International Journal of Algebra, Vol. 3, 2009, no. 3, 149 - 155

On Maximally Completions of $K(x, y)$

Figen Öke and Yasemin Çengellenmiş

Trakya University, Department of Mathematics
22030 - Edirne, Turkey
figenoke@gmail.com

Abstract

Maximally completions of $K(x, y)$ with respect to the certain valuations with rank 1, rank 2 and rank 3 which are extensions of a valuation of K with rank 1 are obtained.

Mathematics Subject Classification: 12F20, 12J10

Keywords: extensions of valuations, transcendental extensions, maximally completions

INTRODUCTION AND SOME PRELIMINARIES

Let K be a field and v be a valuation on K such that $rankv = 1$. All the extensions of v to $K(x)$ were characterized by N. Popescu, A. Zaharescu and V. Alexandru [1, 4, 8]. First and second type residual algebraic free extensions of a valuation v to $K(x)$ with $rank$ 2 were defined in [5]. All extensions of v to $K(x, y)$ with $rank$ 1, $rank$ 2 and $rank$ 3 were studied by F.Öke and H. İşcan in [7].

Throught this paper, v is a valuation of a field K with value group G_v, valuation ring O_v and residue field k_v. $K(x)$ and $K(x, y)$ are rational function fields over K with one and two variables respectively. For any α in O_v, α^* denotes its $v - residue$ in k_v. If $a \in \bar{K}$, then the restriction of $\bar{v}$ to $K(a)$ will be denoted by v_a. Then $\bar{k}_v = k_{\bar{v}}$ is the algebraic closure of k_v and $\bar{G}_v = G_{\bar{v}}$ is the divisible closure of G_v. Let us denote by $\bar{K}$ the algebric closure of K and by $\bar{v}$ a fixed extension of v to $\bar{K}$. Denote by $\tilde{K}$ the completion of K with respect to v and by $\tilde{v}$ the extension of v to $\tilde{K}$. Let K be a field, v be a valuation on K with $rank$ 1 and w be an extension of v to $K(x)$. If k_w is a transcendental extension of k_v then w will be called a residual transcendental (r.t.) extension of to v. If w is a r.t. extension of v to $K(x)$ then there exists a minimal pair $(a, \delta) \in \bar{K} \times G_{\bar{v}}$ respect to K where a is seperable over K.

Let $f = Irr(a, K)$ be a minimal polynomial of a respect to K and $\gamma = w(f)$. If $F \in K[x]$, $F = F_0 + F_1 f + ... + F_n f^n$, $\deg F_i < \deg f$, $i = 0, ..., n$ then w is defined as:

$$w(F) = \inf_i (v_a(F_i(a)) + i\gamma)$$

where v_a is the restriction of $\bar{v}$ to $K(a)$ [1, 4]. Let e be the smallest non-zero positive integer such that $e\gamma \in G_{v_a}$. Then

$$G_w = G_{v_a} + Z\gamma, \quad [G_w : G_v] = e\,[G_{v_a} : G_v]$$

Let $h \in K[x]$ such that $\deg h < \deg f$, $v_a(h(a)) = e\gamma$. Then $r = f^e/h$ is an element of O_w of the smallest order such that $r^* \in k_w$ is transcendental over k_v. Thus the field k_{v_a} can be identified cannonically with the algebraic closure of k_v in k_w and $k_w = k_{v_a}(r^*)$. [1, 4] If w is an extension of v to $K(x)$ and k_w is an algebraic extension of k_v then w is called a residual algebraic (r.a) extension of v. Let w be r.a extension of v to $K(x)$. If G_w/G_v is a torsion group that is every element of G_w/G_v has a finite order then w is called a residual algebraic torsion (r.a.t.) extension of v. If w is a r.a.t. extension of v then there exists a suitable ordered system $(\bar{w}_i)_{i \in I}$ of r.t. extensions of $\bar{v}$ to $\bar{K}(x)$ such that w is the restriction of $\bar{w}$ to $K(x)$ where $\bar{w} = \sup_i(\bar{w}_i)$. That is $w = \sup_i(w_i)$, where w_i is the restriction of $\bar{w}_i$ to $K(x)$ for every $i \in I$. [1] If w is a r.a. extension of v to $K(x)$ and G_w/G_v is not a torsion group then w is called a residual algebraic free (r.a.f.) extension of v. If w is a r.a.f. extension of v to $K(x)$ then $rankw = rankv + 1 = 2$ and $w = w_1 \circ w_2$ where w_1 is a valuation of $K(x)$ and w_2 is a valuation of k_{w_1} such that $rankw_1 = rankw_2 = 1$. If w_1 is trivial on K then it is defined by a monic irreducible polynomial $f \in K[x]$ or is the valuation at infinity. $k_{w_1} = K(a)$ where a is the suitable root of f or $k_{w_1} = K$ if w_1 is a valuation at infinity. w is defined for each $F \in K[x]$, $F = F_0 + F_1 f + ... + F_n f^n$, $\deg F_i < \deg f$, $i = 0, ..., n$ as;

