International Journal of Algebra, Vol. 3, 2009, no. 4, 173 - 179

Groups that Distribute over n-Stars

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Abstract

Suppose $(S, *)$ is a mathematical structure on a set S. As examples, $(S, *)$ might be a topological space on S, a topological group on S, an n-ary operator on S, an n-ary relation on S or a Steiner triple system on S. A similarity mapping on $(S, *)$ is a permutation on S that preserves the structure of $(S, *)$. Such mappings $f : (S, *) \to (S, *)$ are called by different names. As examples, f might be called a homeomorphism or an automorphism on $(S, *)$. See [5] for the details.

Suppose $(S, \cdot)$ is a group on S. For each fixed t in S, the left and right translation on S are the permutations $L_t(x) = t \cdot x$ and $R_t(x) = x \cdot t$ respectively. They are called the left and right translation by t. See [1] and [5].

We say that the group $(S, \cdot)$ left-distributes or right-distributes over $(S, *)$ if respectively for all t in $S, L_t(x) : (S, *) \to (S, *)$ or $R_t(x) : (S, *) \to (S, *)$ is a similarity mapping on $(S, *)$. Naturally occurring examples of this phenomenon are rare. One such example is a topological group in which a group both left-distributes and right-distributes over a topological space. See [5] for more examples.

In this paper, we give a naturally occurring example that involves a double 7 pointed star which is structurally the same as 7 lines in the plane intersecting in $C_2^7 = 21$ distinct points. However, the overwhelming main purpose of this paper is to point out the question of the existence of a group $(S, \cdot)$ that left-distributes or right-distributes over a given structure $(S, *)$. The authors hope this paper will lead to some undergraduate research.

1. Introduction

Suppose $(S, *)$ is a structure on S. In the paper [5] which the reader can easily access electronically, we show how to construct all groups $(S, \cdot)$ on S such that $(S, \cdot)$ left-distributes (or right-distributes) over $(S, *)$ if such a group $(S, \cdot)$ exists. However, usually such a group $(S, \cdot)$ will not exist. Our construction used the group of all similarity mappings on $(S, *)$ which we called $(F, \circ)$, where $\circ$ denotes composition of functions. We showed that a group $(S, \cdot)$ exists such that $(S, \cdot)$ left-distributes over $(S, *)$ if and only if there exists a subgroup $\left(\overline{F}, \circ\right)$ of $(F, \circ)$ such that $\left(\overline{F}, \circ\right)$ is uniquely transitive on S. This means that for every $a, b \in S$, there exists a unique $f \in \overline{F}$ such that $f(a) = b$. If such an $(\overline{F}, \circ)$ exists, then the group $(S, \cdot)$ that left-distributes over $(S, *)$was defined in theorem 3 of [5] as follows. First, we arbitrarily choose $1 \in S$ to be the identity of $(S, \cdot)$. Then we index $\overline{F} = \{f_t \mid t \in S\}$ such that for all $i \in S, f_i(1) = i$. We can do this since $(\overline{F}, \circ)$ is uniquely transitive on S. We can do this by either renaming the functions f of $\overline{F}$ or renaming the elements a of S. A group $(S, \cdot)$ with identity 1 that left-distributes over $(S, *)$ is defined as follows: for every $i, j \in S, i \cdot j = f_i(j)$.

Also, we showed that if the group $(S, \cdot)$ left-distributes over $(S, *)$ then the group $(S, \odot)$ defined by $a \odot b = b \cdot a$ will right-distributes over $(S, *)$. Also, in [5] we dealt with the converse problem in which we started with a given group $(S, \cdot)$ and then constructed certain types of structures on S (such as n-ary operators) so that $(S, \cdot)$ left (or right) distributes over $(S, *)$. These structures were easy to construct. However, they were contrived and almost always very artificial. Furthermore, naturally occurring structures $(S, *)$ such that there exists a group $(S, \cdot)$ that left (or right) distributes over $(S, *)$ are rare. It is analogous to hunting for diamonds, since they are hard to find.

