathfrak{D}}X_i$. Let $\mathbf{K}\in\{\mathbf{SC},\mathbf{QA}\}$. Define

$$f_ix=\{\tau\in V:\mathsf{s}_\tau x\in F_i\}.$$

Then f_i is a homomorphism from $\mathfrak{D}_i$ to the full weak set algebra with unit V [13] (Preserving cylindrifications depend on (+)) Now fix $i\in\{1,2\}$. For $\mathbf{K}\in\{\mathbf{CA},\mathbf{QEA}\}$, let $E_i=\{(k,l):k,l<\beta,\mathsf{d}_{kl}\in F_i\}$. Then $E_1=E_2=E$ is an equivalence relation on β. E is reflexive because $\mathsf{d}_{ii}=1$ and symmetric because $\mathsf{d}_{ij}=\mathsf{d}_{ji}$. E is transitive because F is a filter and for all $k,l,u<\beta$, with $l\notin\{k,u\}$, we have $\mathsf{d}_{kl}.\mathsf{d}_{lu}\leq\mathsf{c}_l(\mathsf{d}_{kl}.\mathsf{d}_{lu})=\mathsf{d}_{ku}$. Let $\sigma,\tau\in V$. Write $\sigma\equiv_E\tau$ iff $(\forall i\in\mu)(\sigma(i),\tau(i))\in E$ and let $\bar{E}=\{(\sigma,\tau)\in{}^2V:\sigma\equiv_E\tau\}$. Let $M=\mu/E$ and for $j\in\beta$, let $q(j)=j/E$. Let W be the weak space ${}^{\mu}M^{(q)}=\{h\in{}^{\beta}M:\{l\in\beta:h_l\neq q_l\}|<\omega\}$. For $h\in W$, write $h=\bar{\tau}$ for $\tau\in V$ such that $\tau(j)/E=h(j)$ for all $j\in\beta$. τ of course may not be unique. Let $\mathfrak{D}_i=\mathfrak{Sg}^{\mathfrak{D}}X_i$. Define f_i from $\mathfrak{D}_i$ to the full weak set algebra $\mathfrak{C}$ with unit W as follows: $f_i(x)=\{\bar{\tau}\in W:\mathsf{s}_\tau x\in F_i\}$, for $x\in\mathfrak{D}_i$.

Then f_i is well defined homomorphism [13]. Without loss of generality, we can assume that $X_1\cup X_2=X$. We have f_1 and f_2 agree on $X_1\cap X_2$. So that $f_1\cup f_2$ defines a function on $X_1\cup X_2$. Since $\{\eta/Cr_\mu^\rho\mathbf{K}_\beta:\eta<\beta\}$ $\mathbf{K}_\beta$-freely generates $\mathfrak{D}=\mathfrak{Fr}_\mu^\rho\mathbf{K}_\beta$, it follows that there is a homomorphism f from $\mathfrak{D}$ to $\mathfrak{C}$ such that $f_1\cup f_2\subseteq f$. We now have $q\in f(a)\cap f(-b)=f(a-b)$. This is so because $\mathsf{s}_{Id}a=a\in F_1$ and $\mathsf{s}_{Id}(-b)=-b\in F_2$. But this contradicts the premise that $a\leq b$. We have proved that $\mathfrak{Fr}_\mu^\rho\mathbf{K}_\alpha$ has the interpolation property when $|\alpha\setminus\rho(i)|\geq\omega$.

(6) We first prove that K has the amalgamation property. Let κ be an arbitrary ordinal > 0. Let $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ be in K and $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorphisms. We want to find an amalgam. Let $\langle a_i : i \in I \rangle$ be an enumeration of $\mathfrak{A}$ and $\langle b_i : i \in J \rangle$ be an enumeration of $\mathfrak{B}$ such that $\langle \mathsf{c}_i : i \in I \cap J \rangle$ is an enumeration of $\mathfrak{C}$ with $f(c_i) = a_i$ and $g(c_i) = b_i$ for all $i \in I \cap J$. Then $\Delta a_i = \Delta b_i$ for all $i \in I \cap J$. Let $k = I \cup J$. Let ξ be a bijection from k onto a cardinal μ. Let $\rho \in {}^{\mu}\mathcal{P}(\kappa + \omega)$ be defined by $\rho\xi i = \Delta a_i$ for $i \in I$ and $\rho\xi j = \Delta b_j$ for $j \in J$. Then ρ is well defined. Let $\beta = \kappa + \omega$ and let $\mathfrak{Fr} = \mathfrak{Fr}_{\mu}^{\rho}\mathbf{K}_{\beta}$. Let $\mathfrak{Fr}^I$ be the subalgebra of $\mathfrak{Fr}$ generated by $\{\xi i / Cr_{\mu}^{\rho}\mathbf{K}_{\beta} : i \in I\}$ and let $\mathfrak{Fr}^J$ be the subalgebra generated by $\{\xi i / Cr_{\mu}^{\rho}\mathbf{K}_{\beta} : i \in J\}$. To avoid cumbersome notation we write ξi instead of $\xi i / Cr_{\mu}^{\rho}\mathbf{K}_{\beta}$ and similarly for ξj. No confusion is likely to ensue. Then there exists a homomorphism from $\mathfrak{Fr}^I$ onto $\mathfrak{A}$ such that $\xi i \mapsto a_i$ $(i \in I)$ and similarly a homomorphism from $\mathfrak{Fr}^J$ into $\mathfrak{B}$ such that $\xi j \mapsto b_j$ $(j \in I)$. Therefore there exist ideals M and N ideals of $\mathfrak{Fr}^I$ and $\mathfrak{Fr}^J$ respectively, and there exist isomorphisms

$$m : \mathfrak{Fr}^I/M \to \mathfrak{A} \text{ and } n : \mathfrak{Fr}^J/N \to \mathfrak{B}$$

such that

$$m(\xi i/M) = a_i \text{ and } (\xi i/N) = b_i.$$

Let $\mathfrak{Fr}^{(I\cap J)}$ denote the subalgebra of $\mathfrak{Fr}$ generated by $\{\xi i : i \in I \cap J\}$. Then there is a homomorphism from $\mathfrak{Fr}^{(I\cap J)}$ into $\mathfrak{Fr}^I/M$ such that $\xi i \mapsto \xi i/M$. Therefore

$$\theta : \mathfrak{Fr}^{(I\cap J)}/M \cap \mathfrak{Fr}^{(I\cap J)} \to (\mathfrak{Fr}^I/M)^{(I\cap J)}$$

defined by

$$\theta\xi i/M \cap \mathfrak{Fr}^{(I\cap J)} = \xi i/M$$

is an isomorphism. Here $(\mathfrak{Fr}^I/M)^{(I\cap J)}$ is the subalgebra of $\mathfrak{Fr}^I/M$ generated by $\{\xi i/M : i \in I \cap J\}$. But

$$\psi : (\mathfrak{Fr}^I/M)^{(I\cap J)} \to \mathfrak{C}$$

defined by

$$\psi(\xi i/M) = c_i$$

is a well defined isomorphism. Thus the map $\psi \circ \theta$ defined by

$$\xi i/M \cap \mathfrak{Fr}^{(I\cap J)} \mapsto c_i$$

is an isomorphism from $\mathfrak{Fr}^{(I\cap J)}/M \cap \mathfrak{Fr}^{(I\cap J)}$ onto $\mathfrak{C}$. Similarly the map

$$\xi i/N \cap \mathfrak{Fr}^{(I\cap J)} \mapsto c_i$$

is an isomorphism from $\mathfrak{Fr}^{(I\cap J)}/N\cap\mathfrak{Fr}^{(I\cap J)}$ onto $\mathfrak{C}$. It follows that

$$M\cap\mathfrak{Fr}^{(I\cap J)}=N\cap\mathfrak{Fr}^{(I\cap J)}.$$

Now let $x\in\mathfrak{Ig}(M\cup N)\cap\mathfrak{Fr}^I$. Then there exist $b\in M$ and $c\in N$ such that $x\leq b+c$. Thus $x-b\leq c$. But $x-b\in\mathfrak{Fr}^{(I)}$ and $c\in\mathfrak{Fr}^J$, it follows that there exists an interpolant $d\in\mathfrak{Fr}^{(I\cap J)}$ such that $x-b\leq d\leq c$. We have $d\in N$ therefore $d\in M$, and since $x\leq d+b$, therefore $x\in M$. It follows that $\mathfrak{Ig}(M\cup N)\cap\mathfrak{Fr}^I=M$ and similarly $\mathfrak{Ig}(M\cup N)\cap\mathfrak{Fr}^J=N$. In particular $P=\mathfrak{Ig}(M\cup N)$ is a proper ideal. Let $\mathfrak{D}=\mathfrak{Fr}/P$. Let $k:\mathfrak{Fr}^I/M\to\mathfrak{Fr}/P$ be defined by $k(a/M)=a/P$ and $h:\mathfrak{Fr}^J/M\to\mathfrak{Fr}/P$ by $h(a/N)=a/P$. Then $k\circ m$ and $h\circ n$ are one to one and $k\circ m\circ f=h\circ n\circ g$.

(7) We now prove that $\mathfrak{Fr}/P$ is actually a superamalgam: i.e. we prove that K has the superamalgamation property. Assume that $k\circ m(a)\leq h\circ n(b)$. There exists $x\in\mathfrak{Fr}^I$ such that $x/P=k(m(a))$ and $m(a)=x/M$. Also there exists $z\in\mathfrak{Fr}^J$ such that $z/P=h(n(b))$ and $n(b)=z/N$. Now $x/P\leq z/P$ hence $x-z\in P$. Therefore there is an $r\in M$ and an $s\in N$ such that $x-r\leq z+s$. Now $x-r\in\mathfrak{Fr}^I$ and $z+s\in\mathfrak{Fr}^J$, it follows that there is an interpolant $u\in\mathfrak{Fr}^{(I\cap J)}$ such that $x-r\leq u\leq z+s$. Let $t\in C$ such that $m\circ f(t)=u/M$ and $n\circ g(t)=u/N$. We have $x/P\leq u/P\leq z/P$. Now $m(f(t))=u/M\geq x/M=m(a)$. Thus $f(t)\geq a$. Similarly $n(g(t))=u/N\leq z/N=n(b)$, hence $g(t)\leq b$. By total symmetry, we are done.

■

Applications

Theorem 8 . *Let $L\subseteq\mathbf{RK}_\alpha$. Consider the following conditions:*

(1) For all $\mathfrak{A}\in L$ for all non-zero x in A, for all finite $\Gamma\subseteq\alpha$, there exist distinct $i,j\in\alpha\smallsetminus\Gamma$, such that $\mathsf{s}_i^jx\neq 0$

(2) For all $\mathfrak{A}\in L$ for every finite sequence ρ without repeating terms and with range included in α, for every non-zero $x\in A$, there is a function h and $k<\alpha$ such that h is an endomorphism of $\mathfrak{Rd}^\rho\mathfrak{A}$, $k\in\alpha\smallsetminus Rg\rho$, $\mathsf{c}_k\circ h=h$ and $h(x)\neq 0$.

(3) (a) If whenever $\mathfrak{A}\in L$, there exists $x\in{}^{|A|}A$ such that if $\rho=\langle\Delta x_i:i<|A|\rangle$, $\mathfrak{D}=\mathfrak{Fr}^\rho_{|A|}\mathbf{K}_\beta$ and $g_\xi=\xi/Cr^\rho_{|A|}\mathbf{K}_\beta$, then $\mathfrak{Sg}^{\mathfrak{Rd}_\alpha\mathfrak{D}}\{g_\xi:\xi<|A|\}\in L$,

(b) If $\mathfrak{A} \in L$, then for any $\mathfrak{B} \in \mathbf{K}_{\alpha+\omega}$ such that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$ and A generates $\mathfrak{B}$, and for any ideal of I of $\mathfrak{B}$, we have $I \subseteq \mathfrak{Ig}^{\mathfrak{B}}(I \cap A)$,[2]

then in (1) and (2) L has AP with respect to $\mathbf{RK}_\alpha$ over L, and in (3) $\mathbf{RK}_\alpha$ has AP with respect to $\mathbf{RK}_\alpha$ over L.

Proof.

- Note that algebras satisfying (2) are representable, cf. [19] Theorem 2.6.50. It is not hard to show that (1) implies (2). Now assume (2). Then the following condition holds for every $\lambda < \omega$: for every $k < \omega$ for every one to one $\rho \in {}^k\alpha$ and every non-zero $x \in A$, there exist σ, h such that $\sigma \in {}^{k+l}\alpha$, σ is one to one, $\rho \subseteq \sigma$, h is an endomorphism of $\mathfrak{Rd}^\rho_k\mathfrak{A}$, $\mathsf{c}_{\sigma_u} \circ h = h$ whenever $k \leq \mu < k + l$ and $h(x) \neq 0$. This can be proved by induction on λ, cf. [19] p. 416. Let $\mathfrak{A}_0$ and $\mathfrak{A}_1 \in L$. We claim that there exist $\mathfrak{B}_0$, $\mathfrak{B}_1 \in \mathbf{K}_{\alpha+\omega}$ $i_0 : \mathfrak{A}_0 \to \mathfrak{Nr}_\alpha\mathfrak{B}_0$ and $i_1 : \mathfrak{A}_1 \to \mathfrak{Nr}_\alpha\mathfrak{B}_1$ such that for every monomorphism $f : \mathfrak{A}_0 \to \mathfrak{A}_1$ there exists a monomorphism $g : \mathfrak{B}_0 \to \mathfrak{B}_1$ such that $g \circ i_0 = i_1 \circ f$. Let R be the set of all ordered quadruples $\langle \rho, n, k, l\rangle$ such that: $\rho \in {}^m\alpha$ is one to one for some $m \in \omega$, $n \in \omega$, k, l are one to one (finite) sequences with

  $$k, l \in {}^n(\alpha \smallsetminus Rg\rho) \text{ and } Rgk \cap Rgl = \emptyset.$$

  For $\rho \in {}^i\alpha$ $(i \in \omega)$ one to one put

  $$X_{\rho,n} = \{\langle \sigma, m, k, l\rangle \in R : \ \rho \subseteq \sigma \text{ and } n \leq m\}.$$

  It is straightforward to check that the set consisting of all the $X_{\rho,n}$'s is closed under finite intersections. Accordingly, we let M be the proper filter of $\wp(R)$ generated by the $X_{\rho,n}$'s so that

  $$M = \{Y \subseteq R : X_{\rho,n} \subseteq Y \text{ for some } \rho \text{ and } n \in \omega\}.$$

  For each $\langle \rho, n, k, l\rangle \in R$, choose a bijection $t(\langle \rho, n, k, l\rangle)$ from $\alpha + \omega$ onto α such that

  $$t(\langle \rho, n, k, l\rangle) \restriction \Gamma \subseteq Id$$

  and

  $$t(\langle \rho, n, k, l\rangle)(\alpha + j) = k_j, \text{ for each } j < n.$$

  Now fix $i \in \{0, 1\}$. Let

  $$\mathbf{F}(\mathfrak{A}_i) = \prod_{\phi \in R} \mathfrak{Rd}^{t(\phi)}\mathfrak{A}_i/M$$

[2] This implies that the lattice of ideals of $\mathfrak{B}$ is isomorphic to that of $\mathfrak{A}$, via $I \mapsto (I \cap A)$.

Here $\mathfrak{Rd}^{t(\phi)}\mathfrak{A}_i$ - the $t(\phi)$ reduct of $\mathfrak{A}_i$ - is a $\mathbf{K}_{\alpha+\omega}$, and so $\mathbf{F}(\mathfrak{A}_i)$ - an ultraproduct of these - is also a $\mathbf{K}_{\alpha+\omega}$. Note too, that for each $\phi \in R$, the algebra $\mathfrak{Rd}^{t\phi}\mathfrak{A}_i$ has universe A_i. Consider a non-zero $x \in A_i$ and $\langle \rho, n, k, l\rangle \in R$. For each $p < n$ for each l_p and each k_p, choose σ such that σ is one to one $\rho \subseteq \sigma$, and $h_{k_p}^{l_p}$ to be in $\mathfrak{Rd}_k^{\rho}\mathfrak{A}$, such that $\mathsf{c}_{\sigma_u} \circ h_{k_p}^{l_p} = h_{k_p}^{l_p}$ whenever $k_p \leq \mu < k_p + l_p$ and and $h_{k_p}^{l_p}(x) \neq 0$. Let j_i be the function from $\mathfrak{A}_i$ into $\mathbf{F}(\mathfrak{A}_i)$ defined to be 0 at 0 and for $x \neq 0$ by,

$$j_i x = \langle (h_{l_0}^{k_0})^{\mathfrak{A}_i} \circ \ldots (h_{l_{n-1}}^{k_{n-1}})^{\mathfrak{A}_i} x : \langle \rho, n, k, l\rangle \in R\rangle / M.$$

Then $j_i \in Ism(\mathfrak{A}_i, \mathfrak{Nr}_\alpha \mathbf{F}(\mathfrak{A}_i))$. Let g be the function from $\mathbf{F}(\mathfrak{A}_0)$ into $\mathbf{F}(\mathfrak{A}_1)$ defined by:

$$g(\langle x_\phi : \phi \in R\rangle / M) = \langle f x_\phi : \phi \in R\rangle / M.$$

Then it is not hard to check that g is well defined and it is the desired "lifting" function. Now we show that L has AP. Let $\mathfrak{C}, \mathfrak{A}, \mathfrak{B} \in L$. Let $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorhisms. Then there exist $\mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{C}^+ \in \mathbf{K}_{\alpha+\omega}$, embeddings $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{A}^+$, $e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha \mathfrak{B}^+$, $e_C : \mathfrak{C} \to \mathfrak{Nr}_\alpha \mathfrak{C}^+$, $\bar{f} : \mathfrak{C}^+ \to \mathfrak{A}^+$ and $\bar{g} : \mathfrak{C}^+ \to \mathfrak{B}^+$ such that $\bar{f} \circ e_A = e_A \circ f$ and $\bar{g} \circ e_B = e_B \circ g$. We can assume that $\mathfrak{Sg}^{\mathfrak{A}^+} e_A(A) = \mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathfrak{C}^+$. (Here we are assuming that $\bar{f}[\mathfrak{Sg}^{\mathfrak{C}^+} e_C(C)] \subseteq \mathfrak{Sg}^{\mathfrak{A}^+}(e_A(A))$ and that $\bar{g}[\mathfrak{Sg}^{\mathfrak{C}^+} e_C(C)] \subseteq \mathfrak{Sg}^{\mathfrak{B}^+}(e_B(B).)$ Now by Theorem 7 L has $SUPAP$, hence there is a $\mathfrak{D}^+$ in L and $k : \mathfrak{A}^+ \to \mathfrak{D}^+$ and $h : \mathfrak{B}^+ \to \mathfrak{D}^+$ such that $k \circ \bar{f} = h \circ \bar{g}$. Let $\mathfrak{D} = \mathfrak{Nr}_\alpha \mathfrak{D}^+$. Then $k \circ e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{D}$ and $h \circ e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha \mathfrak{D}$ are one to one and $k \circ e_A \circ f = h \circ e_B \circ g$. For (1) h_k^l can be taken to be s_k^l.