$$w(F) = \inf_i (i, v'(F_i(a)))$$

where v' is an extension of v to $k_{w_1} = K(a)$. The composite valuation $w = w_1 \circ v'$ of $K(x)$ is called r.a.f. extension of first kind of v [5]. If $O_{w_1} \cap K = O_v$ then w_1 is the r.t extension of v to $K(x)$ and k_{w_1} has a valuation w_2 which is trivial on k_v. Hence w_1 is defined by a minimal pair $(a, \delta) \in \bar{K}xG_{\bar{v}}$. Let f, γ, e, h, v_a, r, $r^* = t$ be as before. Since w_2 is trivial over k_{w_a} then it is defined by an irreducible polynomial $G(t) \in k_{v_a}[t]$ or is the valuation at infinity. If we denote with g the lifting polynomial in $K[x]$ of $G(t) \neq t$ then w is defined as follows: Let $F \in K[x]$, $F = F_0 + F_1 g + ... + F_n g^n$ then

$$w(F) = \inf_i (j(w_1(F_i)) + iw(g))$$

where $w(g)=(w_1(g),1)\in G_{\bar v}\times Q$, j is a map from $G_{\bar v}$ to $G_{\bar v}\times Q$ (ordered lexicographically) defined as $j(d)=(d,0)$ for each $d\in G_{\bar v}$. In this case w is called a r.a.f. extension of v of second kind. In this case there exists a root b of g such that $\bar v_0(b-a)\le\delta$. For any $F\in K[x]$ such that $degF<degf$ one has $F(b)^*=F(a)^*$, $(f(b)^e/h(b))^*=c$ is a root of $G(t)$ and $k_w=k_{v_b}=k_{v_a}(c)$ where v_b is the unique extension of v to $K(b)$. [1, 8] Let K be a field with a discrete valuation v such that $rankv=1$. If K is a complete according to v then K is called local field. The set

$$K\{\{t\}\}=\left\{\sum_{n\in Z}a_nt^n\,|a_n\in K, v(a_{-n})\to\infty\ if\ n\to\infty,\ \{v(a_n)\}\ is\ lower\ bounded\right\}$$

is called the formal Laurent series field such that (K,v) is a local field. If the w_1 is a valuation on $K\{\{t\}\}$ which is defined as follows;

$$w_1\left(\sum_{n\in Z}a_nt^n\right)=\inf_n v(a_n)$$

then $(K\{\{t\}\},w_1)$ is a local field and $k_{w_1}=k_v((t))$. A field F with a valuation w is called an immediate extension of a field K with a valuation v, if F is an extension of K, $G_w=G_v$ and $k_w=k_v$. A field F is called maximally complete with respect to the valuation v if it does not admit any proper immediate extensions. If a field F is complete with respect to a valuation v then extensions of v are easily obtained. Moreover completion of F with respect to v is obtained by using maximally completion of F with respect to v. In [3], the maximally completions of $K(x)$ were determined respect to the valuations which were defined in [5]. Let w be a r.t. extension of a valuation v of K with $rank$ 1 defined by a minimal pair $(a,\delta)\in\bar K\times G_{\bar v}$ and u be a r.t. extension of w to $K(x,y)$ defined by a minimal pair $(b,\mu)\in\bar K\times G_{\bar v}$ as in [7]. In this paper the maximally completion of $K(x,y)$ with respect to u is obtained and then the maximally completions of $K(x,y)$ with respect to the valuations with $rank$ 2 and $rank$ 3 defined by using u are obtained.

