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Very Nearly $\mathfrak{c}$ Rings

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Abstract

Suppose $C \subset R$ are rings with a common ideal $A \neq 0$. Then they have similar properties:
C is prime if and only if R is prime; if they are prime rings then C is PI if and only if R is PI, and they have the same PI class.
Also we can lift from C to R chains of prime ideals not containing A, and we have the properties of LO, GU and INC which relate prime ideals of C not containing A with prime ideals of R.
We want to check which other properties pass from R to C via common ideals.
In a more general case C starts a chain of rings such that the last ring satisfies good properties, and each pair of consecutive rings has a common ideal.

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This paper is based in some of the results of my Ph. D. Thesis dissertation, which was carried out under the supervision of Professor Louis Rowen, at Bar Ilan University, Israel, and was accepted in the year 2007.
The name of my thesis was "Extending Ring-theoretical Properties via Common ideals".

1. *Introduction*

Much of algebraic geometry is based on chains of prime ideals in commutative noetharian rings, in particular affine rings.
Now if a ring C is not noetherian (or affine) but it has a common ideal with an overring R which is noetherian (or affine), then chains of prime ideals in C can be lifted to chains of prime ideals in R, and under some additional conditions ,

some properties of R like cattenarity or finite Krull dimension or Classical Krull dimension , are transmitted to C.

The same thing can occur when the ring C starts a chain of rings, $C \subset R_1 \subset R_2 \subset .... \subset R_l$ where every pair of consecutive rings has a common ideal.

The study of such a case for commutative rings is developed in the Paper: "Ring-Theoretic properties of commutative algebras of invariants" of Issai Kantor and Louis Rowen.

They define such a ring as a "nearly noetherian " or "nearly affine ring", or in general they give the definition of "nearly $\mathfrak{C}$ " rings where $\mathfrak{C}$ is a specific class of rings and the last component of the chain is in $\mathfrak{C}$.

The purpose of my thesis was to study "nearly $\mathfrak{C}$ " rings where $\mathfrak{C}$ is a class of rings such as noetherian, or affine, but now in the noncommutative case, in particular in the case of PI rings. (that is rings satisfying a polynomial identity).

In this paper I study what I defined in my thesis as "Very nearly $\mathfrak{C}$ rings".

Throughout the paper in the remarks I make references to some theorems and propositions of different books and authors.

2. " *Very nearly* $\mathfrak{C}$ *rings* "

<u>Definition 1</u>: "Very nearly $\mathfrak{C}$ rings".

Let $\mathfrak{C}$ denote a class of rings, which has the property that if $C \in \mathfrak{C}$, then, every homomorphic image of C is in $\mathfrak{C}$.

1) A ring C is very nearly $\mathfrak{C}$ of level 0 if $C \in \mathfrak{C}$.
2) Inductively, a ring C is very nearly $\mathfrak{C}$ of level t, with $t > 0$, if there is a ring R, such that:
   i) $C \subset R$.
   ii) R is very nearly $\mathfrak{C}$ of level $< t$.
   iii) R and C have a common ideal $A \neq 0$.
   iv) $\frac{C}{A}$ is very nearly $\mathfrak{C}$ of level $< t$.

<u>Definition 2</u>: " $DIM(C)$ "

We will define a *dimension* for a ring, based on the property of being "very nearly $\mathfrak{C}$ ", where $\mathfrak{C}$ denote a class of rings in which all $C \in \mathfrak{C}$ have a defined dimension $DIM_0(C)$.

We will take $\mathfrak{C}$ as rings with Krull dimension, or rings with Gabriel dimension, depending we work with noetherian rings or affine PI rings.

In both cases we have that if $R \in \mathfrak{C}$ then every homomorphic image of R is in $\mathfrak{C}$.

For a very *nearly* $\mathfrak{C}$ ring C we define a *dimension*:
1) $DIM(C) = DIM_0(C) \ \ \forall C \in \mathfrak{C}$.
2) Assuming we have already defined $DIM_r(W)$ for very nearly $\mathfrak{C}$ rings of minimal level $r \le l$, then:
If C has minimal level $l+1$, we define:

$$DIM_{l+1}(C) = \min_{R_1}\left\{\max_{L}\left\{DIM_r\left(\frac{C}{L}\right)+1, \quad DIM_l\left(R_1\right)\right\}\right\}$$

where L is a common ideal of C and R_1, $\frac{C}{L}$ has level $r < l$ and R_1 has level l, so that $DIM_r\left(\frac{C}{L}\right)$ and $DIM_l\left(R_1\right)$ are defined.

We define: $\boxed{DIM(C) = DIM_{l+1}(C)}$.