By studying the paper [5] it is obvious that intuitively a necessary condition on $(S, *)$ is that the structure of $(S, *)$ must be extremely homogeneous. However, we do not really know how to tell by looking at a structure $(S, *)$ whether it is homogenous enough or not except to say that it is homogeneous enough if and only if there exists a group $(S, \cdot)$ that left (or right) distributes over $(S, *)$.

We now proceed to illustrate this by studying several stars. Intuitively these stars are homogeneous and they look alike, but only one has a group that left (or right) distributes over it. We will purposely omit many of the details since the basic purpose is mostly to illustrate our main idea of finding groups that distribute over a structure and not to write a technical paper on our work.

2. Generalized Stars

Definition 1. Suppose n lines in the plane called $(l_1, l_2, \cdots, l_n)$ intersect each other in $C_n^2 = \frac{n(n-1)}{2}$ distinct points which we call $\left(x_1, x_2, x_3, \cdots, x_{\frac{n(n-1)}{2}}\right)$. If these n lines are the sides of a regular n-gon, then these C_2^n points can be viewed as generalized n-stars; however, we must allow points at infinity when n is even. In Fig. 1 we show the n-stars for $n = 3, 4, 5, 6$ and in Fig. 2 we show the star for $n = 7$.

Let us now define the group of all permutations on $(l_1, l_2, \cdots, l_n)$ using composition of functions. This group which contains $n!$ permutations is called the symmetric group on $(l_1, l_2, \cdots, l_n)$.

Let us also label each x_k so that $x_k = (l_i, l_j) = (i, j), i < j$, where x_k is the intersection of the two lines l_i, l_j. It is not hard to see that each permutation f on $(l_1, l_2, \cdots, l_n)$ defines a corresponding line preserving permutation $\overline{f}$ on $\{x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}\}$. For example, if

$$f = \begin{pmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \\ l_3 & l_1 & l_5 & l_2 & l_4 \end{pmatrix}, \text{ then}$$

$$\overline{f} = \begin{pmatrix} (1,2) & (1,3) & (1,4) & (1,5) & (2,3) & (2,4) & (2,5) & (3,4) & (3,5) & (4,5) \\ (1,3) & (3,5) & (2,3) & (3,4) & (1,5) & (1,2) & (1,4) & (2,5) & (4,5) & (2,4) \end{pmatrix}.$$

The above line preserving permutation is shown in Fig. 3.

It is easy to see that the $n!$ line preserving permutations on $\left(x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}\right)$ is a group using composition of functions. Also, it is reasonably easy to see that this group is isomorphic to the symmetric group on $(l_1, l_2, \cdots, l_n)$ by the isomorphism $\overline{f \circ g} = \overline{f} \circ \overline{g}$ where $\circ$ denotes the composition of functions and $\overline{f}$ is defined above.

3. Groups that Distribute over n-Stars

We say that a permutation $\overline{f} : (x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}) \to (x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}})$ is a similarly mapping on the n-star if and only if $\overline{f}$ maps lines onto lines. Of course, from the paper [5], this means that a group $(\{x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}\}, \cdot)$ with an operator $(\cdot)$ on the set $\left\{x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}\right\}$ left-distributes over the n-star if and only if for all fixed $t \in \left\{x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}\right\}$, the permutation $\{(x_i, tx_i) : x_i \in \{x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}\}\}$ is a line preserving permutation on $\left\{x_1, x_2, \cdots, x_{\frac{n(n-1)}{2}}\right\}$. Right-distribution, of course, is just the dual of this.

Let us now examine the 5 stars shown in Fig. 1 and Fig. 2. The points on each of these five stars intuitively seem to be fairly homogeneous. So it is natural to ask if there exists a group that left (or right) distributes over these 5 stars. Of course, for $n = 3$, the answer is a trivial yes. However, by combining the material in [5], which we stated in the first paragraph of the introduction, with the elementary group-theoretic properties of the line preserving mappings on the stars, we can show that no groups $(S, \cdot)$ exist such that $(S, \cdot)$ left (or right) distributes over the n-stars for $n = 4, 5, 6$.