- Assume (3). Let $\beta = \alpha + \omega$. We first prove the following condition (**): For $\mathfrak{A}, \mathfrak{A}' \in L$, $\mathfrak{B}, \mathfrak{B}' \in \mathbf{K}_\beta$, $e_A, e_{A'}$ embeddings from $\mathfrak{A}, \mathfrak{A}'$ into $\mathfrak{Nr}_\alpha \mathfrak{B}, \mathfrak{Nr}_\alpha \mathfrak{B}'$ respectively, such that $\mathfrak{Sg}^{\mathfrak{B}}(e_A(A)) = \mathfrak{B}$ and $\mathfrak{Sg}^{\mathfrak{B}'}(e_{A'}(A')) = \mathfrak{B}'$, and $i : \mathfrak{A} \longrightarrow \mathfrak{A}'$ an isomorphism, there exists an isomorphism $\bar{i} : \mathfrak{B} \longrightarrow \mathfrak{B}'$ such that $\bar{i} \circ e_A = e_{A'} \circ i$. Let $\mu = |A|$. Let x be a bijection from μ onto A that satisfies the premise of $(3)(a)$. Let y be a bijection from μ onto A', such that $i(x_j) = y_j$ for all $j < \mu$. Let $\rho = \langle \Delta^{(\mathfrak{A})} x_j : j < \mu\rangle$, $\mathfrak{D} = \mathfrak{Fr}_\mu^{(\rho)} \mathbf{K}_\beta$, $g_\xi = \xi / Cr_\mu^{(\rho)} \mathbf{K}_\beta$ for all $\xi < \mu$ and $\mathfrak{C} = \mathfrak{Sg}^{\mathfrak{Rd}_\alpha \mathfrak{D}}\{g_\xi : \xi < \mu\}$. Then $\mathfrak{C} \subseteq \mathfrak{Nr}_\alpha \mathfrak{D}$, C generates $\mathfrak{D}$ and by hypothesis $\mathfrak{C} \in L$. There exist $f \in Hom(\mathfrak{D}, \mathfrak{B})$ and $f' \in Hom(\mathfrak{D}, \mathfrak{B}')$ such that $f(g_\xi) = e_A(x_\xi)$ and $f'(g_\xi) = e_{A'}(y_\xi)$ for all $\xi < \mu$. Note that f and f' are both onto. We now have $e_A \circ i^{-1} \circ e_{A'}^{-1} \circ (f' \upharpoonright \mathfrak{C}) = f \upharpoonright \mathfrak{C}$. Therefore $Kerf' \cap \mathfrak{C} = Kerf \cap \mathfrak{C}$. Hence by $(3)(b)$ $\mathfrak{Ig}(Kerf' \cap \mathfrak{C}) = \mathfrak{Ig}(Kerf \cap \mathfrak{C})$. So, $Kerf' = Kerf$. Let $y \in B$, then there exists $x \in D$ such that $y = f(x)$. Define $\hat{i}(y) = f'(x)$. The map is well defined and is as required. Let $\mathfrak{C} \in L$. let $\mathfrak{A}, \mathfrak{B} \in \mathbf{RK}_\alpha$.

Let $f:\mathfrak{C}\to\mathfrak{A}$ and $g:\mathfrak{C}\to\mathfrak{B}$ be monomorhisms. Then by the Neat Embedding Theorem, there exist $\mathfrak{A}^+,\mathfrak{B}^+,\mathfrak{C}^+\in\mathbf{K}_{\alpha+\omega}$ and embeddings $e_A:\mathfrak{A}\to\mathfrak{Nr}_\alpha\mathfrak{A}^+$, $e_B:\mathfrak{B}\to\mathfrak{Nr}_\alpha\mathfrak{B}^+$ and $e_C:\mathfrak{C}\to\mathfrak{Nr}_\alpha\mathfrak{C}^+$. We can assume that $\mathfrak{Sg}^{\mathfrak{A}^+}e_A(A)=\mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathfrak{C}^+$. Let $f(C)^+=\mathfrak{Sg}^{\mathfrak{A}^+}e_A(f(C))$ and $g(C)^+=\mathfrak{Sg}^{\mathfrak{B}^+}e_B(g(C))$. Then by the above there exist $\bar{f}:\mathfrak{C}^+\to f(C)^+$ and $\bar{g}:\mathfrak{C}^+\to g(C)^+$ such that $(e_A\restriction f(C))\circ f=\bar{f}\circ e_C$ and $(e_B\restriction g(C))\circ g=\bar{g}\circ e_C$. Now again by Theorem 7 L has $SUPAP$, hence there is a $\mathfrak{D}^+$ in K and $k:\mathfrak{A}^+\to\mathfrak{D}^+$ and $h:\mathfrak{B}^+\to\mathfrak{D}^+$ such that $k\circ\bar{f}=h\circ\bar{g}$. Let $\mathfrak{D}=\mathfrak{Nr}_\alpha\mathfrak{D}^+$. Then $k\circ e_A:\mathfrak{A}\to\mathfrak{Nr}_\alpha\mathfrak{D}$ and $h\circ e_B:\mathfrak{B}\to\mathfrak{Nr}_\alpha\mathfrak{D}$ are one to one and $k\circ e_A\circ f=h\circ e_B\circ g$.

■

Consider the following condition:

(*) For all $\mathfrak{A}\in L$ whenever $\mathfrak{B}\in\mathbf{K}_{\alpha+\omega}$ such that $\mathfrak{A}\subseteq\mathfrak{Nr}_\alpha\mathfrak{B}$ then for all $X\subseteq\mathfrak{A}$, $\mathfrak{Sg}^{\mathfrak{A}}X=\mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{B}}X$.

Note that $\mathfrak{Sg}^{\mathfrak{A}}X=\mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{B}}X$, hence the above condition states that forming subalgebras commutes with taking neat reducts. (More succinctly: The subalgebra of the neat reduct is the same as the neat reduct of the subalgebra).

Theorem 9 .

(1) If L satisfies ((*) and [(1) or (2) in Theorem 8], then L has $SUPAP$ with respect to $\mathbf{RK}_\alpha$ over L.

(2) If L satisfies (*) and (3) in Theorem 8 then $\mathbf{RK}_\alpha$ has $SUPAP$ with respect to $\mathbf{RK}_\alpha$ over L.

Proof. We only prove (2). The proof of (1) is completely analgous. Assume (*) and (3). Repeating the above proof for AP, we have for $\mathfrak{C}\in L,\mathfrak{A},\mathfrak{B}\in\mathbf{RK}_\alpha$ $f:\mathfrak{C}\to\mathfrak{A}$ $g:\mathfrak{C}\to\mathfrak{B}$ monomorphisms, there is a $\mathfrak{D}\in\mathfrak{Nr}_\alpha\mathbf{K}_{\alpha+\omega}$ and $m:\mathfrak{A}\to\mathfrak{D}$ $n:\mathfrak{B}\to\mathfrak{D}$ such that $m\circ f=n\circ g$. Here $m=k\circ e_A$ and $n=h\circ e_B$ with k and h are as above. Denote k by m^+ and h by n^+. Now we further want to show that if $m(a)\leq n(b)$, for $a\in A$ and $b\in B$, then there exists $t\in C$ such that $a\leq f(t)$ and $g(t)\leq b$. So let a and b be as indicated . We have $m^+\circ e_A(a)\leq n^+\circ e_B(b)$, so $m^+(e_A(a))\leq n^+(e_B(b))$. Since L has $SUPAP$, there exist $z\in C^+$ such that $e_A(a)\leq\bar{f}(z)$ and $\bar{g}(z)\leq e_B(b)$. Let $\Gamma=\Delta z\smallsetminus\alpha$ and $z'=\mathsf{c}_{(\Gamma)}z$. (Note that Γ is finite.) So, we obtain that $e_A(\mathsf{c}_{(\Gamma)}a)\leq\bar{f}(\mathsf{c}_{(\Gamma)}z)$ and $\bar{g}(\mathsf{c}_{(\Gamma)}z)\leq e_B(\mathsf{c}_{(\Gamma)}b)$. It follows that $e_A(a)\leq\bar{f}(z')$ and $\bar{g}(z')\leq e_B(b)$. Now by hypothesis

$$z'\in\mathfrak{Nr}_\alpha\mathfrak{C}^+=\mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{C}^+}(e_C(C))=e_C(C).$$

So, there exists $t\in C$ with $z'=e_C(t)$. Then we get $e_A(a)\leq\bar{f}(e_C(t))$ and $\bar{g}(e_C(t))\leq e_B(b)$. It follows that $e_A(a)\leq e_A\circ f(t)$ and $e_B\circ g(t)\leq e_B(b)$. Hence, $a\leq f(t)$ and $g(t)\leq b$. We are done. ■

New consequences of the above Theorems

- The classes $\mathbf{RK}_\alpha$ does not satisfy (*), since these classes satisfy 3(a) in Theorem 8 but fails to have SAP [18] and see below. The **CA** part answers a question of Henkin and Monk posed in the introduction of [20] (p.iv item (5)), since failure of (*) can be paraphrased as: There are generating subreducts that are not neat reducts. Compare with Theorem 2.6.67 in [19]. In [12] direct counterexamples are supplied. It is proved therein that for any $\alpha > 1$ there is a semisimple $\mathfrak{A} \in \mathbf{K}_\alpha$ such that the following hold. For every $\beta > \alpha$, there is a $\mathfrak{B} \in \mathbf{K}_\beta$ such that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, A generates $\mathfrak{B}$ but $\mathfrak{A}$ is not even isomorphic to $\mathfrak{Nr}_\alpha\mathfrak{B}$.

- The classes $\mathbf{RCA}_\alpha$ and $\mathbf{RQEA}_\alpha$ for infinite α, do not satisfy (3)(b) in Theorem 8, since these classes satisfy (3)(a) but fail to have AP. Failure of AP for $\mathbf{RCA}_\alpha$ is proved in [25], whereas failure of AP for $\mathbf{RQEA}_\alpha$ is proved below. The **CA** part confirms an unsettled conjecture of Tarski in [19]. cf. op cit top of page 426. That is $\mathbf{Dc}_\alpha$ cannot be replaced in Theorem 2.6.71 of [19] by $\mathbf{RCA}_\alpha$ for infinite α.

- In [16] it is shown that $\mathbf{DKc}_\alpha$'s and monadic generated $\mathbf{K}$'s satisfy (3) in Theorem 8 and (*). In particular, minimal algebras satisifies (3) and (*). It is not trivial to show that **CA**'s and **QEA**'s of positive characteristic $\kappa > 0$ satisfy (1) in Theorem 8 and (*), cf. [19] Theorem 2.6.54. It thus follows from Theorem 9 that **CA**'s and **QEA**'s of positive characteristic $\kappa > 0$, and monadic generated $\mathbf{K}_\alpha$'s have $SUPAP$. (This answers questions of Pigozzi for the **CA** case, cf. [25] p. 336, since $SUPAP$ implies SAP.) It is true that by the above the super amalgam is only representable, but if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}$ is of positive characteristic then so is $\mathfrak{B}$. For monadic generated algebras one takes the subalgebra of the super amalgam generated by the images of the two algebras that lie over the base algebra. This gives a super amalgam that is monadic generated.

- Let $\mathbf{SsK}_\alpha$ denote the class of semisimple algebras in $\mathbf{K}_\alpha$, $\mathbf{ReK}_\alpha$ be the class of algebras satisfying (1) in Theorem 8 and $\mathbf{L}$ be the class of algebras satisfying (2) in Theorem 8. Then $\mathbf{SsK}_\alpha \subseteq \mathbf{ReK}_\alpha \subseteq \mathbf{L} \subseteq \mathbf{RK}_\alpha$. For the first inclusion for **CA**'s cf. [19] Theorem 2.6.50. Generally, let $\mathfrak{A}$ be semisimple. Let x be non zero and $\Gamma \subseteq \alpha$ be finite. Then there is a maximal ideal I of $\mathfrak{A}$ such that $x \notin I$, and so there is a finite subset Δ of α for which $-\mathsf{c}_{(\Delta)}x \in I$. Choose $k, l \in \alpha \sim (\Gamma \cup \Delta)$. Assume that $\mathsf{s}_k^l x = 0$. Then $\mathsf{c}_{(\Delta)}\mathsf{s}_k^l x = \mathsf{s}_k^l \mathsf{c}_{(\Delta)}x = 0$. Hence $-\mathsf{s}_k^l \mathsf{c}_{(\Delta)}x = 1$. But then $\mathsf{s}_k^l - \mathsf{c}_{(\Delta)}x = 1$ so $1 \in I$ which is impossible since I is a proper ideal. $\mathbf{ReK}_\alpha \subseteq \mathbf{L}$ is proved in [19], and so is the last inclusion. The latter follows from the neat embedding Theorem, namely $S\mathfrak{Nr}_\alpha\mathbf{K}_{\alpha+\omega} = \mathbf{RK}_\alpha$.

Now the incusions

$$\mathbf{SsK}_\alpha \subseteq \mathbf{ReK}_\alpha \subseteq \mathbf{L}$$

are proper. To see this let $\mathfrak{A}$ be the full set algebra in the space ${}^\alpha 2$. Then clearly $\mathfrak{A} \in \mathbf{ReK}_\alpha$. Also $\mathfrak{A}$ is not semisimple. Indeed let $X = \{\langle 0 : k < \alpha\rangle\}$. Then X belongs to every maximal ideal of $\mathfrak{A}$. For if $X \notin I$, then there is a finite subset Γ of α such that $\sim \mathsf{c}_{(\Gamma)}X \in I$. Choose $k \in \alpha \sim \Gamma$ and let $\phi = \langle 0 : \mu \in \alpha \sim \{k\}\rangle \cup \langle k, 1\rangle$. Then $\phi \in \sim \mathsf{c}_{(\Gamma)}X$, so $\{\phi\} \in I$. But $X \subseteq \mathsf{c}_k\{\phi\}$, so $X \in I$ which is impossible. Now $\mathbf{ReK}_\alpha \subset \mathbf{L}$. The following example is taken from [19] and adapted to the cases considered herein. If we take $\mathfrak{A}$ to be the full set algebra in the space ${}^\alpha\alpha$, then $\mathsf{s}_k^l(Id \restriction \alpha) = 0$ for every $k, l < \alpha$. Suppose that ρ is a finite one to one sequence with $Rg\rho \subseteq \alpha$ and $X \subseteq {}^\alpha\alpha$, $X \neq 0$. Let $k \in \alpha \setminus Rg\rho$ and choose $\tau \in {}^\alpha\alpha$ such that $k \notin Rg\rho$, $\tau \restriction Rg\rho \subseteq Id$ and τ is one to one. Let

$$h(Y) = \{\phi \in {}^\alpha\alpha : \phi \circ \tau \in Y\}.$$

Then h satisfies the conclusion of (2) in Theorem 8. The class $\mathbf{L}$ does not coincide with the class of representable algebras in the case of $\mathbf{K} \in \{\mathbf{CA}, \mathbf{QEA}\}$ since it has AP with respect to $\mathbf{RK}_\alpha$, while $\mathbf{RK}_\alpha$ fails to have AP. This answers a question of Henkin Monk and Tarski [19] p.417, formulated as problem 2.13. The latter is one of the very few questions that are open in [19], possibly the only one.

- In [11] it is proved using ideas similar to the ones adopted herein, that there exists a semisimple algebra $\mathfrak{A} \in \mathbf{K}_\alpha$ such that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}'$, A generates both $\mathfrak{B}$ and $\mathfrak{B}'$, but $\mathfrak{B}$ and $\mathfrak{B}'$ are not isomorphic via an isomorphism that coincides with the identity map on $\mathfrak{A}$. This confirms an unsettled conjecture of Tarski in [20]. In particular in Theorem 2.6.72 of [19] $\mathbf{Dc}_\alpha$ cannot be replaced by $\mathbf{RCA}_\alpha$ for infinite α. If it could, then it would follow that $\mathbf{RCA}_\alpha$ has AP, which is not the case. To see this let $\mathfrak{C}$, $\mathfrak{A}, \mathfrak{B} \in \mathbf{RCA}_\alpha$. Let $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorphisms. Then by the Neat Embedding Theorem, there exist $\mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{C}^+ \in \mathbf{CA}_{\alpha+\omega}$ and embeddings $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha\mathfrak{A}^+$ $e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha\mathfrak{B}^+$ and $e_C : \mathfrak{C} \to \mathfrak{Nr}_\alpha\mathfrak{C}^+$. We can assume that $\mathfrak{Sg}^{\mathfrak{A}^+}e_A(A) = \mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathfrak{C}^+$. Let $f(C)^+ = \mathfrak{Sg}^{\mathfrak{A}^+}e_A(f(C))$ and $g(C)^+ = \mathfrak{Sg}^{\mathfrak{B}^+}e_B(g(C))$. Then by hypothesis there exist $\bar{f} : \mathfrak{C}^+ \to f(C)^+$ and $\bar{g} : \mathfrak{C}^+ \to g(C)^+$ such that $(e_A \restriction f(C)) \circ f = \bar{f} \circ e_C$ and $(e_B \restriction g(C)) \circ g = \bar{g} \circ e_C$. Then proceed as in the proof of Theorem 9 to show that $\mathbf{RCA}_\alpha$ has AP. In [11], it is shown that there is a semisimple $\mathfrak{C}$, $\mathfrak{A}, \mathfrak{B} \in \mathbf{RCA}_\alpha$ and $f : \mathfrak{C} \to A$ and $g : \mathfrak{C} \to \mathfrak{B}$ for which there is no amalgam. We conjecture that $\mathfrak{C}$ can be chosen to be simple.