Remark for definition 2: If C is very nearly $\mathfrak{C}$ with notation of the definition of DIM ,we can choose a chain with
$DIM\ (R) \le DIM\ (R_1)$ for all possible R_1 containing C. Then

$$\begin{cases} 1) DIM(C) \ge DIM(R). \\ 2)\text{If for some common ideal } L \text{ of } C \text{ and some } R\ ,\ DIM\left(\frac{C}{L}\right)+1 \le DIM(R), \\ \text{then } DIM(C) = DIM(R)\ ,\ DIM\left(\frac{C}{L}\right) < DIM(C). \\ \text{Otherwise } DIM(C) = DIM(R_1) \ge DIM(R) \text{ for some } R_1 \text{ or } DIM(C) = DIM\left(\frac{C}{L}\right)+1 \\ \text{for some common ideal } L \text{ of } C \text{ and one of the } R_1. \text{ But also in these cases } DIM(C) \ge DIM(R). \end{cases}$$

Remark: If a ring is *very* nearly *noetherian*, then it is *very nearly* the class of rings having Krull dimension, and we use Krull dimension for the definition of $DIM(C)$.
If a ring is *very nearly affine* and it is also *PI*, then it is *very* nearly the class of rings having Gabriel dimension, and we use Gabriel dimension for the definition of $DIM(C)$.

Remark 3: *The prime radical*
For any ring R and any ideal I, we define $\sqrt{I}$ as the intersection of all prime ideals containing I .In particular, $\sqrt{I}$ is an ideal, and it is the smallest semiprime ideal in R containing I. (Lam [13]).

Remark 4:
Theorem *: The prime radical of a ring R is precisely the set of strongly nilpotent elements of R. In particular the prime radical of R is nil.*
(From McConnell and Robson, [16]).

Remark 5:
1) *In a right noetherian ring R there exist only finitely many minimal prime ideals,*
and there is a finite product of minimal prime ideals (repetition allowed) that equals zero. (From Goodearl-Warfield [6])

2) Theorem: *If R is a right noetherian ring and S is a* ***nil*** *multiplicatively closed subset of R, then S is nilpotent.*
In particular $\frac{\sqrt{I}}{I}$ is nilpotent: $\frac{\sqrt{I}}{I}=\sqrt{\frac{R}{I}}$ which is a nil ideal , and $\frac{R}{I}$ is noetherian.
(From Goodearl-Warfield [6]).

3) *From these two theorems we conclude that if we have the rings $C \subset R$, and R is noetherian, then the prime radical of C is a nil multiplicatively closed subset of R, then it is nilpotent.*

Remark 6:
1) *Let R be a ring having Krull (or Gabriel) dimension. If N is a nil ideal of R then N is nilpotent.*

2) *The prime radical and the Jaboson radical of a ring R which has Krull (or Gabriel) dimension is a nilpotent ideal.*

3) Lemma *: Let J be an ideal of a ring R with Krull dimension, then there exist prime ideals $P_1,...,P_n$ in Spec(R) such that $J \supset P_1 \cdot .... \cdot P_m$, $J \subset P_i$ for all i=1,..,n .*
(From Nastasescu-Oystaeyen [17])

Remark7:
Proposition**: (Rings with a common ideal: *LO, GU, INC*)**
Suppose the rings $C \subseteq R$ have a common ideal $A \neq 0$.
Then:
1) *LO (P), and $GU(P_1, P)$ hold for any prime ideal P of C such that $P \not\supseteq A$.*
2) *If $P \not\supseteq A$ is a prime ideal* of *C, then INC (a stronger version of INC) holds, that is, if Q_1* and *Q_2 are prime ideals of R such that $Q_1 \cap C \subseteq Q_2 \cap C = P$, then $Q_1 \subseteq Q_2$.*
Suppose the rings $C \subseteq R$ have a common ideal $A \neq 0$.
(A proof of this theorem is in Rowen [20]).

Proposition 8:

Suppose the rings $C \subseteq R$ have a common ideal $A \neq 0$.
1) *If Q is a prime ideal of R such that $A \not\subseteq Q$, then $Q \cap C$ is a prime ideal of C .*
2) *R is prime if and only if C is prime.*

Proof:
1) Denote $P = Q \cap C$.
If $B_1 B_2 \subseteq P \to AB_1 AB_2 \subseteq P,$ then $AB_1 \subseteq P$, or $AB_2 \subseteq P$.
Then for i=1 or i=2 we have $AB_i \subseteq P \subseteq Q \to AB_i R \subseteq Q \underset{A \not\subseteq Q}{\downarrow\to} B_i R \subseteq Q \to B_i \subseteq Q \cap C = P.$

2) Suppose R is a prime ring and $B_1 B_2 = 0$ where B_1 and B_2 are ideals of C . Then $AB_1 AB_2 \subseteq B_1 B_2 = 0$, and AB_1, AB_2 are left ideals of R , a prime ring , then if for example $AB_1 \neq 0$, we have that $AB_2 = 0$, then $ARB_2 = 0$, so $RB_2 = 0$, then $B_2 = 0$.
Now suppose C is a prime ring and: $B_1 B_2 = 0$, where B_1 and B_2 are ideals of R. Then $AB_1 AB_2 \subseteq B_1 B_2 = 0$, and AB_1, AB_2 are left ideals of C , a prime ring , then if for example $AB_1 \neq 0$, we have that $AB_2 = 0$, then $ACB_2 = 0$, so $CB_2 = 0$, then $B_2 = 0$.