THE MAXIMALLY COMPLETIONS OF $K(x,y)$

Let w be a residual transcendental extension of v to $K(x)$ which is defined by a minimal pair $(a,\delta)\in\bar K\times G_{\bar v}$. We assume that f,γ,e,h and v_a are in preliminaries. Let u be a residual transcendental extension of w to $K(x,y)$ by minimal pair $(b,\mu)\in\bar K\times G_{\bar v}$. Each polynomial $F\in K[x,y]$ can be written as $F=\sum_{i,j}F_{ij}f^ig^j$ where $\deg_x F_{ij}(x,y)<\deg f$, $\deg_y F_{ij}(x,y)<\deg g$, $F_{ij}\in K[x,y]$ and g is a minimal polynomial of b on K. If u is defined as follows

$$u(F)=\inf_{ij}(v_{a,b}(F_{ij}(a,b))+i\gamma+j\rho) \tag{1}$$

then u is a valuation on $K[x,y]$ and u can be extended uniquelly to $K(x,y)$ where $\rho = u(g)$ and $v_{a,b} = v \underset{K(a,b)}{|}$. Let $e' = e'(\rho, K(b))$ be at least positive integer such that $e'\rho \in G_{v_b}$. We suppose that $h' \in K[y]$ is a polynomial which is satisfied $\deg h' < \deg g$ and $u(h'(b)) = e'\rho$ Hence $G_u = G_{v_{a,b}} + Z\gamma + Z\rho$ and $k_u = k_{v_{a,b}}(r^*, s^*)$ where $s = g^{e'}/h'$[7].

THEOREM 2.1: Let v be a valuation on K, u be an extension of v to $K(x,y)$ which is defined in (1). So the maximally completion of $K(x,y)$ according to u is the field $\widetilde{K(a,b)}\{\{r\}\}\{\{s\}\}(x,y)$.

Proof: Any element of ring $\widetilde{K(a,b)}\{\{r\}\}\{\{s\}\}[x,y]$ is written uniquelly as

$$\alpha = \sum_{i=1}^{ne-1}\sum_{j=1}^{me'-1}\left(\sum_{n\in Z}\sum_{m\in Z} a_{mn} r^m s^n\right) x^i y^j$$

where $a_{mn} \in \widetilde{K(a,b)}$. If this equation is reordered, then we have

$$\alpha = \sum_{m\in Z}\sum_{n\in Z}\left(c_{mn}(x,y)\right) r^m s^n$$

where $c_{mn}(x,y) \in \widetilde{K(a,b)}[x,y]$, $\deg c_{mn}(x,b) \leq ne-1$, $\deg c_{mn}(a,y) \leq me'-1$.

If u is defined as

$$\tilde{u}(\alpha) = \inf_{m,n}\left(u\left(c_{mn}(x,y)\right)\right)$$

then $\tilde{u}$ satisfies all valuation conditions and can be uniquelly extended to

$\widetilde{K(a,b)}\{\{r\}\}\{\{s\}\}(x,y)$.

It is clear that $G_{\tilde{u}} = G_u$ and so $G_{\tilde{u}} = G_{v_{a,b}} + Z\gamma + Z\rho$. If $\alpha \in O_{\tilde{u}}$ then $\alpha^* \in k_{v_{a,b}}(r^*, s^*)$ and therefore $k_{\tilde{u}} = k_{v_{a,b}}(r^*, s^*)$. Hence the field

$\widetilde{K(a,b)}\{\{r\}\}\{\{s\}\}(x,y)$ is the maximally completion of $K(x,y)$ in according to u.

Let w be a r.t extension of v to $K(x)$ which is defined by a minimal pair $(a,\delta) \in \bar{K} \times G_{\bar{v}}$. We assume that f, γ, v_a, k_w be as before. Let u_1 be a r.t extension of w to $K(x,y)$ which is defined by a minimal pair $(b,\mu) \in \bar{K} \times G_{\bar{v}}$ and v_a, $v_{a,b}$, r, s, γ, ρ, e, e' be as above. Let we choose $P \in k_{v_{a,b}}[s^*]$, $P \neq s^*$ monic, irreducible polinomial and let $p \in K[y]$ be a lifting polynomial of P. We shall denote by $\eta = u(f) = (\gamma, 0)$, $\beta = \mathrm{u(p)} = (\mathrm{w}_1(p), 1)$, φ: $\mathrm{G}_{\bar{v}} \to G_{\bar{v}} \times Q$ which is defined by $\varphi(t) = (t,0)$ for each $t \in G_{\bar{v}}$. Then each polynomial $F \in K[x,y]$ is written uniquelly as

$$F = \sum_{i,j\ finite} F_{ij} f^i p^j,\ F_{ij} \in K[x,y]$$

$$\deg_x(F_{\rm ij}(x,y)) < \deg f, \ \ \deg_y(F_{\rm ij}(x,y)) < \deg p$$

Define u as;

$$u(F) = \inf_{i,j}(\varphi(w(F_{i,j}) + i\eta + j\beta) \tag{2}$$

u satisfies the valuation conditions on $K[x, y]$ and can be extended to $K(x, y)$ as uniquely. u is the r.a.f extension of second kind of w to $K(x, y)$. [7]

THEOREM 2.2: The maximally completion of $K(x, y)$ according to valuation u which is defined in (2) is the field $\widetilde{K(a,b)}\{\{r\}\}\{\{t\}\}(x)$.