- We highlight the property (**) in the proof of theorem 8.

Definition . *Let $\mathfrak{A} \in \mathbf{RK}_\alpha$. Then $\mathfrak{A}$ has the $UNEP$ (short for unique neat embedding property) if for all $\mathfrak{A}' \in \mathbf{K}_\alpha$, $\mathfrak{B}$, $\mathfrak{B}' \in \mathbf{K}_{\alpha+\omega}$, isomorphism $i : \mathfrak{A} \to \mathfrak{A}'$, embeddings $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha\mathfrak{B}$ and $e_{A'} : \mathfrak{A}' \to \mathfrak{Nr}_\alpha\mathfrak{B}'$ such that $\mathfrak{Sg}^{\mathfrak{B}} e_A(A) = \mathfrak{B}$ and $\mathfrak{Sg}^{\mathfrak{B}'} e_{A'}(A)' = \mathfrak{B}'$, there exists an isomorphism $\bar{i} : \mathfrak{B} \to \mathfrak{B}'$ such that $\bar{i} \circ e_A = e_{A'} \circ i$.*

And we highlight the property in (*):

Definition . *Let $\mathfrak{A} \in \mathbf{RK}_\alpha$. Then $\mathfrak{A}$ has the NS property (short for neat reducts commuting with forming subalgebras) if for all $\mathfrak{B} \in \mathbf{K}_{\alpha+\omega}$ if $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$ then for all $X \subseteq A$, $\mathfrak{Sg}^{\mathfrak{A}} X = \mathfrak{Nr}_\alpha \mathfrak{Sg}^{\mathfrak{B}} X$.*

Let us examine closely the above conditions. At first glance the Definition of $UNEP$ might seem complicated, but in fact it is a slight generalization of a very simple and indeed "natural" property. Let $\mathfrak{A} \in \mathbf{K}_\alpha$. Let $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$ with $\mathfrak{B} \in \mathbf{K}_{\alpha+\omega}$. Call $\mathfrak{B}$ an ω dilation of $\mathfrak{A}$. If further A generates $\mathfrak{B}$ (using the $\alpha + \omega$ operations of $\mathfrak{B}$) call $\mathfrak{B}$ a *minimal* ω dilation of $\mathfrak{A}$. In this case, one might expect that $\mathfrak{A}$ has some controlle over $\mathfrak{B}$. In fact, Definition 1 implies that any two minimal ω dilations of $\mathfrak{A}$ are in fact *isomorphic*. Furthermore this isomorphism can be chosen to fix A. This follows from the special case when $\mathfrak{A} = \mathfrak{A}'$, and i and $e_A = e_{A'}$ are the inclusion maps. So, the $UNEP$ says that $\mathfrak{A}$ determines essentially the structure of its minimal ω dilations.

Now for Definition of NS. Again let $\mathfrak{B}$ be an ω dilation of $\mathfrak{A} \in \mathbf{K}_\alpha$ so that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B} \in \mathbf{K}_{\alpha+\omega}$. Let $X \subseteq A$. Form the subalgebra of $\mathfrak{A}$ generated by X and form the subalgebra of $\mathfrak{B}$ generated by X. Then, in principal, in the second process of generation, new α dimensional elements can be generated. Definition 2 excludes this possibility. It says that if we take the set of α dimensional elements of $\mathfrak{Sg}^{\mathfrak{B}} X$ (i.e we form $\mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{B}} X$), then we come back exactly to where we started namely to $\mathfrak{Sg}^{\mathfrak{A}} X$ (and not to a bigger algebra). No new α dimensional elements are generated (even in the presence of ω extra dimensions). Note that in this case, we have

$$\mathfrak{Sg}^{\mathfrak{A}} X = \mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{B}} X = \mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{B}} X.$$

Let K be a class of algebras having a boolean reduct. $\mathfrak{A}_0 \in K$ is in the amalgamation base of K if for all $\mathfrak{A}_1, \mathfrak{A}_2 \in K$ and monomorphisms $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$, $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ there exist $\mathfrak{D} \in K$ and monomorphisms $m_1 : \mathfrak{A}_1 \to \mathfrak{D}$ and $m_2 : \mathfrak{A}_2 \to \mathfrak{D}$ such that $m_1 \circ i_1 = m_2 \circ i_2$. If in addition, $(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y))$ where $\{j, k\} = \{1, 2\}$, then we say that $\mathfrak{A}_0$ lies in the super amalgamation base of K. Observe that K has the (super) amalgamation property $((SUP)AP)$, if the (super) amalgamation base of K coincides with K.

Let $APbase(K)$ be the class of algebras that lie in the amalgamation base of K and $SUPAPbase(K)$ the class of algebras that lie in the super amalgam base of K. One can easily destill from the proof of Theorem 8, the following:

Theorem . *Let α be an ordinal. Let $\mathfrak{A} \in \mathbf{RK}_\alpha$.*

(i) If $\mathfrak{A}$ has $UNEP$, then $\mathfrak{A} \in APbase(\mathbf{RK}_\alpha)$.

(ii) If $\mathfrak{A}$ has $UNEP$ and NS, then $\mathfrak{A} \in SUPAPbase(\mathbf{RK}_\alpha)$.

The above theorem answers a question of Monk, cf [1] p. 739. Note that for infinite α, $\mathbf{DKc}_\alpha \subseteq SUPAPbase(\mathbf{RK}_\alpha)$. And for $\mathbf{K} \in \{\mathbf{CA}, \mathbf{QEA}\}$ if PK_α denotes the algebras of positive characteristic, of dimension α, then for $\alpha > 1$, we have

$$PK_\alpha \subseteq SUPAPbase(\mathbf{K}_\alpha) \cap \mathbf{RK}_\alpha \subseteq SUPAPbase(\mathbf{RK}_\alpha).$$

In the appendix we show that $SUPAPbase(\mathbf{RCA}_\omega) \nsubseteq APbase(\mathbf{CA}_\omega)$ answering another question of Monk.

- We recall different forms of the interpolation property from [25]. We write $A^{(X)}$ for $Sg^{\mathfrak{A}}X$, i.e the universe of the subalgebra of $\mathfrak{A}$ generated by X.

Definition . *Let $\mathfrak{A} \in \mathbf{K}_\alpha$ and $X \subseteq A$ such that $A^{(X)} = A$.*

(1) $\mathfrak{A}$ has the strong interpolation property, SIP for short with respect to X, or simply SIP, if for any $Y, Z \subseteq X$:
(*) Under the hypothesis that $x \in A^{(Y)}, z \in A^{(Z)}$ and $x \leq z$, then we can always find a $y \in A^{(Y \cap Z)}$ such that

$$x \leq y \leq z.$$

(2) $\mathfrak{A}$ has the strong restricted IP, if (*) is satisfied for any pair of disjoint *subsets Y, Z of X.*

(3) $\mathfrak{A}$ has the interpolation property, IP for short, if for any Y, $Z \subseteq X$:
(**) Under the hypothesis $x \in A^{(Y)}, z \in A^{(Z)}$ and $x \leq z$, then we can always find a $y \in A^{(Y \cap Z)}$ and a finite $\Gamma \subseteq \alpha$ such that

$$x \leq y \leq \mathsf{c}_{(\Gamma)} z.$$

(4) $\mathfrak{A}$ *has the restricted* IP, *if (**) is satisfied for any pair of disjoint subsets* Y, Z *of* X.

Now $\mathfrak{Fr}_\omega\mathbf{RCA}_\alpha$ has the strong restricted interpolation property. Indeed, let $\mathfrak{A} = \mathfrak{Fr}_\beta\mathbf{RCA}_\omega$ and let $X_1, X_2 \subseteq \beta$ be disjoint sets. We can asssume without loss of generality that $X_1 \cup X_2 = \beta$. Assume that $a \in \mathfrak{A}_1 = \mathfrak{Sg}^{\mathfrak{A}}X_1$ and $b \in \mathfrak{A}_2 = \mathfrak{Sg}^{\mathfrak{A}}X_2$ such that $a \leq b$. We want to find an interpolant. Since $X_1 \cap X_2 = \emptyset$, we have $\mathfrak{A}_0 = \mathfrak{Sg}^{\mathfrak{A}}(X_1 \cap X_2) \in \mathbf{Mn}_\alpha$ and it embeds in $\mathfrak{Sg}^{\mathfrak{A}}X_1$ and $\mathfrak{Sg}^{\mathfrak{A}}X_2$, respectively via the inclusion maps i_0 and i_1 say. Since $\mathbf{Mn}_\alpha \subseteq \mathbf{Dc}_\alpha$ it follows fromthe above that there is a $\mathfrak{D} \in \mathbf{RCA}_\alpha$, a monomorphism m_1 from $\mathfrak{A}_1$ into $\mathfrak{D}$ and a monomorphism m_2 from $\mathfrak{A}_2$ into $\mathfrak{D}$ such that $m_1 \circ i_1 = m_2 \circ i_2$, and $(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \implies (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y))$ where $\{j, k\} = \{1, 2\}$. Now since $\mathfrak{A}$ is free, there exists a homomorphism $f : \mathfrak{A} \to \mathfrak{D}$ such that $f \upharpoonright \mathfrak{A}_1 = m_1$ and $f \upharpoonright \mathfrak{A}_2 = m_2$. Since $f(a) \leq f(b)$ it follows that $m_1(a) \leq m_2(b)$. Hence there exists $z \in \mathfrak{A}_0$ such that $a \leq z \leq b$. Notice that $i_1 z = i_2 z = z$. This answers a question of Pigozzi in [25]. The **SC QA** and **QEA** cases are analgous. (Note that for **SC** and **QA** the class of minimal algebras coincide with the class consisting of the two element discrete algebra $\{0, 1\}$).

- Recently we showed [16] that unlike the **CA** and **QEA** cases $\mathbf{RQA}_\alpha$ and $\mathbf{RSC}_\alpha$ has AP. In fact such algebras satisfy (**) in the proof of Theorem 8, i.e such algebras have the $UNEP$. However such algebras do not satisfy (*), nor 3(b) in Theorem 8. This gives that for $V \in \{\mathbf{RSC}_\alpha, \mathbf{RQEA}_\alpha\}$, $\mathfrak{Fr}_\omega V$ has the interpolation property but not the strong interpolation property. The latter follows from the fact that these classes lack strong amalgamation [18] and see below. (See also [25] Remark 2.1.17). We note that (*) implies (3b). To see this assume (*). Let $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$ and A generates $\mathfrak{B}$. Let I be an ideal in $\mathfrak{B}$. Then $\Delta x \sim \alpha$ is finite for evey $x \in B$. We have $A = Nr_\alpha B$ and so $\mathsf{c}_{(\Delta x \sim \alpha)}x \in A$, hence is in $I \cap A$. Since $x \leq \mathsf{c}_{(\Delta x \sim \alpha)}x$, we get that $x \in \mathfrak{Ig}^{\mathfrak{B}}(I \cap A)$.
- The following table was published in [18] where the the classes of algebras in the left column and notions in the top row are defined. In the following $K \in \{\mathbf{SC}, \mathbf{CA}, \mathbf{QA}, \mathbf{QEA}\}$.

Table 1:

	strong AP	strong AP w.r.t K_ω	AP	AP w.r.t RK_ω	AP for simple algebras	strong EP	EP	EP for simple algebras	SUP AP	ES
LfK_ω	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
DcK_ω	no	yes	no	yes	no	no	no	no	no	yes
ScK_ω	no	no	?	?	?	yes	yes	yes	no	no
ReK_ω	no	no	?	?	?	yes	yes	yes	no	no
RK_ω	no	no	?	?	?	yes	yes	yes	no	no
SA_ω	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
PA_ω	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes

Let us attack the question marks:

- It is not hard to show that simple and semisimple algebras have AP. Let us first consider simple algebras. We have a representable amalgam $\mathfrak{C}$. The following ensures that $\mathfrak{C}$ can be chosen to be simple. Let $\mathfrak{A} \subseteq \mathfrak{C}$. Suppose that $\mathfrak{A}$ is simple, and let I be an ideal of $\mathfrak{C}$. Then $\mathfrak{A}$ can be embedded in $\mathfrak{C}/I$. Indeed, $I \cap A$ is an ideal of $\mathfrak{A}$, and is proper since $1 \notin I$. Hence $I \cap A = \{0\}$ because $\mathfrak{A}$ is simple. But then $\mathfrak{A} \cong \mathfrak{A}/(I \cap A)$ and the latter embeds in $\mathfrak{C}/I$. From the above we can obtain that the class of semisimple algebras has AP. To see this assume that $\mathfrak{C}$, $\mathfrak{A}$ and $\mathfrak{B}$ are semisimple and let $h : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be one to one homomorphisms. Then there exist a system $\langle \mathfrak{C}_l : l \in I \rangle$ of simple algebras and $k : \mathfrak{A} \to \prod_{l \in I} \mathfrak{C}_l$ such that $\pi_l \circ k(\mathfrak{A}) = \mathfrak{C}_l$ for each l where π_l is the projection map. Now fix $l \in I$. let

$$t_l = \pi_l \circ k \circ h \circ g^{-1}.$$

Then $t_l : g(\mathfrak{C}) \to t_l(g(C))$ is an onto homomorphism and $i_{g(C)} : g(C) \to B$. There exist simple $\mathfrak{D}_l$, $j_l : \mathfrak{B} \to \mathfrak{D}_l$ onto and $i_l : t_l(g(C)) \to \mathfrak{D}_l$ one to one such that

$$j_l \circ i_{g(C)} = i_l \circ t_l.$$

Indeed, let M be a maximal ideal of $g(\mathfrak{C})$ such that $g(\mathfrak{C})/M \cong t_l(g(\mathfrak{C}))$ via $\bar{t}_l(a) = t_l(a)/M$. Then $i_{g(\mathfrak{C})}(M)$ is an ideal in $\mathfrak{B}$. Let N be a maximal ideal of $\mathfrak{B}$ such that $N \cap i_{g(\mathfrak{C})}(g(\mathfrak{C})) = i_{g(\mathfrak{C})}(M)$. Let $\mathfrak{D}_l = \mathfrak{B}/N$ and define $j_l(b) = b/N$. Then $j_l : \mathfrak{B} \to \mathfrak{D}_l$. Let $a \in t_l(g(\mathfrak{C}))$. Choose $x \in t_l^{-1}(a)$ and define $i_l(a) = j_l \circ i_{g(\mathfrak{C})}(x)$. Then i_l is well defined; one to one and is as required. Now $\mathfrak{C}_l$ and $\mathfrak{D}_l$ amalgamate over $t_l(g(\mathfrak{C}))$, hence there exist $\mathfrak{F}_l$ simple and m_l, n_l such that $n_l \circ i_l = m_l$. Define

$$m(a) = \langle m_l \circ \pi_l \circ k(a) : l \in I \rangle$$

and

$$n(b) = \langle n_l \circ j_l(b) : l \in I \rangle$$

Then for $l \in I$ we have

$$(m(h(c))_l = m_l \circ \pi_l \circ k \circ h(c) = m_l \circ \pi_l \circ k \circ h \circ g^{-1} \circ g(c)$$

$$= m_l \circ t_l \circ g(c) = n_l \circ i_l \circ t_l \circ g(c)$$

$$= n_l \circ j_l \circ g(c) = n(g(c))_l$$

Therefore $m \circ h = n \circ g$.

- Now $\mathbf{ReK}_\omega$ also has AP. To see this first one finds an amalgam $\mathfrak{D} \in \mathbf{RK}_\alpha$, then one uses the technique in Theorem 2.2.20 on [25] to form an appropriate quotient that is in $\mathbf{ReK}_\alpha$. In more detail, let $\mathfrak{C}, \mathfrak{A}, \mathfrak{B} \in \mathbf{ReK}_\alpha$ and $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorphisms. Then there exist $\mathfrak{D} \in \mathbf{RSC}_\alpha$, $m : \mathfrak{A} \to \mathfrak{D}$ and $n : \mathfrak{B} \to \mathfrak{D}$ such that $m \circ f = n \circ g$. Define a transfinite sequence of ideals M_i where i ranges over arbitrary ordinals. $M_0 = \{0\}$ and for each $i > 0$ M_i is the set of all elements $x \in D$ such that for some finite $\Gamma \subseteq \alpha$ and all distinct $k, l < \alpha \sim \Gamma$, $\mathsf{s}_k^l x \in \bigcup_{j<i} M_j$. Then M_i is an ideal in $\mathfrak{D}$ and $M_i \cap m(A) = M_i \cap n(B) = \{0\}$ for every i. Since $M_i \subseteq M_j$ whenever $i < j$ it follows that $M_i = M_{i+1}$ for sufficiently large i, if β is the least such ordinal with this property, then $\mathfrak{C}/M_\beta$ is in $\mathbf{ReK}_\alpha$ and is the required amalgam.