This is fairly easy to show for $n = 4, 5$ and is a little harder to show for $n = 6$. For $n = 4$, we know that there are $C_2^4 = 6$ points called $\{x_1, x_2, \cdots, x_6\}$. Let $(F, \circ)$ be the group of all line preserving mappings on the 6 elements of the 4-star. Also, suppose there exists a subgroup $(\overline{F}, \circ)$ of $(F, \circ)$ such that $(\overline{F}, \circ)$ is uniquely transitive on $\{x_1, x_2, \cdots, x_6\}$. Of course, $\overline{F}$ must have exactly $|\overline{F}| = 6$ permutations since $(\overline{F}, \circ)$ is uniquely transitive. By the Sylow theorems of group theory, we know that there exists $\overline{f} \in \overline{F}$ such that $\overline{f} \neq i$ and $\overline{f} \circ \overline{f} = i$, the identity permutation on $\{x_1, x_2, \cdots, x_6\}$. By the isomorphism $\overline{f \circ g} = \overline{f} \circ \overline{g}$ stated earlier, this means that there exists a permutation f on $(l_1, l_2, l_3, l_4, l_5, l_6)$ such that $f \neq I, f \circ f = I$, the identity permutation on $(l_1, l_2, \cdots, l_6)$, and such that $f, \overline{f}$ correspond to each other as defined earlier. Since $f \neq I, f \circ f = I$, this means that there exist $l_i, l_j, i < j$, such that $f(l_i) = l_j$ and $f(l_j) = l_i$. But by the way that $\overline{f}$ is defined from f, this means that $\overline{f}(i, j) = (i, j)$ since $f(\{l_i, l_j\}) = \{l_i, l_j\}$. But since $\overline{f} \neq i$ and $i(i, j) = (i, j)$, this means that $(\overline{F}, \circ)$ cannot possibly be uniquely transitive on $\{x_1, x_2, \cdots, x_6\}$ since $\overline{f}(i, j) = i(i, j) = (i, j)$. The proof for the 5-star is identical to this proof. The proof for the 6-star is a little harder and is omitted. This brings us to the 7-star. By analogy with $n = 4, 5, 6$, we would expect to find that no group $(S, \cdot)$ exists on the $C_2^7 = 21$ points of the 7-star such that $(S, \cdot)$ left (or right) distributes over the 7-star. But this analogy is wrong, and this is very surprising to us since the 7-star does not seem any more homogeneous than the $4, 5, 6$ stars. By simple experimentation that combines the construction of [5] and the isomorphism $\overline{f \circ g} = \overline{f} \circ \overline{g}$ with elementary group theory, we have constructed the following group which left-distributes over the 7-star. The 21 points of the 7-star are labelled as in Fig. 2.

We defined the group $(\{1, 2, 3, \cdots, 21\}, \cdot)$ as follows. We omit the details of the derivation of $(\{1, 2, 3, \cdots, 21\}, \cdot)$, but we point out that from the Sylow theorems we knew that $(\{1, 2, 3, \cdots, 21\}, \cdot)$ must have cyclic subgroups of 7 and 3 elements. This fact was our main machinery. We then generated a group $\left(\left\{x^a \cdot y^b : a \in \{0, 1, 2, 3, \cdots, 6\}, b \in \{0, 1, 2\}\right\}, \cdot\right)$ as follows where we called the members of the group $x^a \cdot y^b$ with $a \in \{0, 1, 2, \cdots, 6\}, b \in \{0, 1, 2\}$. First, we agreed that $yx = x^2 y$. Also, $x^7 = x^0 = 1, y^3 = y^0 = 1$.