Proof: According to the [3] the maximally completion of $K(x, y)$ is $\widetilde{K(x,c)}\{\{t\}\}$ where c is a root of $p(y)$ and P goes onto t. Here $(\widetilde{K(x,c)}, \tilde{w}_c)$ is the completion of $(K(x, c), w_c)$ where w_c is an extension of w to $K(x, c)$. According to the [2] $\widetilde{K(x,c)} = \widetilde{K(a,c)}\{\{r\}\}(x)$. then the maximally completion of $K(x, y)$ with respect to u is $\widetilde{K(a,b)}\{\{r\}\}\{\{t\}\}(x)$. Moreover for each $\sum_{n\in \mathrm{Z}} A_n t^n \in \widetilde{K(x,c)}\{\{t\}\}$ the valuation $\tilde{u}_c$ defined as

$$\tilde{u}_c(\sum_{n\in \mathrm{Z}} A_n t^n) = (\inf_n \tilde{w}_c(A_n), n_0)$$

where n_0 is the smallest integer number such that the inf on the first component is reached. To define the extension of v to $\widetilde{K(a,c)}\{\{r\}\}\{\{t\}\}(x)$ each $\alpha \in \widetilde{K(a,c)}\{\{r\}\}\{\{t\}\}[x]$ is written as

$$\alpha = \sum_{i=1}^{ne-1}\left(\sum_{n\in Z}\left(\sum_{m\in Z} a_{mn} r^m\right) t^n\right) x^i, \; a_{mn} \in \widetilde{K(a,c)}$$

If this equation is reordered then it is obtained that

$$\alpha = \sum_{m\in \mathrm{Z}}\left(\sum_{n\in \mathrm{Z}} (M_{mn}(x) t^n)\right) r^m, \; M_{mn}(x) \in \widetilde{K(a,c)}[x], \quad \deg M_{mn}(x) \le ne - 1.$$

Define

$$\tilde{u}(\alpha) = \inf_m(\tilde{u}(\sum_{n\in Z} M_{mn}(x) t^n))$$

u satisfies all valuation conditions on $\widetilde{K(a,c)}\{\{r\}\}\{\{t\}\}[x]$ and can be uniquely extended to $\widetilde{K(a,c)}\{\{r\}\}\{\{t\}\}(x)$. u is Gauss extension of u to $\widetilde{K(a,c)}\{\{t\}\}(x)$. Then it is seen that the value group of u is $G_{\tilde{u}} = G_{u_1} \times Q$ (lexicographically ordered) and the residue field is $k_{\tilde{u}} = k_{\tilde{u}_{a,c}}(r^*)$. Since there is no immediate extension of $K(x, y)$ bigger than $\widetilde{K(a,c)}\{\{r\}\}\{\{t\}\}(x)$ it is maximally completion of $K(x, y)$ with respect to u.

Let v be a valuation on K, w be a residual transcendental extension of v to $K(x)$ which is defined by a minimal pair $(a,\delta) \in \bar{K} \times G_{\bar{v}}$ and f, γ, e, h be in preliminaries. We assume that u_1 is a residual transcendental extension of w to $K(x,y)$ which is defined by a minimal pair $(b,\mu) \in \bar{K} \times G_{\bar{v}}$. Let v_a, $v_{a,b}$, r, s, γ, ρ, e, e' be as above. We assume that lifting of monic, irreducible polynomial $P \in k_{v_b}[s^*]$, $P \neq s^*\, is\; p \in K[y]$ and lifting of monic, irreducible polynomial $Q \in k_{v_b}[r^*]$, $Q \neq r^*\, is\; q \in K[x]$. Each polynomial $F \in K[x,y]$ can be uniquelly written as $F = \sum_{i,j} F_{ij} p^i q^j$. Let c, d be suitable roots of p and q respectively. Define u as

$$u(F) = \inf_{i,j}(\psi\left(v_{c,d}\left(F_{ij}(c,d)\right)\right) + i\eta + j\beta) \tag{3}$$

Then u satisfies all valuation conditions on $K[x,y]$ and it can be extended to $K(x,y)$ where ψ is a map which is defined as $\psi : QG_v \to QG_v \times Q \times Q$, $\psi(d) = (d,0,0)$ for all $d \in QG_v$, $\eta = u(f) = (\gamma,0,0)$, $\eta = u(p) = (u_1(p),1,0)$, $\beta = u(q) = (u_1(q),0,1)$. Then $G_u = \psi(QG_v) + Z\eta + Z\beta$ and $k_u = k_{v_{c,d}}$.