- One can use that dimension restricted free algebras have the strong interpolation property to show that (semi)-simple algebras have AP as follows. The argument used is simple, yet subtle and it does clarify matters. Now c_i represents the existential quantifier $\exists v_i$. Set $\mathsf{c}_i^\partial x = -\mathsf{c}_i - x$. Then c_i^∂ represents the universal quantifier $\forall x_i$. If $\Gamma = \{i_0, \ldots i_{n-1}\}$, Then $\mathsf{c}_{(\Gamma)}^\partial$ denotes $\mathsf{c}_{i_0}^\partial \ldots \mathsf{c}_{i_{n-1}}^\partial$. For a term τ, $var(\tau)$ denotes the set of variables in τ. Assume that K is a class of $\mathbf{K}$ algebras, containing $\mathbf{RK}_\alpha$ such that $\sigma \le \tau$ is an inclusion which holds in every $\mathfrak{A} \in K$. Then we show that there exists a term π with $var(\pi) \subseteq var(\sigma) \cap var(\tau)$ and

$$\mathsf{c}_{(\Delta)}^{(\partial)}\sigma \le \pi \le \mathsf{c}_{(\Delta)}\tau.$$

Let $\mathbf{L}_\alpha$ be the language of $\mathbf{K}_\alpha$ and for $\eta \le \omega$, let $\mathbf{L}_\alpha^{(\eta)}$ or simply $\mathbf{L}^{(\eta)}$ be the language of $\mathbf{K}_{\alpha+\eta}$. We write $\mathbf{L}$ for $\mathbf{L}^{(0)}$. We can assume that $\mathbf{L}^{(\eta)} \subseteq \mathbf{L}^{(\chi)}$ for $\eta \le \chi \le \omega$. Now since $\mathfrak{Nr}_\alpha \mathbf{K}_{\alpha+\omega} \subseteq \mathbf{RK}_\alpha \subseteq K$, we see that for every $\mathfrak{B} \in \mathbf{K}_{\alpha+\omega}$ the inclusion $\tau \le \sigma$ is satisfied in $\mathfrak{B}$ by every assignment of variables to elements of $\mathfrak{B}$- if we restrict ourselves to those assignments which take as values only elements whose dimensions sets are included in α. That is $\sigma \le \tau$ is satisfied by every assignment whose range is included in $\mathfrak{Nr}_\alpha\mathfrak{B}$. From the fact that dimension restricted free algebras have the interpolation property, we conclude that there is a term π of $\mathbf{L}^{(\omega)}$ which contains only occurrences of variables which occur in both σ and τ and which satisfies the following condition for every $\mathfrak{B} \in \mathbf{K}_{\alpha+\omega}$.

$$\text{The two inclusions } \sigma \leq \pi \text{ and } \pi \leq \tau \text{ are satisfied in } \mathfrak{B} \text{ by every assignment whose range is included in } \mathfrak{Nr}_\alpha \mathfrak{B}. \tag{5}$$

Now π may contain occurrences of cylindrification symbols, substitutions and diagonal symbols which have as indices ordinals$\geq \alpha$, but this obviously can happen for only a finite number of such ordinals. (π cannot in general be chosen so that the indices of all cylindrifications, substitutions and diagonal symbols occurring in it are $< \alpha$ for **CA** and **QEA**, however for **SC** and **QA** it can!) For each $\xi \leq \omega$ set $\Sigma^{(\xi)}$ equal to the set of all instances of the axiom schemas which define $\mathbf{K}_{\alpha+\xi}$'s. Then

$$\Sigma^{(\xi)} \subseteq \Sigma^{(\eta)} \text{ whenever } \xi \leq \eta \leq \omega \tag{6}$$

and

$$\Sigma^{(\omega)} = \bigcup_{k<\omega} \Sigma^{(k)}. \tag{7}$$

We now extend each of the language $\mathbf{L}^{(\xi)}$, $\xi < \omega$, by adjoining to its set of non-logical constants all the terms of a fixed ω-terms sequence $a = \langle a_0, a_1, a_2, \ldots \rangle$ of distinct individual constants symbols. Let σ', τ' and π' be the terms of the language extending $\mathbf{L}^{(\omega)}$ that are obtained respectively from σ, τ and π by replacing each variable v_k in all of its occurrences by the corresponding constant symbol a_k. For each $k < \omega$ let $\Pi^{(k)}$ be the set of all identities of the form

$$c_\mu a_v = a_v$$

where $\alpha \leq \mu < \alpha + k$ and $v < \omega$; of course we have

$$\Pi^{(k)} \subseteq \Pi^{(\lambda)} \text{ whenever } k \leq \lambda < \omega \tag{8}$$

To say that (1) holds for every $\mathfrak{B} \in \mathbf{K}_{\alpha+\omega}$ means exactly that the double inclusion $\sigma' \leq \pi' \leq \tau'$ is a consequence of $\Sigma^{(\omega)} \cup \bigcup_{k<\omega} \Pi^{(k)}$, i.e., that

$$\Sigma^{(\omega)} \cup \bigcup_{k<\omega} \Pi^{(k)} \models (\sigma' \leq \pi') \wedge (\pi' \leq \tau')$$

Therefore, by the compactness theorem of first order logic there is a finite subset θ of $\Sigma^{(\omega)} \cup \bigcup_{k<\omega} \Pi^{(k)}$ such that

$$\theta \models (\sigma' \leq \pi') \wedge (\pi' \leq \tau'). \tag{9}$$

In view of (2)- (4) we see that that there is an ordinal $\delta < \omega$ such that $\theta \subseteq \Sigma^{(\delta)} \bigcup \Pi^{(\delta)}$. Hence if we assume (as we obviously may in light of (3) and (4)) that the indices of all cylindrifications, substitutions and diagonal symbols occurring in π are ordinals $< \alpha + \delta$, we can conclude from (5) that condition (1) holds forevery $\mathfrak{B} \in \mathbf{K}_{\alpha+\omega}$. Choose two distinct sets $\Gamma, \triangle \subseteq \alpha$ such that $|\Gamma| = |\triangle| = \delta$ and such that neither Γ nor $\triangle$ contains an ordinal which appears as the index of a cylindricalification or a substitution or a diagonal symbol occurring in σ, τ, or π (recall that $\omega \leq \alpha$). Let μ, v be two sequences of length δ which enumerate the elements of $\triangle$, respectively (and hence are necessarily one-one).

Let $\bar{\sigma}$ and $\bar{\tau}$ be the terms of $\mathbf{L}$ that are obtained from σ and τ, respectively, by replacing each variable v_k in all its occurrences by the term

$$\mathsf{s}_{v_0}^{\mu_0}\mathsf{s}_{v_1}^{\mu_1}\ldots\mathsf{s}_{v_{\delta-1}}^{\mu_{\delta-1}}v_k.$$

Let $\bar{\pi}$ be obtained from π by making these same replacements in addition to the following ones : every index λ of a cylindrification or diagonal symbol occurring in πsuch that $\lambda \in (\alpha+\delta) \sim \alpha$ is changed to an ordinal in Γ; the only restriction on how these changes are made is that different ordinals are to be substituted for different ordinals. Notice that $\bar{\pi}$, as well as $\bar{\sigma}$ and $\bar{\tau}$, is a term of $\mathbf{L}$. Finally, let ρ be any one-one function from $\alpha+\delta$ onto α such that $\delta\xi = \xi$ for every $\xi < \alpha$ which appears as an index of a cylindrification , substitution, or diagonal symbol occurring in σ, τ, or π and such that

$$\rho(k+k) = \mu_k \text{ for every } k < \delta.$$

Then for every $\mathfrak{C} \in \mathbf{K}_\alpha$ we have,

$$\mathfrak{Rd}^{(\rho)}\mathfrak{C} \in \mathbf{K}_{\alpha+\delta} \tag{10}$$

and

$$\mathsf{s}_{v_0}^{\mu_0}\mathsf{s}_{v_1}^{\mu_1}....\mathsf{s}_{v_{\delta-1}}^{\mu_{\delta-1}}x \in \mathfrak{Nr}_\alpha\mathfrak{Rd}^{(\rho)}\mathfrak{C} \text{ for every } x \in C \tag{11}$$

We have been seen that (1) holds for every $\mathfrak{B} \in \mathbf{K}_{\alpha+\delta}$. Hence, since (6) and (7) hold for every $\mathbf{K}_\alpha\mathfrak{C}$, we conclude that the inclusions $\bar{\sigma} \leq \bar{\pi}$ and $\bar{\pi} \leq \bar{\tau}$ are identically satisfied in every $\mathbf{K}_\alpha$. On the other hand, since neither Γ nor $\triangle$ contains an ordinal which is the index of a cylindrification, substitution or diagonal symbol occurring in σ, τ, or π we readily see that the equations

$$\bar{\sigma} = \mathsf{s}_{v_0}^{\mu_0}\mathsf{s}_{v_1}^{\mu_1}....\mathsf{s}_{v_{\delta-1}}^{\mu_{\delta-1}}\sigma$$

and

$$\bar{\tau} = \mathsf{s}^{\mu_0}_{v_0}\mathsf{s}^{\mu_1}_{v_1}....\mathsf{s}^{\mu_{\delta-1}}_{v_{\delta-1}}\tau$$

are both identically satisfied in every $\mathbf{K}_\alpha$. Combining these results we get that

$$\mathsf{s}^{\mu_0}_{v_0}\mathsf{s}^{\mu_1}_{v_1}....\mathsf{s}^{\mu_{\delta-1}}_{v_{\delta-1}}\sigma \leq \bar{\pi} \leq \mathsf{s}^{\mu_0}_{v_0}\mathsf{s}^{\mu_1}_{v_1}....\mathsf{s}^{\mu_{\delta-1}}_{v_{\delta-1}}\tau$$

is identically satisfied in every $\mathbf{K}_\alpha$ and hence, in particular, in every member of K; we easily show that the same is true of

$$\mathsf{c}^{\partial}_{\mu_0}\mathsf{c}^{\partial}_{\mu_1}\ldots\mathsf{c}^{\partial}_{\mu_{\delta-1}}\sigma \leq \bar{\pi} \leq \mathsf{c}_{\mu_0}c_{\mu_1}\ldots\mathsf{c}_{\mu_{\delta-1}}\tau.$$

Therefor, since $\bar{\pi}$ is a term of $\mathbf{L}$ and it contains like π only occurrences of variables which occur at the sametime in both σ and τ we have shown that the inclusion $\sigma \leq \tau$ can indeed be interpolated relative to$K$. It follows therefore that the free $\mathbf{RK}_\alpha$ algebras have the weak interpolation property. We need to make the definition explicit at this point of our investigation. $\mathfrak{A} = \mathfrak{Sg}^A X$ has the weak interpolation property if for any Y, $Z \subseteq X$: Under the hypothesis that $x \in \mathfrak{Sg}^{\mathfrak{A}}Y, z \in \mathfrak{Sg}^{\mathfrak{A}}Z$ and $x \leq z$, then we can always find a $y \in \mathfrak{Sg}^{\mathfrak{A}}(Y \cap Z)$ and a finite $\Gamma \subseteq \alpha$ such that

$$\mathsf{c}^{\partial}_{(\Gamma)}x \leq y \leq \mathsf{c}_{(\Gamma)}z.$$

Now let $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ be in K and $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorphisms. We want to find a simple amalgam. Let $\langle a_i : i \in I\rangle$ be an enumeration of $\mathfrak{A}$ and $\langle b_i : i \in J\rangle$ be an enumeration of $\mathfrak{B}$ such that $\langle \mathsf{c}_i : i \in I \cap J\rangle$ is an enumeration of $\mathfrak{C}$ with $f(c_i) = a_i$ and $g(c_i) = b_i$ for all $i \in I \cap J$. Let $X = I \cup J$ and $\mathfrak{Fr} = \mathfrak{Fr}_X$. Let $\mathfrak{Fr}^I$ be the subalgebra of $\mathfrak{Fr}$ generated by I and let $\mathfrak{Fr}^J$ be the subalgebra generated by J. There exists a homomorphism from $\mathfrak{Fr}^I$ onto $\mathfrak{A}$ such that $\xi i \mapsto a_i$ $(i \in I)$ and similarly a homomorphism from $\mathfrak{Fr}^J$ into $\mathfrak{B}$ such that $\xi j \mapsto b_j$ $(j \in I)$. Therefore there exist ideals M and N ideals of $\mathfrak{Fr}^I$ and $\mathfrak{Fr}^J$ respectively, and there exist isomorphisms

$$m : \mathfrak{Fr}^I/M \to \mathfrak{A} \text{ and } n : \mathfrak{Fr}^J/N \to \mathfrak{B}$$

such that

$$m(\xi i/M) = a_i \text{ and } (\xi i/N) = b_i.$$

It is easy to show that

$$M \cap \mathfrak{Fr}^{(I\cap J)} = N \cap \mathfrak{Fr}^{(I\cap J)}.$$

Then we claim that $\mathfrak{Ig}^{\mathfrak{A}}(M \cup N) \neq A$, for assume otherwise. Then there exist $x \in M$ and $z \in N$ such that $x + z = 1$, hence $-x \leq z$. So there is $y \in A^{(X_1 \cap X_2)}$ $\Gamma \subseteq \alpha$ such that

$$\mathsf{c}^{\partial}_{(\Gamma)} - x \leq y \leq \mathsf{c}_{(\Gamma)}z$$

Hence

$$-y \leq \mathsf{c}_{(\Gamma)}x \in \mathfrak{Ig}^{\mathfrak{A}^{(X_1)}}\{x\} \subseteq M$$

and

$$y \leq \mathsf{c}_{(\Gamma)}z \in \mathfrak{Ig}^{\mathfrak{A}^{(X_2)}}\{z\} \subseteq N$$

hence

$$-y \in M \cap A^{(X_1 \cap X_2)} \text{ and } y \in N \cap A^{(X_1 \cap X_2)}$$

Therefore

$$-y + y = 1 \in M$$

which cannot happen. It follows that $\mathfrak{Ig}(M \cup N)$ is proper, and by maximality of M and N we have

$$P \cap A^{(X_1)} = M, \quad P \cap A^{(X_2)} = N$$

for any maximal ideal P containing $\mathfrak{Ig}(M \cup N)$. Then, for any such P, $\mathfrak{Fr}/P$ is a a simple amalgam. Indeed, Let $k : \mathfrak{Fr}^I/M \to \mathfrak{Fr}/P$ be defined by $k(a/M) = a/P$ and $h : \mathfrak{Fr}^J/M \to \mathfrak{Fr}/P$ by $h(a/N) = a/P$. Then $k \circ m$ and $h \circ n$ are one to one and $k \circ m \circ f = h \circ n \circ g$.

- We use a construction of Judit Madarasz to show that semisimple algebras do not have the strong amalgamation property with respect to $\mathbf{K}_\alpha$. It follows that $\mathbf{ReK}_\alpha$ does not have AP with respect to $\mathbf{K}_\alpha$ as well. For $\mathbf{K} \in \{\mathbf{SC}, \mathbf{QA}, \mathbf{QEA}\}$ this is a new result. We shall first construct two weak cylindric set algebras $\mathfrak{B} \subseteq \mathfrak{A}$ such that $B \neq A$ and the inclusion $\mathfrak{B} \subseteq \mathfrak{A}$ is an $\mathbf{RCA}_\omega$ epimorphism i.e is right cancellitive map.(The proof for arbitrary infinite ordinals are the same). Then we show how to modify the construction to get the required. Our exposition will be somewhat sketchy. Let $U_0, U_1, \ldots U_n \ldots n < \omega$ be a system of mutually disjoint finite sets such that

$$|U_0| = 3 \text{ and } |U_{i+1}| = 2 \text{ for } i \in \omega.$$

Let

$$U = \bigcup\{U_i : i \in \omega\},$$

and let

$$T^+ = U_0 \times U_1 \times U_i \ldots \times .$$

Then $T^+ \subseteq {}^\omega U$. Now fix $q \in T^+$, and let V be the following weak space

$$V = {}^\omega U^{(q)}.$$

Let $T = V \cap T^+$, and let $\mathfrak{C}$ be the full weak cylindric set algebra with unit V. Then $T \in C$. Now we *split* T into two relations R and $T \smallsetminus R$ as follows: Suppose that $U_0 = \{a, b, c\}$ with a, b, c pairwise distinct. Define

$$X = \{s \in T : s_0 = a \text{ and } |\{i \in \omega : s_i \neq q_i\}| \text{ is even }\}$$

and

$$Y = \{s \in T : s_0 \in \{b, c\} \text{ and } |\{i \in \omega : s_i \neq q_i\}| \text{ is odd }\}.$$

For the sake of brevity, we often refer to the set $\{i \in \omega : s_i \neq q_i\}$ as *the q-deviation of s,* since it measures how much s "devaites" from q. Thus we can write X and Y more concisely as:

$$X = \{a\} \times \{s : \text{ the } q \text{ deviation of } s \text{ is even}\}$$

$$Y = \{b, c\} \times \{s : \text{ the } q \text{ deviation of } s \text{ is odd}\}.$$

Now we define

$$R = X \cup Y \text{ and } R^- = T \smallsetminus R.$$

Let

$$\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{C}}\{R\} \text{ and } \mathfrak{A} = \mathfrak{Sg}^{\mathfrak{C}}\{R, X\}.$$

We now have our weak set algebras. We proceed to show that $\mathfrak{B}$ is a proper subalgebra of $\mathfrak{A}$ and that the inclusion $\mathfrak{B} \subseteq \mathfrak{A}$ is an $\mathbf{RCA}_\omega$ epimorphism. Let $\mathfrak{B}_0 = \mathfrak{Sg}^{\mathfrak{C}}\{T\}$. Then T is an atom in $\mathfrak{B}_0$. Indeed, let s, z be two sequences in T. Then there exists a permutation $\sigma : U \to U$ of U taking s to z and fixing T. That is

$$\sigma \circ s = z \text{ and } T = \{\sigma \circ p : p \in T\}.$$

If σ is a permutation of U fixing T, then σ fixes all elements generated by T because, as easily checked, the operations considered are permutation invariant. Thus if $a \in B_0$ and $s \in a \cap T$ then $T \subseteq a$, showing that T is an atom as desired. Then the boolean reduct of $\mathfrak{B}$ is the boolean subalgebra of $\mathfrak{C}$ generated by $B_0 \cup \{R, -R\}$. In particular, R and R^- are atoms in B. Let us see why. It is easy to see that R and R^- are *cylindrically equivalent to* T, meaning that (1*)

$$\text{For all } i \in \omega, \text{ we have } \mathsf{C}_i R = \mathsf{C}_i R^- = \mathsf{C}_i T.$$

Now let

$$Z = \{a \cup r_1 \cup \ldots r_{n-1} : a \in B_0, n \in \omega, r_i \in \{R, R^-\} \text{ for all } i \in n\}.$$

Then $Z = B$. $Z \subseteq B$ because $R^- \in B$. This follows from the fact that R generates T, in the sense that $\mathsf{C}_0 R \cap \mathsf{C}_1 R = T$, thus T hence $R^- = T \smallsetminus R$ is in B. Now for the opposite inclusion. For that, it is enough to show that Z is a subuniverse of $\mathfrak{C}$, i.e. that Z is closed under the operations of $\mathfrak{C}$, because Z contains the generators of $\mathfrak{B}$. Clearly Z is closed under finite unions and it contains diagonal elements because B_0 contains the diagonal elements. We proceed to check cylindrifications. Towards this end, let $i \in \omega$. Recall that a typical element of Z is of the form

$$a \cup r_1 \ldots \cup r_n,$$

but because of the additivity of cylindrifications it suffices to check those elements of the form

$$a \cup r.$$

Now in this case, we have

$$\mathsf{C}_i(a \cup r) = \mathsf{C}_i a \cup \mathsf{C}_i r.$$

Now $\mathsf{C}_i a \in B_0$ and we have by (1^*) that

$$\mathsf{C}_i r = \mathsf{C}_i T \text{ whenever } r \in \{R, R^-\}$$

which also is in B_0. Now we check that Z is closed under the boolean complement. First of all notice that

$$V \smallsetminus [a \cup r_1 \cup r_n] = (V \smallsetminus a) \cap (V \smallsetminus r_1) \cap (V \smallsetminus r_n.)$$

Now the required result readily follows by observing that

$$V \smallsetminus r = [V \smallsetminus T] \cup (T \smallsetminus r),$$

that

$$a \cap r \text{ is either equal to } \emptyset \text{ or } r$$

if $a \in B_0$ and $r \leq T$, because T is an atom of B_0. Now it is easy to check that R and R^- are atoms in B. Now we have actually proved that $\mathfrak{B}$ is a proper subalgebra of $\mathfrak{A}$, because $X \in A$ and $X \notin B$, since X is a proper subset of R and R is an atom in $\mathfrak{B}$. Now we show that the inclsuion $\mathfrak{B} \subseteq \mathfrak{A}$ is an epimorphism in the category $\mathbf{RCA}_\omega$. Let $\mathfrak{D} \in \mathbf{RCA}_\omega$. Assume that g, h are homomorphisms from $\mathfrak{A} \to \mathfrak{D}$ such that g and h agree on $\mathfrak{B}$. We want to show that $h = g$. Without loss of generality, we can and will suppose that $\mathfrak{D}$ is subdirectly irreducible i.e. that $\mathfrak{D}$ is a weak cylindric set algebra. Now we investigate what the image $g(\mathfrak{A})$ looks like within $\mathfrak{D}$. First of all, we claim that the unit $1^{\mathfrak{D}}$ of $\mathfrak{D}$ must be of the form $W = {}^\omega U'^{(q')}$, for some infinite set U', and some one to one sequence q'. This is because $T \leq V \smallsetminus \mathsf{D}_{ij}$ for all $i < j \in \omega$, and so $g(T) \leq g(V) \smallsetminus \mathsf{D}_{ij}$. Furthermore,

$$g(T) = U'_0 \times U'_1 \times \ldots \cap W$$

where $|U_0|' = 3$ and $|U_{i+1}|' = 2$. This follows from the fact that $T = \mathsf{C}_i T \cap \mathsf{C}_j T$ for all $i < j < \omega$.