Using the rule $yx = x^2y$, we see that $\left(x^a \cdot y^b\right) \cdot \left(x^{\overline{a}} \cdot y^{\overline{b}}\right) = x^{a+\overline{a}2^b} \cdot y^{b+\overline{b}}$ where $a + \overline{a}2^b$ is computed modulo 7 and $b + \overline{b}$ is computed modulo 3. This is a group. The group elements can be translated into integers as follows:

$$
\begin{array}{ccc}
x^7 = x^0 = 1 & x^0y = y = 8 & x^0y^2 = y^2 = 15 \\
x = 2 & xy = 9 & xy^2 = 16 \\
x^2 = 3 & x^2y = 10 & x^2y^2 = 17 \\
x^3 = 4 & x^3y = 11 & x^3y^2 = 18 \\
x^4 = 5 & x^4y = 12 & x^4y^2 = 19 \\
x^5 = 6 & x^5y = 13 & x^5y^2 = 20 \\
x^6 = 7 & x^6y = 14 & x^6y^2 = 21.
\end{array}
$$

This defines the group given in the appendix where $a \cdot b$ is given in standard matrix notation. That is, $a \cdot b$ is the entry at the intersection of the a^{th} row from the top and the b^{th} column from the left. The following are examples: $13 \cdot 17 = (x^5y) \cdot (x^2y^2) = x^{5+2\cdot2} \cdot y^{1+2} = x^9y^3 = x^2 = 3$. Also, $15 \cdot 11 = (x^0y^2) \cdot (x^3y) = x^{0+3\cdot2^2} \cdot y^{2+1} = x^{12} \cdot y^3 = x^5 = 6$.

This group $(\{1, 2, 3, \cdots, 21\} \cdot)$ left-distributes over the 7-star of Fig. 2 and this is easy to verify. For example, we see that $\{19, 12, 7, 1, 14, 15\}$ are 6 colinear points. Thus, for all $a \in \{1, 2, 3, \cdots, 21\}$, $a \cdot \{19, 12, 7, 1, 14, 15\}$ must be 6 colinear points. As examples, $11 \cdot \{19, 12, 7, 1, 14, 15\} = \{5, 19, 9, 11, 16, 4\}$ and $9 \cdot \{19, 12, 7, 1, 14, 15\} = \{3, 17, 14, 9, 21, 2\}$ must be sets of 6 colinear points, and they are. This seems amazing to us.

We can show that there are exactly $7!/2 = 2520$ distinct groups that left (or right) distribute over the 7-star and all of these groups are isomorphic to the group given here. Also, in general, if p is any odd prime, then there exists a group that left (or right) distributes over the p-star if and only if p is congruent to 3 modulo 4.

4. Conclusion The main point of this paper is for the reader to see that the group in Figure 4 left-distributes over the 7-star shown in Figure 2.