THEOREM 2.3: Let v be a valuation on K, u_1 be the extension of v to $K(x,y)$ which is defined in (3). Then the maximally completion of $K(x,y)$ with respect to u is the field $\widetilde{K(a,c)}\{\{t\}\}\{\{z\}\}$.

Proof: u is the third kind r.a.f. extension of v to $K(x,y)$.[7] Let $u_1 \circ u_2 = u'$. u' is the second kind r.a.f extension of w to $K(x,y)$. The maximally completion of $K(x,y)$ with respect to u' is $\widetilde{K(x,c)}\{\{t\}\}$ where $\widetilde{K(x,c)}$ is the completion of $K(x,c)$ with respect to the w_c extension w_c of w to $K(x,c)$ and P goes to t. Using the above theorem and [3] it is obtained that the maximally completion of $K(x,y)$ with respect to $u = u_1 \circ u_2 \circ u_3$ is $\widetilde{K(c,d)}\{\{t\}\}\{\{z\}\}$ where $\widetilde{K(x,c)}$ is the completion of $K(c,d)$ with respect to the $v_{c,d} = \bar{v}\big|_{K(c,d)}$ and Q goes to z. Each $\alpha \in \widetilde{K(c,d)}\{\{t\}\}\{\{z\}\}$ is written as

$$\alpha = \sum_{m \in Z}\left(\sum_{n \in \mathrm{Z}} a_{mn} t^n\right) z^m,\;\; a_{mn} \in \widetilde{K(c,d)}$$

where $\widetilde{K(x,c)}$ is the completion of $K(c,d)$ with respect to $v_{c,d}$ which is the restriction of $\bar{v}$ to $K(x,c)$. Define

$$\tilde{u}(\alpha) = (\inf_m \tilde{u}'\left(\sum_n a_{mn} t^n\right), m_0)$$

where m_0 is the smallest integer number such that the infimum on the first component is reached. The value group of $\tilde{u}$ is $G_{\tilde{v}_{c,d}} = G_{u_1} \times Q \times Q$ (lexicographically ordered) and the residue field is $k_{\tilde{u}} = k_{\tilde{v}_{c,d}}$. Since there is no immediate extension of bigger than $K(c, \tilde{d})\{\{t\}\}\{\{z\}\}$ it is maximally completion of $K(x, y)$ with respect to u.

Aknowlegment: The authors are thankful to Prof. Hülya ISCAN for helpful suggestion on the improvement of the manuscript.

References

[1] V. Alexandru, N. Popescu, A. Zaharescu, *All valuations on* $K(x)$, (J.Math.Kyoto Univ, **30**, 1990), pp. 281-296

[2] V. Alexandru, A. Popescu, N. Popescu, *Completion of r.t. extensions of local fields (II)* (Sem .Mat.Univ.Padova, **Vol 100**, 1998), pp 57-66.

[3] V. Alexandru, N. Popescu, *Some elementary remarks about n-local fields*, (Math. Univ. Padova, **Vol 91** , 1994), pp 17-25.

[4] V. Alexandru, N. Popescu, A. Zaharescu, *A theorem of characterization of residual transcendental extensions of a valuation*, (J. Math. Kyoto Univ, **28**, 1988), pp 579-592.

[5] S. K. Khanduja, U. Garg, *Rank two valuations of* $K(x)$, (Mathematica, **37**, 1990), pp 97-105.

[6] Sudesh K. Khanduja, *On valuations of* $K(x)$, (Proceedings of the Edinburgh Mathematical Society, **35**, 1992), pp 419-426.

[7] F. Öke, H. İşcan, *An Introduction to Extensions of Valuations on* K *to* $K(x, y)$, (Journal of the Indian Math. Soc, **Vol 69, 1-4**, 2002), pp 33-44.

[8] V. Alexandru, A. Zaharescu, *On a class of valuations on* $K(x)$, (An. St. Univ. Ovidius Constantza, **2**, 1994), pp 120-136.

[9] O.F.G. Schilling, *The Theory of Valuations*, (AMS Surveys, **Nr. 4**, Providence, Rhode Island, 1950)