Assume first that $g(R) \neq 0$. We are going to show that $R' = g(R)$ is similar to R, i.e. there are a', b', c' and q" such that

$$(*) \quad R' = (\{a'\} \times \{s \in T' : \text{ the } q\text{" deviation of } s \text{ is even}\}$$

$$\cup(\{b', c'\} \times \{s \in T'' : \text{ the } q'' \text{ deviation of } s \text{ is odd}\}) \cap W.$$

For s be an α-ary sequence and $i \in \alpha$, we write $s(i|u)$ for the sequence that coincides with s on $\alpha \smallsetminus \{i\}$ and whose value at i is equal to u. Now the proof of $(*)$ goes as follows. Let $0 < i < \omega$. By

$$\mathsf{C}_i R = \mathsf{C}_i(T \smallsetminus R) = \mathsf{C}_i T,$$

$$0 < R < T \text{ and } |U_i'| = 2$$

we get that the same equations hold for R', T' and so for every $p \in R'$, the sequence $p(i|u)$ where $u \in U_i'$, $u \neq p_i$, is in $T' \smallsetminus R'$, and the same is true for R' in place of $T' \smallsetminus R'$. By using this repeatedly for all $0 < i < \omega$, we get

$$(**) \text{ For every } p \in R' \text{ and } s \in T', \text{ if } s_0 = p_0 \text{ then } s \in R'$$

$$\text{if and only if the } p\text{-deviation of } s \text{ is even.}$$

Fix now $p \in T'$ arbitrarily, and let

$$x(p) = \{u \in U_0' : p(0|u) \in R'\}$$

and

$$y(p) = \{u \in U_0' : p(0|u) \notin R'\}.$$

By $\mathsf{C}_0 R' = \mathsf{C}_0(T \smallsetminus R')$, we have that $x(p), y(p)$ form a partition of U_0' to two non-empty sets. Thus one of $x(p)$ and $y(p)$ is a one element set and the other is a two element set. We may assume possibly, after renaming if necessary, that $x(p) = \{a'\}$ and that $y(p) = \{b', c'\}$. We choose q" to be p. Then $(*)$ is satisfied (by $(**)$.) Finally, we may assume that $q' = q$". But then we may assume that $\mathfrak{D} = \mathfrak{C}$, i.e. that $\mathfrak{D}$ is the full set algebra with unit V, that $g = Id$, and that $h : A \to C$ with $h \restriction B = Id$. (Note that this assumption is only in case $g(R) \neq 0$, we will return to the case when $g(R) = 0$ later on.) Now

$$h(R) = R \geq h(X) \text{ and } h(R) \geq h(Y).$$

We want to show that $h(X) = X$. The following equations to be used in the sequel are easy to check:

$$(2^*) \quad \mathsf{C}_0\mathsf{C}_1 X = \mathsf{C}_0\mathsf{C}_1 T,$$

$$(3^*)\quad \mathsf{C}_i(\mathsf{C}_jX - X)\cap R\subseteq X,\ \text{for } i,j\neq 0,$$
$$(4^*)\quad \mathsf{C}_1X\cap \mathsf{S}^0_1\mathsf{C}_1(X)\subseteq \mathsf{D}_{01},$$
$$(5^*)\quad R\cap \mathsf{C}_0X = X.$$

We now prove that

$$X' = h(X) = X.$$

Frst of all, we have $X'\neq 0$ by (2^*) because $h(T) = T\neq 0$. Let $s\in X'$ be arbitrary. We show that

(+) $z\in X'$ for all $z\in T$ such that $z_0 = s_0$ and the s deviation of z is even.

Indeed, let $i,j\in\omega$ be distinct non-zero elements, and $p\in X'$ be arbitrary. Let $u\in U_j, u\neq p_j$. Then $p(j|u)\in R^-$ by $\mathsf{C}_jR = \mathsf{C}_jR^-$ and $|U_j| = 2$. In particular, $p(j|u)\in \mathsf{C}_jX\smallsetminus X$. Let $v\in U_i$ and $v\neq p_i$. Then $p(j|u)(i|v)\in R$ by $\mathsf{C}_iR^- = \mathsf{C}_iR$ and $|U_i| = 2$. Then $p(j|u)(i|v)\in X'$ by (3^*). This proves (+). Next we show

$$(++)\ s_0 = z_0 \text{ for all } s,z\in X'$$

Indeed let $u\in U_1$ be such that the s-deviation of $z(1|u)$ is even. Then $\langle s_0,u,z_2,z_3\dots\rangle\in X'$ by $s\in X'$ and (+). Then $w = \langle s_0,z_0,z_2,z_3\dots\rangle\in D_{01}\cap C_1X'$, because $z\in X'$. Thus

$$w\in \mathsf{C}_0(\mathsf{D}_{01}\cap \mathsf{C}_1X') = \mathsf{S}^0_1\mathsf{C}_1X'.$$

By (4^*) we get that $w\in \mathsf{D}_{01}$ i.e. $s_0 = z_0$. (++) is proved. By (+) and (++) and by the definition of R we have that

$$(+++)\quad X' = \{s\in R : s_0 = e\} \text{ for some } e\in U_0.$$

Let us now show that $e = a$. Assume that $e = b$ and let $s\in X'$ be arbitrary. Then $s(0|c)\in R\smallsetminus X'$ by $(+++)$. But $s(0|c)\in \mathsf{C}_0X'$. This contradicts (5^*). The case that $e = c$ is entirely analagous. Then $e = a$ must be the case and this proves that

$$h(X) = X' = X.$$

Now suppose that $h(R) = 0$. Then we have

$$0 = h(R) = g(R) = g(T) = h(T).$$

and since $X\leq R$ we have

$$h(X) = g(X) = 0.$$

But the again in this case we have that $h = g$ since they agree on a generating set of $\mathfrak{A}$. Let $\mathbf{Df}_\omega$ denote the class of diagonal free $\mathbf{CA}_\omega$'s. It is also the case that the inlcusion $\mathfrak{Rd}_{Df}\mathfrak{B} \subset \mathfrak{Rd}_{Df}\mathfrak{A}$ is a $\mathbf{Df}_\omega$ epimorphism. The proof can now be refined to give that $\mathbf{SsK}_\omega$ does not have SAP with respect to $\mathbf{K}_\omega$. Let $U, U_0, U_1, \ldots q$ be as in the above proof. Let $n \in \omega$. We define

$$U_n = U_0 \times U_1 \times \ldots U_{n-1}$$

$$T_n = \{s \in {}^\omega U_n : (\forall i < n) s_i \in U_i\}$$

$$X_n = \{s \in T_n : s_0 = a \text{ and } |\{0 < i < n : s_i \neq q_i\}| \text{ is even }\}$$

$$Y_n = \{s \in T_n : s_0 \in \{b, c\} \text{ and } |\{0 < i < n : s_i \neq q_i\}| \text{ is odd }\}$$

$$R_n = X_n \cup Y_n$$

$\mathfrak{C}_n$ is the full quasi- polyadic set algebra with base U_n

$$\mathfrak{C} = \prod \langle \mathfrak{C}_{n+2} : n \in \omega \rangle$$

$$\mathbf{R} = \langle R_{n+2} : n \in \omega \rangle$$

$$\mathbf{X} = \langle X_{n+2} : n \in \omega \rangle$$

$$\mathfrak{B} = \mathfrak{Sg}^{\mathfrak{C}}\{\mathbf{R}\}$$

$$\mathfrak{A} = \mathfrak{Sg}^{\mathfrak{C}}\{\mathbf{R}, \mathbf{X}\}$$

Then by adapting the above proof it can be shown that $\mathfrak{B}$, $\mathfrak{A}$ are semisimple quasipolydic equality algebras such that their $\mathbf{Df}$ reducts are also semisimple, and the inclusion $\mathfrak{Rd}_{Df}\mathfrak{B} \subseteq \mathfrak{Rd}_{Df}\mathfrak{A}$ is a $\mathbf{Df}_\omega$ epimorphism that is not surjective. Indeed we have $\mathbf{X} \notin \mathfrak{B}$. Now let $\mathbf{K} \in \{\mathbf{SC}, \mathbf{CA}, \mathbf{QEA}, \mathbf{QA}\}$. Then $\mathfrak{Rd}_K\mathfrak{B}$ embeds in $\mathfrak{Rd}_K\mathfrak{A}$ via the inclusion map i say. But there is no $\mathfrak{D} \in \mathbf{K}_\alpha$ and $m : \mathfrak{Rd}_K\mathfrak{A} \to \mathfrak{D}$ and $n : \mathfrak{Rd}_K\mathfrak{A} \to \mathfrak{D}$ such that $m \circ i(B) = m(A) \cap n(A)$, for else m and n would agree on $\mathfrak{B}$ and $m(\mathbf{X}) \neq n(\mathbf{X})$.

Now adapting techniques of Pigozzi, we now prove the following new result:

Theorem 10 . $\mathbf{RQEA}_\omega$ *does not have even AP with respect to* $\mathbf{QEA}_\omega$. *In particular, any K such that* $\mathbf{RQEA}_\omega \subseteq K \subseteq \mathbf{QEA}_\omega$ *fails to have AP.*

Proof. We find it appropriate to denote $\mathfrak{Sg}^{\mathfrak{A}}X$ by $\mathfrak{A}^{(X)}$ and (its universe) $Sg^A X$ by $A^{(X)}$. Now seeking a contradiction assume that $\mathbf{RQEA}_\omega$ has AP with respect to $\mathbf{QEA}_\omega$. Let $\mathfrak{A} = \mathfrak{Fr}_4\mathbf{QEA}_\omega$. (Any cardinal ≥ 4 would do but we stick to 4 to be definite). Let r, s and t be defined as follows:

$$r = \mathsf{c}_0(x \cdot \mathsf{c}_1 y) \cdot \mathsf{c}_0(x \cdot -\mathsf{c}_1 y),$$

$$s = \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z \cdot \mathsf{s}^0_1\mathsf{c}_1 z \cdot -\mathsf{d}_{01}) + \mathsf{c}_0(x \cdot -\mathsf{c}_1 z)$$

$$t = \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 w \cdot \mathsf{s}^0_1\mathsf{c}_1 w - d_{01}) + \mathsf{c}_0(x \cdot -\mathsf{c}_1 w)$$

where $x, y, z,$ and w are the first four free generators of $\mathfrak{A}$. Then $r \leq s \cdot t$. Indeed let

$$a = x \cdot \mathsf{c}_1 y \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z),$$

$$b = x \cdot -\mathsf{c}_1 y \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z)$$

Then we have

$$\begin{aligned} \mathsf{c}_1 a \cdot \mathsf{c}_1 b &\leq \mathsf{c}_1(x \cdot \mathsf{c}_1 y) \cdot \mathsf{c}_1(x \cdot -\mathsf{c}_1 y) \quad \text{by [19]1.2.7} \\ &= \mathsf{c}_1 x \cdot \mathsf{c}_1 y \cdot \mathsf{c}_1 x \cdot -\mathsf{c}_1 y \quad \text{by [19] 1.2.11} \end{aligned}$$

and so

$$\mathsf{c}_1 a \cdot \mathsf{c}_1 b = 0 \tag{12}$$

From the inclusion $x \cdot -\mathsf{c}_1 z \leq \mathsf{c}_0(x \cdot -\mathsf{c}_1 z)$ we get

$$x \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z) \leq \mathsf{c}_1 z.$$

Thus $a, b \leq \mathsf{c}_1 z$ and hence, by [19] 1.2.9,

$$\mathsf{c}_1 a, \mathsf{c}_1 b \leq \mathsf{c}_1 z \tag{13}$$

We now compute:

$$\begin{aligned} \mathsf{c}_0 a \cdot \mathsf{c}_0 b &\leq \mathsf{c}_0\mathsf{c}_1 a \cdot \mathsf{c}_0\mathsf{c}_1 b \quad \text{by [19] 1.2.7} \\ &= \mathsf{c}_0\mathsf{c}_1 a \cdot \mathsf{c}_1\mathsf{s}^0_1\mathsf{c}_1 b \quad \text{by [19] 1.5.8 (i),} \quad \text{[19] 1.5.9 (i)} \\ &= \mathsf{c}_1(\mathsf{c}_0\mathsf{c}_1 a \cdot \mathsf{s}^0_1\mathsf{c}_1 b) \\ &= \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 a \cdot \mathsf{s}^0_1\mathsf{c}_1 b) \\ &= \mathsf{c}_0\mathsf{c}_1[\mathsf{c}_1 a \cdot \mathsf{s}^0_1\mathsf{c}_1 b \cdot (-\mathsf{d}_{01} + \mathsf{d}_{01}) \\ &= \mathsf{c}_0\mathsf{c}_1[(\mathsf{c}_1 a \cdot \mathsf{s}^0_1\mathsf{c}_1 b \cdot -\mathsf{d}_{01}) + (\mathsf{c}_1 a \cdot \cdot^0_1 \mathsf{c}_1 b \cdot \mathsf{d}_{01})] \\ &= \mathsf{c}_0\mathsf{c}_1[(\mathsf{c}_1 a \cdot \mathsf{s}^0_1\mathsf{c}_1 b \cdot -\mathsf{d}_{01}) + (\mathsf{c}_1 a \cdot \mathsf{c}_1 b \cdot \mathsf{d}_{01})] \quad \text{by [19] 1.5.5} \\ &= \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 a \cdot \mathsf{s}^0_1\mathsf{c}_1 b \cdot -\mathsf{d}_{01}) \quad \text{by (12)} \\ &\leq \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z \cdot \mathsf{s}^0_1\mathsf{c}_1 z \cdot -\mathsf{d}_{01}) \quad \text{by (13), [19] 1.2.7} \end{aligned}$$

We have proved that

$$\mathsf{c}_0[x \cdot \mathsf{c}_1 y \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z)] \cdot \mathsf{c}_0[x \cdot -\mathsf{c}_1 y \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z)] \leq \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z \cdot \mathsf{s}^0_1\mathsf{c}_1 z \cdot -\mathsf{d}_{01}).$$

In view of [19] 1.2.11 and axiom (C_3) of [19] 1.1.1 this gives

$$\mathsf{c}_0(x \cdot \mathsf{c}_1 y) \cdot \mathsf{c}_0(x \cdot -\mathsf{c}_1 y) \cdot -\mathsf{c}_0(x \cdot -\mathsf{c}_1 z) \leq \mathsf{c}_0\mathsf{c}_1(\mathsf{c}_1 z \cdot \mathsf{s}^0_1\mathsf{c}_1 z \cdot -\mathsf{d}_{01}).$$

The conclusion now follows. Let $X_1=\{x,y\}$. and $X_2=\{x,z,w\}$. Then

$$\mathfrak{A}^{(X_1\cap X_2)}=\mathfrak{Sg}^{\mathfrak{A}}\{x\}. \tag{14}$$

We have

$$r\in A^{(X_1)} \text{ and } s,t\in A^{(X_2)}. \tag{15}$$

Let R be an ideal of $\mathfrak{A}$ such that

$$\mathfrak{A}/R\cong\mathfrak{Fr}_4\mathbf{RQEA}_\omega \tag{16}$$

Since $r\leq s\cdot t$ we have

$$r\in\mathfrak{Ig}^{\mathfrak{A}}\{s\cdot t\}\cap A^{(X_1).} \tag{17}$$

Let

$$M=\mathfrak{Ig}^{\mathfrak{A}^{(X_2)}}[\{s\cdot t\}\cup(R\cap A^{(X_2)})]; \tag{18}$$

$$N=\mathfrak{Ig}^{\mathfrak{A}^{(X_1)}}[(M\cap A^{(X_1\cap X_2)})\cup(R\cap A^{(X_1)})]. \tag{19}$$