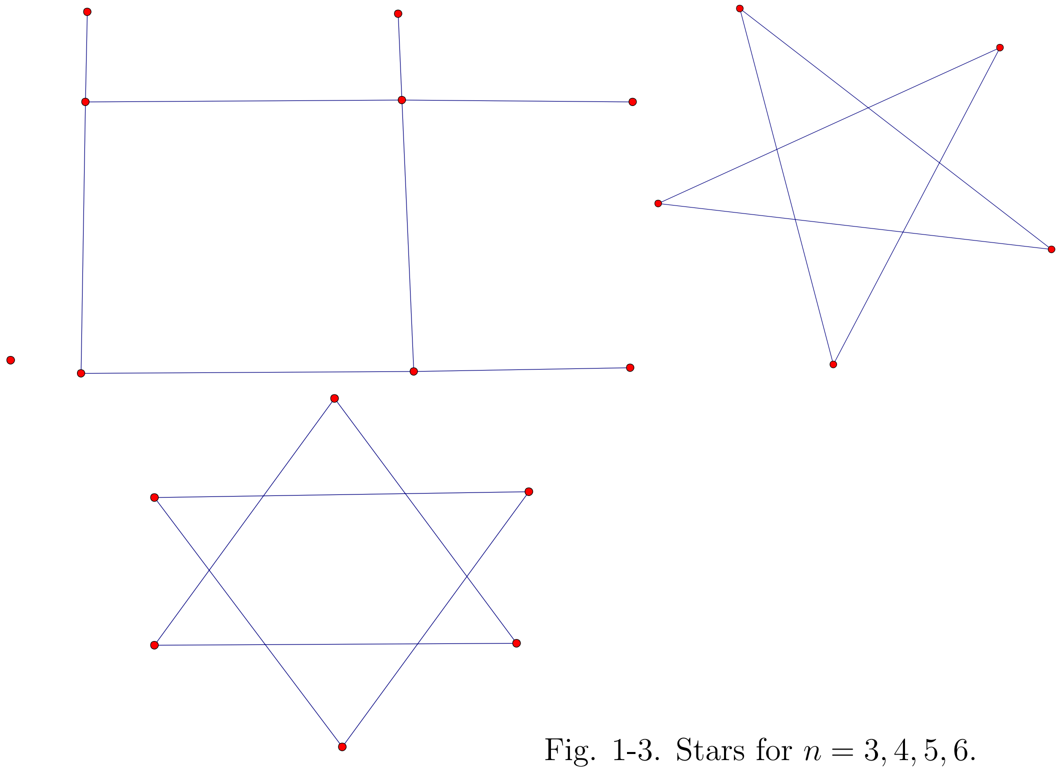

Fig. 1-3. Stars for $n = 3, 4, 5, 6$.

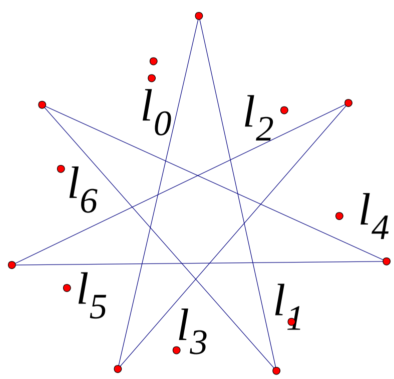

Fig. 4. A double 7-star.

Appendix

·	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	2	3	4	5	6	7	1	9	10	11	12	13	14	8	16	17	18	19	20	21	15
3	3	4	5	6	7	1	2	10	11	12	13	14	8	9	17	18	19	20	21	15	16
4	4	5	6	7	1	2	3	11	12	13	14	8	9	10	18	19	20	21	15	16	17
5	5	6	7	1	2	3	4	12	13	14	8	9	10	11	19	20	21	15	16	17	18
6	6	7	1	2	3	4	5	13	14	8	9	10	11	12	20	21	15	16	17	18	19
7	7	1	2	3	4	5	6	14	8	9	10	11	12	13	21	15	16	17	18	19	20
8	8	10	12	14	9	11	13	15	17	19	21	16	18	20	1	3	5	7	2	4	6
9	9	11	13	8	10	12	14	16	18	20	15	17	19	21	2	4	6	1	3	5	7
10	10	12	14	9	11	13	8	17	19	21	16	18	20	15	3	5	7	2	4	6	1
11	11	13	8	10	12	14	9	18	20	15	17	19	21	16	4	6	1	3	5	7	2
12	12	14	9	11	13	8	10	19	21	16	18	20	15	17	5	7	2	4	6	1	3
13	13	8	10	12	14	9	11	20	15	17	19	21	16	18	6	1	3	5	7	2	4
14	14	9	11	13	8	10	12	21	16	18	20	15	17	19	7	2	4	6	1	3	5
15	15	19	16	20	17	21	18	1	5	2	6	3	7	4	8	12	9	13	10	14	11
16	16	20	17	21	18	15	19	2	6	3	7	4	1	5	9	13	10	14	11	8	12
17	17	21	18	15	19	16	20	3	7	4	1	5	2	6	10	14	11	8	12	9	13
18	18	15	19	16	20	17	21	4	1	5	2	6	3	7	11	8	12	9	13	10	14
19	19	16	21	17	21	18	15	5	2	6	3	7	4	1	12	9	13	10	14	11	8
20	20	17	21	18	15	19	16	6	3	7	4	1	5	2	13	10	14	11	8	12	9
21	21	18	15	19	16	20	17	7	4	1	5	2	6	3	14	11	8	12	9	13	10

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Received: December 7, 2007