Then we have

$$R\cap A^{(X_2)}\subseteq M \text{ and } R\cap A^{(X_1)}\subseteq N \tag{20}$$

From the first of these inclusions we get

$$M\cap A^{(X_1\cap X_2)}\supseteq(R\cap A^{(X_2)})\cap A^{(X_1\cap X_2)}=(R\cap A^{(X_1)})\cap A^{(X_1\cap X_2)}.$$

By (18) we have

$$N\cap A^{(X_1\cap X_2)}=M\cap A^{(X_1\cap X_2)}.$$

From this we get

$$(\mathfrak{A}^{(X_2)}/M)^{(X_1\cap X_2)}\cong\mathfrak{A}^{(X_1\cap X_2)}/M\cap A^{(X_1\cap X_2)} \tag{21}$$

$$=\mathfrak{A}^{(X_1\cap X_2)}/N\cap A^{(X_1\cap X_2)}\cong(\mathfrak{A}^{(X_1)}/N)^{(X_1\cap X_2)}$$

From (17) and (19) we have $\mathfrak{A}^{(X_2)}/M$ is in $\mathbf{RQEA}_\omega$. By a similar argument $\mathfrak{A}^{(X_1)}/N$ is in $\mathbf{RQEA}_\omega$. By our assumption, there is an amalgam, i.e. there is a $\mathfrak{B}$, a $Y=\{y_0,y_1,y_2,y_3\}$ generating $\mathfrak{B}$ and

$$\mathfrak{B}^{(Y_1)}\cong\mathfrak{A}^{(X_1)}/N,\quad \mathfrak{B}^{(Y_2)}\cong\mathfrak{A}^{(X_2)}/M.$$

Here $Y_1 = \{y_0, y_1\}$ and $Y_2 = \{y_0, y_2, y_3\}$. Let P be the ideal of $\mathfrak{A}$ such that $\mathfrak{A}/P \cong \mathfrak{B}$. Then

$$\mathfrak{A}^{(X_2)}/P \cap A^{(X_2)} \cong (\mathfrak{A}/P)^{(X_2)} \cong \mathfrak{B}^{(Y_2)} \cong \mathfrak{A}^{(X_2)}/M.$$

Thus

$$P \cap A^{(X_1)} = N \tag{22}$$

and

$$P \cap A^{(X_2)} = M. \tag{23}$$

In view of (17), (21) we have $s.t \in P$ and hence by (16) $r \in P$. Consequently from (14) and (21) we get $r \in N$. From (20) there exists elements

$$u \in M \cap A^{(X_1 \cap X_2)} \tag{24}$$

and $b \in R$ such that

$$r \leq u + b. \tag{25}$$

Since $u \in M$, there is a finite $\Gamma \subseteq \omega$ and $c \in R$ such that

$$u \leq \mathsf{c}_{(\Gamma)}(s \cdot t) + c.$$

Let $\{x', y', z', w'\}$ be the first four generators of $\mathfrak{D} = \mathfrak{Fr}_4\mathbf{RQEA}_\omega$. Let h be the homomorphism from $\mathfrak{A}$ to $\mathfrak{D}$ be such that $h(i) = i'$ for $i \in \{x, y, w, z\}$. Notice that $kerh = R$. Then $h(b) = h(c) = 0$. It follows that

$$h(r) \leq h(u) \leq \mathsf{c}_{(\Gamma)}(h(s).h(t).)$$

Let $r' = h(r)$, $u' = h(u)$, $s' = h(s)$ and $t' = h(t)$. Let

$$\mathfrak{B} = (\wp({}^\omega\omega), \cup, \cap, \sim, \emptyset, {}^\omega\omega, \mathsf{C}_\kappa, \mathsf{D}_{\kappa\lambda}, \mathsf{S}_{[\kappa,\lambda]})_{\kappa,\lambda<\omega}$$

that is $\mathfrak{B}$ is the full set algebra in the space ${}^\omega\omega$. Let E be the set of all equivalence relations on ω, and for each $R \in E$ set

$$X_R = \{\varphi : \varphi \in {}^\omega\omega \text{ and, for all } \xi, \eta < \omega, \varphi_\xi = \varphi_\eta \text{ iff } \xi R \eta\}.$$

Let

$$C = \{\bigcup_{R \in L} X_R : L \subseteq E\}.$$

C is clearly closed under the formation of arbitrary unions, and since

$$\sim \bigcup_{R\in L} X_L = \bigcup_{R\in E\sim L} X_R$$

for every $L \subseteq E$, we see that C is closed under the formation of complements with respect to ${}^\omega\omega$. Thus C is a Boolean subuniverse (indeed, a complete Boolean subuniverse) of $\mathfrak{B}$; moreover, it is obvious that

$$X_R \quad \text{is an atom of} \quad (C, \cup, \cap, \sim, 0, {}^\omega\omega) \quad \text{for each} \quad R \in E. \tag{26}$$

For all $\kappa, \lambda < \omega$ we have $\mathsf{D}_{\kappa\lambda} = \bigcup\{X_R : (\kappa, \lambda) \in R \in E\}$ and hence $\mathsf{D}_{\kappa\lambda} \in B$. Also,

$$\mathsf{C}_\kappa X_R = \bigcup\{X_S : S \in E, {}^2(\omega \sim \{\kappa\}) \cap S = {}^2(\omega \sim \{\kappa\}) \cap R\}$$

for any $\kappa < \omega$ and $R \in E$. Thus, because C_κ is completely additive we see that C is closed under the operation C_κ for every $\kappa < \omega$. It is easy to show that C is closed under *arbitrary* substitutions. For any $\tau \in {}^\omega\omega$,

$$\mathsf{S}_\tau X_R = \bigcup\{X_S : S \in E, \forall i, j < \omega(iRj \longleftrightarrow \tau(i)S\tau(j)\}.$$

The set on the right may of course be empty. Therefore, we have shown that

$$C \quad \text{is a subuniverse of} \quad \mathfrak{B}. \tag{27}$$

We now show that there is a subset Y of ${}^\omega\omega$ such that

$$\begin{array}{l} X_{Id} \cap f(r') \neq 0 \ \text{ for every } \ f \in Hom(\mathfrak{D}, \mathfrak{B}) \\ \text{such that } \ f(x') = X_{Id} \ \text{ and } \ f(y') = Y, \end{array} \tag{28}$$

and also that for every $\Gamma \subseteq \omega$, there are subsets Z, W of ${}^\omega\omega$ such that

$$\begin{array}{l} X_{Id} \sim \mathsf{C}_{(\Gamma)} g(s' \cdot t') \neq 0 \ \text{ for every } \ g \in Hom(\mathfrak{D}, \mathfrak{B}) \\ \text{such that } \ g(x') = X_{Id}, g(z') = Z \ \text{ and } \ g(w') = W. \end{array} \tag{29}$$

Here $Hom(\mathfrak{A}, \mathfrak{B})$ stands for the set of all homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. Let $\sigma \in {}^\omega\omega$ be such that $\sigma_0 = 0$, and $\sigma_\kappa = \kappa + 1$ for every non-zero $\kappa < \omega$. Let $\tau = \sigma \restriction (\omega \sim \{0\}) \cup \{(0, 1)\}$. Then $\sigma, \tau \in X_{Id}$. Take

$$Y = \{\sigma\}.$$

Then

$$\sigma \in X_{Id} \cap \mathsf{C}_1 Y \quad \text{and} \quad \tau \in X_{Id} \sim \mathsf{C}_1 Y$$

and hence

$$\sigma \in \mathsf{C}_0(X_{Id} \cap \mathsf{C}_1 Y) \cap \mathsf{C}_0(X_{Id} \sim \mathsf{C}_1 Y). \tag{30}$$

Therefore, we have $\sigma \in f(r)$ for every $f \in Hom(\mathfrak{A}, \mathfrak{B})$ such that $f(x) = X_{Id}$ and $f(y) = Y$, and that (28) holds. We now want to show that for any given finite $\Gamma \subseteq \omega$ and $H \subseteq G$, there exist sets $Z, W \subseteq {}^{\omega}\omega$ such that (29) holds; it is clear that no generality is lost if we assume that $0, 1 \in \Gamma$, so we make this assumption. Take

$$Z = \{\varphi : \varphi \in X_{Id}, \varphi_0 < \varphi_1\} \cap \mathsf{C}_{(\Gamma)}\{Id\}$$

and

$$W = \{\varphi : \varphi \in X_{Id}, \varphi_0 > \varphi_1\} \cap \mathsf{C}_{(\Gamma)}\{Id\}.$$

It can be shown that

$$Id \in X_{Id} \sim \mathsf{C}_{(\Gamma)} g(s \cdot t) \tag{31}$$

for any $g \in Hom(\mathfrak{A}, \mathfrak{B})$ such that $g(x) = X_{Id}$, $g(z) = Z$, and $g(w) = W$. Now there exists a finite $\Gamma \subseteq \omega$ and an interpolant $u' \in \mathfrak{D}^{(\{x'\})}$, that is

$$r' \leq u' \leq \mathsf{c}_{(\Gamma)}(s' \cdot t').$$

There also exist $Y, Z, W \subseteq {}^{\omega}\omega$ such that (28) and (29) hold. Take any $h \in Hom(\mathfrak{D}, \mathfrak{B})$ such that $h(x') = X_{Id}$, $h(y') = Y$, $h(z') = Z$, and $h(w') = W$. This is possible by the freeness of $\mathfrak{D}$. Then using the fact that $X_{Id} \cap h(r')$ is non-empty by (28) we get

$$X_{Id} \cap h(u') = h(x' \cdot u') \supseteq h(x' \cdot r') \neq 0.$$

And using the fact that $X_{Id} \sim \mathsf{C}_{(\Gamma)} h(s' \cdot t')$ is non-empty by (29) we get

$$X_{Id} \sim hu' = h(x' \cdot -u') \supseteq h(x' \cdot -\mathsf{c}_{(\Gamma)}(s' \cdot t')) \neq 0.$$

However, in view of (26), it is impossible for X_{Id} to intersect both $h(u)$ and its complement since $h(u) \in C$ and X_{Id} is an atom; to see that $h(u)$ is indeed contained in C recall that $u' \in \mathfrak{D}^{(\{x'\})}$, and then observe that because of (27) and the fact that $X_{Id} \in C$ we must have $h[\mathfrak{D}^{(\{x'\})}] \subseteq C$. ∎

Today the above table appearing in [18] with question marks with last row added for cylindric and quasipolyadic equality algebras of positive characterstic $\kappa > 0$ (by noting that $SUPAP$ implies everything else) looks like this:

Table 2:

	strong AP	strong AP w.r.t RK_ω	AP	AP w.r.t RK_ω	AP for simple algebras	strong EP	EP	EP for simple algebras	SUP AP	ES
LfK_ω	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
DcK_ω	no	yes	no	yes	no	no	no	no	no	yes
ScK_ω	no	no	yes	yes	yes	yes	yes	yes	no	no
ReK_ω	no	no	yes	yes	yes	yes	yes	yes	no	no
RK_ω	no	no	yes only for **SC** and **QA**	yes only for **SC** and **QA**	yes	yes	yes	yes	no	no
SA_ω	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
PA_ω	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
CA′*s* **QEA**'s of +ve chara.	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes

- The notion of independent algebras over a class of algebras K is defined in [25] Definition 1.1.6. Free algebras and dimension restricted free algebras are instances of this notion. Let $\mathfrak{A}$ be an algebra and $\mathcal{C}$ be a set of congruences on subalgebras of $\mathfrak{A}$. The (restricted) congruence property CP relative to $\mathcal{C}$ is defined in [25] Definition 1.2.7. When $\mathcal{C}$ is the set of all congruences on subalgebras of $\mathfrak{A}$ then (restricted) CP relative to $\mathcal{C}$ is just (restricted)CP. Two algebras $\mathfrak{A}$ and $\mathfrak{B}$ are compatible if they have isomorphic minimal subalgebras, $\mathfrak{C}$ covers $\mathfrak{A}$ and $\mathfrak{B}$ if both $\mathfrak{A}$ and $\mathfrak{B}$ are homomorphic images of $\mathfrak{C}$. We have the following result due to Pigozzi relating the local property of the CP for algebras in a class K and the global amalgamation property for the class K. Let K be a class of algebras such that $SK = HK = K$. If every compatible pair of algebras in K is covered by a member of K with the (restricted) CP, then K has $(EP)AP$. Conversely if K has $(EP)AP$, then every K independently generated in algebra has the (restricted) CP. If we restrict our attention to (restricted) CP relative to all maximal congruences we get (EP) AP for simple algebras. The strong interpolaton property, restricted interpolation property, weak interpolation property and weak restricted interpolation property are defined above in the course of our discussion, see also [25]. The following is proved by Pigozzi: $\mathfrak{A}$ has the (restricted) CP interpolation property iff it if it has (restricted) IP. Let U be the class of all sets M such that M is a *maximal* proper ideal of some subalgebra of $\mathfrak{A}$. Identifying ideals with congruences, $\mathfrak{A}$ has the (restricted) CP relative to U iff it has the (restricted) weak IP. One makes the transfer from amalgamation properties of subvariaties of the class $\mathbf{RK}_\alpha$ to interpolation properties in free algebras by the following Theorem the proof of which can be destilled from the literature:

Theorem 11 . *Let $K \subseteq \mathbf{K}_\alpha$ be a variety.*

(i) K has AP if and only if the free K algebras have the interpolation property

(ii) K has AP for simple algebras if and only if the free K algebras have the weak interpolation property

(iii) K has EP if and only if the free K algebras have the restricted interpolation property.

(iv) K has $SUPAP$ if and only if the free K algebras have strong interpolation property

Summarizing our results concerning free algebras in tabular form, we now have:

Table 3:

	strong IP	IP	weak IP	strong restricted IP	restricted IP	weak restricted IP
$Fr_\omega^\rho K_\omega$ which is in Lf_ω	yes	yes	yes	yes	yes	yes
$Fr_\omega^\rho K_\omega$ which is in DKc_ω	yes	yes	yes	yes	yes	yes
$Fr_\omega RK_\omega$	no	yes only for **SC** and **QA**	yes	yes	yes	yes
$Fr_\omega L$, where L is the class of algebras of positive characteristic in **CA** or **QEA**	yes	yes	yes	yes	yes	yes

Appendix

Let $APbase(K)$ denote the amalgamation base of K and $SUPAPbase(K)$ denote the super amalgamation base of K. We say that $\mathfrak{A}_0 \in K$ is in the strong amalgamation base of K, breifly $\mathfrak{A}_0 \in SAPbase(K)$ if for all $\mathfrak{A}_1, \mathfrak{A}_2 \in K$ and monomorphisms $i_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ $i_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$ there exist $\mathfrak{D} \in K$ and monomorphisms $m_1 : \mathfrak{A}_1 \to \mathfrak{D}$ and $m_2 : \mathfrak{A}_2 \to \mathfrak{D}$ such that $m_1 \circ i_1 = m_2 \circ i_2$ and $m_1(A_1) \cap m_2(A_2) = m_2 \circ i_2(A_0)$. Then, it is easy to see that

$$SUPAPbase(K) \subseteq SAPbase(K) \subseteq APbase(K).$$

In this appendix, we use an unpublished construction of Andras Simon and Judit Madárasz to show that $SUPAPbase(\mathbf{RCA}_\omega) \nsubseteq APbase(\mathbf{CA}_\omega) \cap \mathbf{RCA}_\omega$. We will show that $\mathbf{Mn}_\omega \nsubseteq APbase(\mathbf{CA}_\omega)$. In particular, we have

$$AP(\mathbf{RCA}_\omega) \nsubseteq APbase(\mathbf{CA}_\omega),$$

and

$$SAPbase(\mathbf{RCA}_\omega) \nsubseteq SAPbase(\mathbf{CA}_\omega.)$$

This anwers two questions of Monk.

Definition 1 . *Let α be an ordinal. A structure*

$$\mathfrak{B} = (B, T_i, E_{ij})_{i,j<\alpha}$$

with binary relation T_i and unary relations E_{ij} is a cylindric atom structure of dimension α if the following conditions hold for all $i, j, k < \alpha$:

(i) T_i *is an equivalence relation on* B

(ii) $T_i|T_j = T_j|T_i$

(iii) $E_{ii} = B$

(iv) $E_{ij} = T_k(E_{ik} \cap E_{kj})$ *if* $k \notin \{i, j\}$

(v) $T_j \cap (E_{ij} \times E_{ij}) \subseteq Id$ *if* $i \neq j$.

$\mathfrak{Ca}_\alpha$ is the class of cylindric atom structures of dimension α. The complex algebra $\mathfrak{Cm}\mathfrak{B}$ of an atom structure $(B, T_i, E_{i,j})$ is the algebra

$$(\wp(B), \cap, \cup, \sim, T_i^*, E_{ij})_{i,j<\alpha}$$

where for $X \subseteq B$,

$$T_i^* X = \{y \in B : (\exists x \in X)(xT_iy)\}$$

The proof of the following is tedious but routine.

Theorem 2 . *If* $\mathfrak{B} \in \mathfrak{Ca}_\alpha$ *iff* $\mathfrak{Cm}\mathfrak{B} \in \mathbf{CA}_\alpha$

Proof. [19] Theorem 2.7.40. ■

Definition 6 . *Let $\alpha \geq 3$ be an ordinal. For any structure $\mathfrak{B} = (B, T_i, E_{ij})$ with binary relations T_i and unary relations E_{ij} we introduce the binary relations ${}_kK^{\mathfrak{B}}_{ij}$ (for $i, j, k < \alpha$) as follows*

$$a\ {}_kK_{ij}b \text{ iff } a \in E_{jk} \wedge (\exists c \in E_{ij})(\exists d \in E_{ik})aT_jcT_idT_kb.$$

Lemma 3 . *Let $\alpha \geq 3$, $i, j, k \in \alpha$, $\mathfrak{B} \in \mathfrak{Ca}_\alpha$ and let $x, y \in C$ for some $\mathfrak{C} \subseteq \mathfrak{Cm}\mathfrak{B}$. Then*

$$_kK^*_{ij}x = {}_k\mathsf{s}(i, j)x$$

K_{01} stands for ${}_2K_{01}$. K_{01} is called the accessibility relation corresponding to $\breve{}$.

We now recall a method by which one can construct non-representable algebras. This method, called twisting, is due to Henkin.

Definition 4 . *Let $\mathfrak{B} = (B, T_i, E_{ij})_{i,j<\alpha} \in \mathfrak{Ca}_\alpha$, $t \in \alpha$ and $\xi \in {}^IB$ for some set I, and suppose that*

$$(\xi_i, \xi_j) \in T_t \quad \text{for all distinct} \quad i, j \in I \tag{32}$$

$$\xi_i \notin E_{jk} \text{ for all } i \in I \text{ and all distinct } j, k < \alpha \text{ such that } t \notin \{j, k\}. \tag{33}$$

For $i \in I$, let Ξ_i denote the T_t-class of ξ_i, let π be a permutation of I, and for all $i \in I$, let Ξ_i be partitioned into Ξ'_i and Ξ''_i. Assume that for all $i \in I$ and $j < \alpha, j \neq t$,

$$\begin{array}{ll} dom(T_j \cap (\Xi'_i \times \Xi'_{\pi i})) \supseteq \Xi'_i, & ran(T_j \cap (\Xi'_i \times \Xi'_{\pi i})) \supseteq \Xi'_{\pi i} \\ dom(T_j \cap (\Xi''_i \times \Xi''_{\pi i})) \supseteq \Xi''_i, & ran(T_j \cap (\Xi''_i \times \Xi''_{\pi i})) \supseteq \Xi''_{\pi i} \end{array} \tag{34}$$

Then we form a new relational structure $\mathfrak{B}' = (B, T'_i, E_{ij})_{i,j<\alpha}$ by letting T'_i be the equivalence relation on B with equivalence classes x/T_t for $x \in B \sim \bigcup_{i\in I} \Xi_i$, together with the classes $\Xi'_i \cup \Xi''_{\pi i}(i \in I)$, and $T'_i = T_i$ if $i \neq t$. We say that $\mathfrak{B}'(\mathfrak{C}m\mathfrak{B}')$ is a twisted version of $\mathfrak{B}(\mathfrak{Cm}\mathfrak{B})$.

We omit the proofs of Theorem 5, Lemma 6 and Lemma 7 which can be found in [30].

Theorem 5 . *Let $\alpha > 2$ and let $\mathfrak{B}'$ be the result of twisting $\mathfrak{B} \in \mathfrak{Ca}_\alpha$. Then $\mathfrak{B}' \in \mathfrak{Ca}_\alpha$.*

The next lemma is used to simplify computations in a twisted algebra. It says that the effect of twisting the tth cylinder of an algebra is invisible if c_t

is buried deep enough in a term. (That cylindrifications other than c_t and the diagonals are not affected by twisting dimension t is clear from the definition.)

Lemma 6 . *Let $\alpha > 2$ and let $\mathfrak{B}'$ be the result of twisting the dimension t of $\mathfrak{B} \in \mathfrak{Ca}_\alpha$. Then the term functions $\mathsf{c}_i\mathsf{c}_t, \mathsf{c}_t\mathsf{c}_i$ and $\mathsf{s}_i^t\mathsf{c}_j$ are the same in $\mathfrak{Cm}\mathfrak{B}'$ as in $\mathfrak{Cm}\mathfrak{B}$, provided $|\{t, i, j\}| = 3$.*

The following lemma describes the behaviour of $\breve{} = {}_2\mathsf{s}(0, 1)$ in a twisted algebra.

Lemma 7 . *Let $\mathfrak{B} \in \mathfrak{Ca}_\alpha, \alpha > 2$, and let $\mathfrak{B}'$ be the result of twisting T_1 in $\mathfrak{B}$, as in 2.23. Let K'_{01} denote the accessibility relation of $\breve{}$ in $\mathfrak{B}'$. Then for all and $a, b \in B$, $aK'_{01}b$ iff one of the following (mutually exclusive) conditions holds:*

(i) $aK_{01}b$, *and either* $a \notin \bigcup_{i\in I} \Xi_i$ *or* $(\exists i \in I)(\{a, \mathsf{d}_{01} \cdot \mathsf{c}_1 a\} \subseteq \Xi'_i \vee \{a, \mathsf{d}_{01} \cdot \mathsf{c}_1 a\} \subseteq \Xi''_i)$;

(ii) $\mathsf{d}_{12} \cdot \mathsf{c}_1 \xi_{\pi i} K_{01} b, a \in E_{12} \cap \Xi'_i$ *and* $\mathsf{d}_{01} \cdot \mathsf{c}_1 a \in \Xi''_i$ *for some* $i \in I$;

(iii) $\mathsf{d}_{12} \cdot \mathsf{c}_1 \xi_{\pi^{-1} i} K_{01} b, a \in E_{12} \cap \Xi''_i$ *and* $\mathsf{d}_{01} \cdot \mathsf{c}_1 a \in \Xi'_i$ *for some* $i \in I$.

We shall now construct non representable algebras (using twisting) that separates $SUPAPbase(\mathbf{RCA}_\omega)$ and $APbase\mathbf{CA}_\omega \cap \mathbf{RCA}_\omega$ We will "twist" certain representable algebras defined as follows:

Definition 8 . *Let α be an ordinal, and let $\mathfrak{A}$ be the full set algebra on a set U. If G is a group of permutations of U, then each element g of G induces a permutation $s \mapsto g \circ s$ of U sequences of a fixed length. Call two U sequences s and z G-equivalent (written $s \equiv_G z$) if $z = g \circ s$ for some $g \in G$. If s is a U sequence, $s^- = s/\equiv_G = \{g \circ s : g \in G\}$ denotes the orbit of s under the permutation of G.*

$$\mathfrak{Fr}_G\mathfrak{A} = \{X \subseteq {}^\alpha U : (\forall g \in G) X = \{g \circ s : s \in X\}\}.$$

Clearly $\mathfrak{Fr}_G\mathfrak{A}$ is an atomic subalgebra of the full cylindric set algebra $\mathfrak{A}$ with universe $\wp({}^\alpha U)$ and its atoms are exactly the relations s^-, where $s \in {}^\alpha U$. Furthermore, two atoms s^- and z^- are T_i related in the atom structure of $\mathfrak{Fr}_G\mathfrak{A}$ if $s \upharpoonright (\alpha \sim \{i\}) \equiv_G z \upharpoonright (\alpha \sim \{i\})$ and $s^- \in E_{ij}$ if $s_i = s_j$.

While the lemma 7 is fairly general, it is not very convenient to use. We will usually choose the parameters in twisting so that the following special version applies.

Corollary 9 . *Let $\mathfrak{B} \in \mathfrak{Ca}_\alpha, \alpha > 2$, and let $\mathfrak{B}'$ be the result of twisting T_1 in $\mathfrak{B}$. Assume that $\xi_i \in E_{12} \cap \Xi'_i$ for all $i \in I$. Then*

(i)

$$aK'_{01}b \Longleftrightarrow \begin{cases} aK_{01}b, \quad \textit{and} \quad (\forall i \in I)(a = \xi_i \Rightarrow \Xi''_i \cap E_{01} = \phi), \textit{or} \\ a = \xi_i \quad \textit{and} \quad \xi_{\pi i} K_{01} b \;\; \textit{for some} \;\; i \in I \; \textit{such that} \; \Xi''_i \cap E_{01} \neq \phi; \end{cases}$$

for all $a, b \in B$*;*

(ii) for all $a \in B$ *there is at most one* $i \in I$ *such that* $aT_2\xi_i$*, and*

$$(\breve{\mathsf{c}_2} a)' = \begin{cases} (\breve{\mathsf{c}_2} a) \textit{ if } (\forall i \in I)(aT_2\xi_i \Rightarrow \Xi''_i \cap E_{01} = \phi) \\ (\breve{\mathsf{c}_2} \xi_{\pi i}) \textit{ if } aT_2\xi_i \textit{ and } \Xi''_i \cap E_{01} \neq \phi \end{cases}$$

in the complex algebra $\mathfrak{Cm}\mathfrak{B}'$ *of* $\mathfrak{B}'$*.*

Proof. (i) We show how the conditions of Lemma 7 and Corollary 9 correspond under the assumptions of the latter. So assume that $\xi_i \in E_{12} \cap \Xi'_i$ for all $i \in I$. Then condition (iii) of the lemma never holds, since $a \in E_{12} \cap \Xi'_i$ implies $a = \xi_i \in \Xi'_i$ by 2.6(v). We claim that the first condition in the corollary corresponds to (i) of the lemma, that is, $aK_{01}b$, and $(\forall i \in I)(a = \xi_i \Rightarrow \Xi''_i \cap E_{01} = \phi) \Longleftrightarrow aK_{01}b$, and either $a \notin \bigcup_{i \in I} \Xi_i$ or $(\exists i \in I)(\{a, \mathsf{d}_{01} \cdot \mathsf{c}_1 a\} \subseteq \Xi'_i \vee \{a, \mathsf{d}_{01} \cdot \mathsf{c}_1 a\} \subseteq \Xi''_i)$. $(\Rightarrow)$ By $aK_{01}b, a \in E_{12}$; hence if $a \in \bigcup_{i \in I} \Xi_i$, then $a \in \xi_i$ for some $i \in I$. Then $\Xi''_i \cap E_{01} = \phi$ and hence $\{a, \mathsf{d}_{01} \cdot \mathsf{c}_1 a\} \subseteq \Xi'_i$.

$(\Leftarrow)$ If $a = \xi_i$ for some $i \in I$, then $a \in \Xi'_i$, whence $\mathsf{d}_{01} \cdot \mathsf{c}_1 a \in \Xi'_i$ and from this, $\Xi''_i \cap E_{01} = \phi$.

Finally, to see that the second condition in the corollary corresponds to (ii) of the lemma, it suffices to show that $a = \xi_i, \xi_{\pi i} K_{01} b$ and $\Xi''_i \cap E_{01} \neq \phi \Longleftrightarrow \mathsf{d}_{12} \cdot \mathsf{c}_1 \xi_{\pi i} K_{01} b, a \in E_{12} \cap \Xi'_i$, and $\mathsf{d}_{01} \cdot \mathsf{c}_1 a \in \Xi''_i$ for all $i \in I$. Now $a\xi_i$ iff $a \in E_{12} \cap \Xi'_i$ and $\xi_{\pi i} K_{01} b$ iff $\mathsf{d}_{12} \cdot \mathsf{c}_1 \xi_{\pi i} K_{01} b$ since $\xi_{\pi i} = \mathsf{c}_1 \xi_{\pi i}$. Clearly, $\mathsf{d}_{01} \cdot \mathsf{c}_1 a \in \Xi''_i$ implies $\Xi''_i \cap E_{01} \neq \phi$; conversely, since $\mathsf{d}_{01} \cdot \mathsf{c}_1 a$ is the unique element of $E_{01} \cap \Xi'_i$, if $\Xi''_i \cap E_{01} \neq \phi$ then $\mathsf{d}_{01} \cdot \mathsf{c}_1 a \in \Xi''_i \cap E_{01}$. The proof of (ii) follows from (i). ■

Theorem 10 . *Let* $\mathfrak{D}$ *the full set algebra with unit* ${}^\omega\omega$*. Let* $\mathfrak{M}$ *be its minimal subalgeba. Then* $\mathfrak{M} \in SUPAPbase(\mathbf{RCA}_\omega) \sim APbase(\mathbf{CA}_\omega)$

Proof. We construct a non-representable algebra $\mathfrak{C}$ whose minimal subalgebra is isomorphic to $\mathfrak{M}$, and such that $\mathfrak{C}$ and $\mathfrak{D}$ cannot be embedded in a common algebra. First of all it is easy to see that $\mathfrak{M}$ is simple and of characteristic 0.

For $0 < k < \omega$, let π_k be the permutation $(01)(2k, 2k+1)$ of $U = \omega$, G the subgroup of $Sym(U)$ generated by $\{\pi_k : 0 < k < \omega\}$, and let $\mathfrak{A} = \mathfrak{Fr}_G\mathfrak{E}$, where $\mathfrak{E}$ is the powerset algebra of dimension ω on U.

Fact . *Let $H = \{X \subseteq \omega : |X|$ is even$\}$. Then H is closed under $\oplus$ (symmetric difference), and $(H, \oplus)$ is a group isomorphic to G.*

Proof. If $X, Y \subseteq \omega$ and $X \sim Y$ are even, then $X \cap Y$ is even, because $X = (X \sim Y) \cup (X \cap Y)$; hence $Y \sim X$ must be even, since $Y = (Y \sim X) \cup (X \cap Y)$. As $X \oplus Y = (X \sim Y) \cup (Y \sim X)$, this proves the first part of the claim. Let f be the map $X \mapsto \prod_{k \in X}(2k, 2k+1)$ from H into $SymU$. It is not hard to check that f is a one-one group homomorphism; and its range contains G, because $\pi_k = f(\{0, k\})$ for all $0 < k < \omega$. We show by induction on the size of $X \in H$ that f maps into G. $f(\emptyset) = 1^G$, and if $|X| = 2$, then either $X = \{0, k\}$ for some $k > 0$, and then $f(X) = \pi_k$, or $X = \{k, l\}$, with $k, l > 0, k \neq l$, and then $f(X) = \pi_k \pi_l$. Let $n > 0$, and suppose we know inductively that $f(X) \in G$ for all $X \in H$ of size $2n$, and let $X \in H$ be of size $2(n+1)$. Then $X = X' \cap \{k, l\}$ for some $X' \in H$ and $k \neq l$ such that $0 \notin \{k, l\}$. Hence

$$f(X) = f(X' \oplus \{k, l\}) = f(X')f(\{k, l\}) = f(X')\pi_k\pi_l \in G.$$

Thus f is an isomorphism from H to G. ■

From the proven fact it follows that G is abelian and it contains exactly those permutations of α which "invert" an even number of blocks (where a block is a set $\{2k, 2k+1\}$) and leaves the rest of α alone. For $n \in \alpha$, let $\xi_0^n = (0, n, 2, 3, 4, 6 \ldots)^-$. Let $\xi_1^n = (0, n, 3, 4, 6, \ldots), \xi_0 = \xi_0^2$ and $\xi_1 = \xi_1^3$. Note that $\xi_i^n = \xi_j^m$ iff $j = i$ and $m = n$: this is because there is no $\pi \in G$ which fixes all even numbers except 2. It is clear that the twisting conditions (T1) and (T2) hold for $t = 1$ and $I = \{0, 1\}$, and that

$$\Xi_0 = \xi_0/T_1 = \{\xi_0^n : n \in \omega\}$$

and

$$\Xi_1 = \xi_1/T_1 = \{\xi_1^n : n \in \omega\}.$$

Next we claim that

$$T_0 \cap (\Xi_0 \times \Xi_1) = \{(\xi_0^m, \xi_1^n) : m = n > 3 \ \ \textit{or} \ \ \{m, n\} \in \{\{0, 1\}, \{2, 3\}\}\}, \quad (35)$$

$$T_2 \cap (\Xi_0 \times \Xi_1) = \{(\xi_0^m, \xi_1^n) : m = n\} \quad (36)$$

if $\omega > i \geq 2$, then

$$T_{i+1} \cap (\Xi_0 \times \Xi_1) = \{(\xi_0^m, \xi_1^n) : m = n \notin \{2, 3, 2i, 2i+1\} \ \ \textit{or} \ \ \{m, n\} \in \{\{2, 3\}, \{2i, 2i+1\}\}\}, \quad (37)$$

The first equality holds because $\xi_0^m T_0 \xi_1^n$ iff $(m, 2, 4, 6, \ldots) \equiv_G (n, 3, 4, 6, \ldots)$, and the only element of G that fixes $2k$ for all $k > 1$ and sends 2 to 3 is

π_1. Hence from $\pi_1(m) = n$ it follows that either $m = n > 3$ or $\{m, n\} \in \{\{0, 1\}, \{2, 3\}\}$. The second equality holds because $\xi_0^m T_2 \xi_1^n$ iff $(0, m, 4, 6, \dots) \equiv_G (0, n, 4, 6, \dots)$, and the identity is the only $\pi \in G$ for which $\pi(0) = 0$ and $\pi(2k) = 2k$ for all $k > 1$. The last equality holds, because if $i \geq 2$, then $\xi_0^m T_{i+1} \xi_1^n$ iff $(0, m, 2, 6, \dots) \equiv_G (0, n, 3, 6, \dots)$ (if $i = 2$) or $(0, m, 2, \dots, 2(i-1), 2(i+1), 2(i+2), \dots) \equiv_G (0, n, 3, \dots 2(i-1), 2(i+1), 2(i+2), \dots)$ (if $i > 2$), and $\{(23)(2i2i+1)\} = \pi_1\pi_i$ is the only $\pi \in G$ with $\pi(2) = 3$ and $\pi(2k) = 2k$ for all $k \in \alpha \sim \{1, i\}$. Hence either $m = n \notin \{2, 3, 2i, 2i+1\}$ or $\{m, n\} \in \{\{2, 3\}, \{2i, 2i+1\}\}$.

From these equalities it is clear that if $\Xi_i (i = 0, 1)$ are partitioned into

$$\Xi_i' = \{\xi_i^2, \xi_i^3\}, \qquad \Xi_i'' = \{\xi_i^n : n \in \omega \sim \{2, 3\}\},$$

then condition (T3) holds. Let $\mathfrak{C}$ denote the $\mathbf{CA}_\omega$ we get by twisting $\mathfrak{A}$ according to this partition and the permutation (01). Then, by Lemma 7 in $\mathfrak{C}$ we have

$$(\mathsf{c}_2\xi_i)\breve{} = \mathsf{c}_2\xi_j \text{ if } \{i, j\} = \{0, 1\}.$$

For a $\mathbf{CA}_\omega$, let $a_0 = 1$ and for $1 \leq k < \omega$, let $a_k = \mathsf{c}_0 \dots \mathsf{c}_{k-1} \prod_{i<j<k} -\mathsf{d}_{ij}$. Let $\mathfrak{C}_0$ be the minimal subalgebra of $\mathfrak{C}$. Then by [19], 2.1.17, we have $\mathfrak{Nr}_0\mathfrak{C}_0 = \mathfrak{Sg}^{Bl\mathfrak{C}_0}\{a_k : k < \omega\}$. But for all $k < \omega$ $a_k^{\mathfrak{A}} = 1$ hence by Lemma 7 $a_k^{\mathfrak{C}_0} = 1$. Hence $\mathfrak{Nr}_0\mathfrak{C}_0$ is isomorphic to the two element boolean algebra and by [19] 2.4.63(i) $\mathfrak{C}_0$ is of characteristic 0. Hence by [19], 2.5.30 $\mathfrak{C}_0 \cong \mathfrak{M}$. Now we prove some properties of the elements ξ_0, ξ_1 in $\mathfrak{C}$.

$$\mathsf{c}_2\xi_0 \cdot \mathsf{c}_2\xi_1 = 0 \tag{38}$$

Otherwise we'd have $\xi_0 T_2 \xi_1$, i.e. $(0, 2, 4, 6, \dots) \equiv_G (0, 3, 4, 6, \dots)$, contradicting the fact that there is no $\pi \in G$ which inverts exactly one block.

$$\mathsf{c}_2(\xi_0 + \xi_1) = \mathsf{c}_0\mathsf{c}_2\xi_0 \cdot \mathsf{c}_1\mathsf{c}_2\xi_0 \tag{39}$$

$(\leq)\mathsf{c}_2\xi_0 \leq RHS$ is obvious. $\xi_1 \leq \mathsf{c}_0\xi_0$ by (35), and hence $\mathsf{c}_2\xi_1 \leq \mathsf{c}_0\mathsf{c}_2\xi_0$. Finally, $\xi_1 \leq \mathsf{c}_2\xi_0^3$ by (36), and $\xi_0^3 \leq \mathsf{c}_1\xi_0$, since $\xi_0^3, \xi_0 = \xi_0^2 \in \Xi_0'$; hence $\mathsf{c}_2\xi_1 \leq \mathsf{c}_2\xi_0^3 \leq \mathsf{c}_2\mathsf{c}_2\xi_0$. Thus $\mathsf{c}_2\xi_1 \leq RHS$.

$(\geq)$ First note that the term function $\mathsf{c}_1\mathsf{c}_2$ is the same in $\mathfrak{C}$ as in $\mathfrak{A}$ by Lemma 7, so it is enough to check $\geq$ in $\mathfrak{A}$. Let $s \in {}^\alpha\alpha$ and assume that $s^- \leq \mathsf{c}_0\mathsf{c}_2\xi_0 \cdot \mathsf{c}_1\mathsf{c}_2\xi_0$. By $s^- \leq \mathsf{c}_0\mathsf{c}_2\xi_0$, $s \restriction \omega \sim \{0, 2\} \equiv_G (2, 4, 6, \dots)$. We can assume that $s \restriction \omega \sim \{0, 2\} = (2, 4, 6, \dots)$ (since s^- contains such a sequence). So $s = (s_0, 2, s_2, 4, 6, \dots)$. From this, and $s^- \in \mathsf{c}_1\mathsf{c}_2\xi_0$ we get

$$(s_0, 4, 6, \dots) = s \restriction \alpha \sim \{1, 2\} \equiv_G (0, 4, 6, \dots),$$

so there is a $\sigma \in G$ with $\sigma(s_0) = 0$ and $\sigma(2k) = 2k$ for all $k \geq 2$. Hence σ is either the identity or π_1. In the first case we have $s_0 = 0$, and hence $s^- =$

$(0, 2, s_2, 4, 6, \dots)^- T_2 \xi_0$ and in the second $s_0 = 1$ and $s^- = (1, 2, s_2, 4, 6, \dots)^- = (0, 3, \pi_1 s_2, 4, 6, \dots)^- T_2 \xi_1$. Hence $s^- \le \mathsf{c}_2\xi_0 + \mathsf{c}_2\xi_1$, completing the proof of (39).

$$\mathsf{c}_1\mathsf{c}_2\xi_0 = \mathsf{c}_1\mathsf{c}_2\xi_1 \text{ and } \mathsf{c}_0\mathsf{c}_2\xi_0 = \mathsf{c}_0\mathsf{c}_2\xi_1 \tag{40}$$

These equations clearly hold in $\mathfrak{A}$ (the first is shown by the identity permutation, and the second by π_1), and since $\mathsf{c}_0, \mathsf{c}_2$ and $\mathsf{c}_1\mathsf{c}_2$ in $\mathfrak{C}$ are the same as in $\mathfrak{A}$, they continue to hold in $\mathfrak{C}$. Intuitively, (41) below says that "$\mathfrak{C}$ thinks that the 0th projection (or domain) of ξ_0 is of cardinality ≤ 2".

$$\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_1\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \le \mathsf{d}_{01} + \mathsf{d}_{02} + \mathsf{d}_{12} \tag{41}$$

Since c_0 and $\mathsf{c}_1\mathsf{c}_2$ in $\mathfrak{C}$ are the same as in $\mathfrak{A}$, it is enough to check that this holds in $\mathfrak{A}$. Let $s \in {}^\alpha\alpha$ and suppose that $s^- \le \mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_1\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0$ and $\mathsf{s}^- \not\le \mathsf{d}_{01} + \mathsf{d}_{02}$. Then $s \upharpoonright \omega \sim \{1,2\} \equiv_G (0, 4, 6, \dots) \cup Id$, and there are $z, w \in {}^\omega\omega$ such that $z_0 = z_1, z \upharpoonright \omega \sim \{0\} \equiv_G s \upharpoonright \omega \sim \{0\}, z \upharpoonright \omega \sim \{1,2\} \equiv_G (0, 4, 6, \dots)$, and $w_0 = w_2, w \upharpoonright \omega \sim \{0\} \equiv_G s \upharpoonright \omega \sim \{0\}, w \upharpoonright \omega \sim \{1,2\} \equiv_G (0, 4, 6, \dots)$. We may assume $s \upharpoonright \omega \sim \{1,2\} = (0, 4, 6, \dots), z \upharpoonright \omega \sim \{1,2\} = (0, 4, 6, \dots)$ and $w \upharpoonright \omega \sim \{1,2\} = (0, 4, 6, \dots)$. So $s = (0, s_1, s_2, 4, 6, \dots), z = (0, 0, z_2, 4, 6, \dots)$, and $w = (0, w_1, 0, 4, 6, \dots)$ and there are $\sigma, \pi \in G$ such that $\sigma(s_1) = 0 = (s_2)$ and $\sigma(2k) = 2k = \pi(2k)$ for all $k \ge 2$. From this last equality it follows that $\{\sigma, \pi\} \subseteq \{Id, \pi_1\}$. But by the assumption $s^- \not\le \mathsf{d}_{01} + \mathsf{d}_{02}, 0 \notin \{s_1, s_2\}$, whence $Id \notin \{\sigma, \pi\}$. Hence $\sigma = \pi_1 = \pi$ and thus $s_1 = 1 = s_2$, that is, $s^- \le \mathsf{d}_{12}$. A similar argument shows that $\mathsf{c}_0\mathsf{c}_2\xi_0 \cdot \mathsf{s}^1_0\mathsf{c}_0\mathsf{c}_2\xi_0 \cdot \mathsf{s}^1_2\mathsf{c}_0\mathsf{c}_2\xi_0 \le \mathsf{d}_{01} + \mathsf{d}_{02} + \mathsf{d}_{12}$, i.e., that "$\mathfrak{C}$ thinks that the 1st projection of ξ_0 is of cardinality ≤ 2", but we will not need this in the sequel.

Now let $\mathfrak{D}$ be the full set algebra with universe $\wp({}^\omega\omega)$, and let r be the ordering $<$ of ω, that is, $r = \{s \in {}^\omega\omega :: s_0 < s_1\}$. We will show that there is no $\mathfrak{E} \in \mathbf{CA}_\omega$ which embeds both $\mathfrak{C}$ and $\mathfrak{D}$. Now suppose, seeking a contradiction, that $\mathfrak{C}$ and $\mathfrak{D}$ can be jointly embedded in a $\mathbf{CA}_\omega$, $\mathfrak{E}$, say. We can of course assume that $\mathfrak{C}$ and $\mathfrak{D}$ are subalgebras of $\mathfrak{E}$. Unless otherwise stated, the definitions and statements in the rest of the proof all refer to the algebra $\mathfrak{E}$. Let

$$\mu = \mathsf{c}_1\mathsf{c}_2\xi_0 \cdot -\mathsf{c}_1(\mathsf{s}^0_1\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r).$$

Our first goal is to show that

$$\mu \text{ is non-zero.} \tag{42}$$

$$0 < \mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot -\mathsf{d}_{02} \tag{43}$$

Of course we only need to see that this holds in $\mathfrak{C}$ and hence in $\mathfrak{A}$; and there we have

$$(0, 2, 1, 4, 6, \dots)^- T_2 (0, 2, 2, 4, 6, \dots)^- = \xi_0 T_1 \xi_0$$

and

$$(0,2,1,4,6,\dots)^{-}T_0(1,2,1,4,6,\dots)^{-}T_2(1,2,3,4,6,\dots)^{-}$$
$$= (0,3,2,4,6,\dots)^{-}T_1\xi_0$$

and since $(0,2,1,4,6,\dots)^{-} \leq \mathsf{d}_{02}$ and $(1,2,1,4,6,\dots)^{-} \leq \mathsf{d}_{02}$, this shows that $0 < (0,2,1,4,6,\dots)^{-} \leq \mathsf{c}_2\mathsf{c}_1\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_2\mathsf{c}_1\xi_0 \cdot -\mathsf{d}_{02} = \mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot -\mathsf{d}_{02}$. Let $r_{02} = \mathsf{s}^1_2 r$ and $r_{20} = \mathsf{s}^1_0\mathsf{s}^0_2 r$.

$$-\mathsf{d}_{02} \leq r_{02} + r_{20} \text{ and } r_{20} \cdot r \leq -(\mathsf{d}_{01} + \mathsf{d}_{02} + \mathsf{d}_{12}) \tag{44}$$

It suffices to check these statements in $\mathfrak{D}$; and there, since $r_{02} = \{s \in {}^{\omega}\omega : s_0 < s_2\}$ and $r_{20} = \{s \in {}^{\omega}\omega : s_0 > s_2\}$, the first is true by the trichotomy of $<$, and the second by the irreflexivity and transitivity of $<$ on ω.

$$0 < \mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{20} \tag{45}$$

For suppose that $\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{20} = 0$. Then

$$\begin{aligned} 0 &= {}_1\mathsf{s}(2,0)\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot {}_1\mathsf{s}(2,0)\mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot {}_1\mathsf{s}(2,0)r_{20} \quad \text{by 1.11(v)} \\ &= \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{02} \end{aligned}$$

because ${}_1\mathsf{s}(2,0)\mathsf{c}_1\mathsf{c}_2\xi_0 = \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0$,

$$\begin{aligned} {}_1\mathsf{s}(2,0)\mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 &= \mathsf{s}^2_0\mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \\ &= \mathsf{c}_1\mathsf{c}_2\xi_0 \end{aligned}$$

and

$${}_1s(2,0)r_{20} = \mathsf{s}^1_2\mathsf{s}^2_0\mathsf{s}^0_1\mathsf{s}^1_0\mathsf{s}^0_2 r = \mathsf{s}^1_2\mathsf{s}^2_0\mathsf{s}^0_2 r = \mathsf{s}^1_2 r = r_{02}.$$

But this is a contradiction, since

$$\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{20} = 0$$

implies

$$\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{02} > 0$$

by (43) and (44). Hence to prove (42) it is enough to show that

$$\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{20} \leq \mathsf{c}_1\mathsf{c}_2\xi_0 \cdot -\mathsf{c}_1(\mathsf{s}^0_1\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r),$$

or, equivalently, that

$$\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{20} \cdot \mathsf{c}_1(\mathsf{s}^0_1\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r) = 0.$$

Since $\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{20}$ is c_1-closed the latter statement is equivalent to

$$\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot \mathsf{s}^0_2\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r_{20} \cdot \mathsf{s}^0_1\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r = 0,$$

and this is an obvious consequence of (41) and (44). Our next goal is to show that

$$\mu \cdot \mathsf{s}_2^0\mu \leq \mathsf{d}_{02}. \tag{46}$$

This follows from (47) below, because applying s_2^1 to both sides of (47) we get $\mathsf{s}_2^1\mu \cdot \mathsf{s}_2^1\mathsf{s}_1^0\mu \leq \mathsf{s}_2^1\mathsf{d}_{01}$ and $\mathsf{s}_2^1\mu = \mu$.

$$\begin{aligned}
\mathsf{s}_2^1\mathsf{s}_1^0\mu &= \mathsf{s}_2^1\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2\mu \\
&= \mathsf{s}_2^1\mathsf{c}_2\mathsf{s}_1^0\mathsf{c}_1\mu \\
&= \mathsf{s}_2^1\mathsf{s}_0^2\mathsf{c}_2\mathsf{s}_1^0\mathsf{c}_1\mu \\
&= \mathsf{s}_2^1\mathsf{s}_0^2\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2\mu \\
&= {}_1\mathsf{s}(2,0)\mathsf{c}_1\mathsf{c}_2\mu \\
&= \mathsf{s}_2^0\mathsf{c}_1\mathsf{c}_2\mu \\
&= \mathsf{s}_2^0\mu
\end{aligned}$$

and $\mathsf{s}_2^1\mathsf{d}_{01} = \mathsf{d}_{02}$. Now we prove

$$\mu \cdot \mathsf{s}_1^0\mu \leq \mathsf{d}_{01}. \tag{47}$$

Since $-\mathsf{d}_{01} \leq r + \breve{r}$ (in $\mathfrak{D}$ and hence in $\mathfrak{E}$), to see that (47) holds it suffices to check that $\mu \cdot \mathsf{s}_1^0\mu \cdot r = 0$ and $\mu \cdot \mathsf{s}_1^0\mu \cdot \breve{r} = 0$. The first of these statements is true because

$$\begin{aligned}
\mu \cdot \mathsf{s}_1^0\mu \cdot r &\leq \mathsf{c}_1(\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r) \cdot \mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r \\
&\leq \mathsf{c}_1(\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r) \cdot \mathsf{c}_1(\mathsf{s}_1^0\mathsf{c}_1\mathsf{c}_2\xi_0 \cdot r) \\
&= 0,
\end{aligned}$$

and the second statement follows from the first, because ${}_2\mathsf{s}(0,1)\mu = \mathsf{s}_1^0\mu$ and ${}_2\mathsf{s}(0,1)\mathsf{s}_1^0\mu = \mathsf{s}_0^1\mathsf{s}_1^0\mu = \mu$. The last statement about $\mathfrak{E}$ we need is

$$\mu \cdot \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu) \leq \mathsf{c}_2\xi_0 \tag{48}$$

To see that this holds, assume for contradiction that $\mu \cdot -\mathsf{c}_2\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu) \neq 0$; then $0 \neq \mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu) = [\mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu \cdot \mathsf{d}_{02})] + [\mathsf{d}_{02} \cdot \mu - \mathsf{c}_2\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu \cdot -\mathsf{d}_{02})]$ and since

$$\begin{aligned}
\mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu) &= \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu \cdot (\mathsf{d}_{02} + -\mathsf{d}_{02})) \\
&= \mathsf{c}_0((\mathsf{c}_2\xi_0 \cdot \mu \cdot \mathsf{d}_{02}) + (\mathsf{c}_2\xi_0.\mu \cdot -\mathsf{d}_{02})) \\
&= \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu \cdot \mathsf{d}_{02}) + \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu \cdot -\mathsf{d}_{02}).
\end{aligned}$$

But this is a contradiction, since

$$\mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\xi_0.\mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu \cdot \mathsf{d}_{02}) \leq \mathsf{c}_0(\mathsf{d}_{02} \cdot -\mathsf{c}_2\xi_0) \cdot -\mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mathsf{d}_{02}) = 0.$$

by (C7) and $\mathsf{d}_{02} \cdot \mu - \mathsf{c}_2\xi_0 \cdot \mathsf{c}_0(\mathsf{c}_2\xi_0 \cdot \mu \cdot -\mathsf{d}_{02}) = 0$, for otherwise we would have

$$\begin{aligned} 0 &< \mathsf{c}_0(\mathsf{d}_{02} \cdot \mu \cdot -\mathsf{c}_2\xi_0) \cdot \mathsf{c}_2\xi_0 \cdot \mu \cdot -\mathsf{d}_{02} \\ &\leq \mathsf{c}_0(\mathsf{d}_{02} \cdot \mu) \cdot \mu - \mathsf{d}_{02} \\ &= \mathsf{s}_2^0\mu \cdot \mu \cdot -\mathsf{d}_{02} \\ &= 0. \end{aligned}$$

Now let $y = \mu \cdot \mathsf{c}_2\xi_0$. Then $y \neq 0$, since that would imply $0 = \mu \cdot \mathsf{c}_1\mathsf{c}_2\xi_0 = \mu$ and the definition of (in particular, that it is c_1-closed), contradicting (42). If we succeed in showing that $\breve{y} = y$, then we have proved the theorem, for then $0 < y \cdot \breve{y} \leq \mathsf{c}_2\xi_0.(\mathsf{c}_2\xi_0)\breve{} = \mathsf{c}_2\xi_0 \cdot \mathsf{c}_2\xi_1 = 0$ by (38), whence our assumption that there is a $\mathbf{CA}_\omega$ containing both $\mathfrak{C}$ and $\mathfrak{D}$ as subalgebras led to a contradiction. We start showing $\breve{y} = y$ by proving first that $y = \mathsf{c}_0 y \cdot \mathsf{c}_1 y$. Of course, only the $\geq$ part needs proof; here is how it goes:

$$\begin{aligned} \mathsf{c}_0 y \cdot \mathsf{c}_1 y &= \mathsf{c}_0(\mu.\mathsf{c}_2\xi_0) \cdot \mathsf{c}_1(\mu \cdot \mathsf{c}_2\xi_0) \\ &= \mathsf{c}_0(\mu \cdot \mathsf{c}_2\xi_0) \cdot \mu \cdot \mathsf{c}_1\mathsf{c}_2\xi_0 \\ &= \mathsf{c}_0(\mu \cdot \mathsf{c}_2\xi_0) \cdot \mu \\ &\leq \mathsf{c}_2\xi_0 \cdot \mu \\ &= y. \end{aligned}$$

To see that $y = \breve{y}$, it suffices to show that ${}_2\mathsf{s}(0,1)\mathsf{c}_i y = {}_2\mathsf{s}(1,0)\mathsf{c}_i y$ for $i = 0, 1$. But this is immediate since $y = \mathsf{c}_2 y$. ■

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