Proposition 9:

If $C \subset R$ are PI algebras with a common ideal $A \neq 0$, with R prime. Then C and R have the same ring of quotients

Proof:
$Z(C) \subseteq Z(R)$, then $\mathfrak{Q}(C) \subseteq \mathfrak{Q}(R)$.) If the PI class of them is *n*, then they have the common ideal $H = \{ \ g_n(a, c_2, ..., c_t) : a \in A, c_i \in C\}^+ \neq 0$ between $Z(C)$ and $Z(R)$. Suppose $\frac{r}{s} \in \mathfrak{Q}(R)$ and $0 \neq a \in H$. Then $\frac{ra}{sa}$ where $ra \in A \subseteq C$ and $sa \in Z(R).H \subseteq H \subseteq Z(C)$. Then $\mathfrak{Q}(R) \subseteq \mathfrak{Q}(C)$ and then $\mathfrak{Q}(R) = \mathfrak{Q}(C)$.

Proposition 10:
Suppose the ring C is very nearly noetherian, then:
i) *If $I \lhd C$, then, there are only a finite number of prime ideals of C minimal over I .*
ii) *C satisfies ACC on prime ideals.*

Proof:
In the proof *A* and *R* are as in the definition 1.

i) Replacing I by $\sqrt{I}$, we can write $\sqrt{I}=\bigcap_i P_i$ for suitable prime ideals in C containing *I*. Since every prime ideal containing I contains a minimal prime over *I* one may replace P_i all by prime ideals minimal over I.

Let $\mathfrak{P}'$ denote those P_i containing A, and $\mathfrak{P}''$ denote those P_i not containing A. Define :

$$I'=\bigcap_i \{P_i\in\mathfrak{P}'\} \qquad I''=\bigcap_i \{P_i\in\mathfrak{P}''\}$$

We claim both sets $\mathfrak{P}'$ and $\mathfrak{P}''$ are finite. Indeed , $\dfrac{I'}{A}=\bigcap\left\{\dfrac{P}{A}:P\in\mathfrak{P}'\right\}$,

which by induction on the level, is a finite intersection of prime ideals $\dfrac{P_1}{A},\ldots,\dfrac{P_k}{A}$, since the level of $\dfrac{C}{A}$ is smaller than the level of *C*.

(And for level 1 of *C*, $\dfrac{C}{A}$ is noetherian, then there are only finitely many primes minimal over I, and containing A).

On the other hand, if we take a prime ideal Q_i of R lying over $P_i\in\mathfrak{P}''$, then by induction on the level $\bigcap Q_i$ is a finite intersection of prime ideals $Q_1'\cap\ldots\cap Q_m'$.

Then: $I''=C\cap(\bigcap Q_i)=C\cap(\bigcap_{i=1}^{m}Q_i')=\bigcap_{i=1}^{m}T_i$, where $T_i=C\cap Q_i'$

That is I'' is a finite intersection of prime ideals of C.

Any I -minimal prime contains $P_1\cdots P_k\cdot Q_1\cdots Q_m$, and thus one of these, so by minimality is one of these.

ii) Suppose we have an infinite ascending chain of prime ideals of C.

If one of them contains A, then passing to $\dfrac{C}{A}$ we get a contradiction (by induction on the level). (For level one, we have that $\dfrac{C}{A}$ is noetherian).

So we may assume each does not contain A.

But then applying LO, and GO yields an infinite ascending chain of prime ideals in R, which is a contradiction by induction on the level .

Remark : We can prove the same proposition under the hypothesis " *very nearly* $\mathfrak{C}$ " for $\mathfrak{C}$ the class of rings having Krull (or Gabriel) dimension.

3. Ore extension of a very nearly Noetherian ring, and the dimension of the Ore extension

Remark 11:

Proposition: *Let R be a right Noetherian ring, σ an automorphism, and δ a σ-derivation.*
Then $\kappa(R) \leq \kappa(R[x;\sigma,\delta]) \leq \kappa(R)+1$*; in particular, if δ =0, then* $\kappa(R[x;\sigma]) = \kappa(R)+1$.

Remark: If R is noetherian, then $G(R) = \kappa(R)+1$. And $R[x;\sigma,\delta]$ is also noetherian, then $\kappa(R)+1 \leq \kappa(R[x;\sigma,\delta])+1 \leq \kappa(R)+2$ implying that $G(R) \leq G(R[x;\sigma,\delta]) \leq G(R)+1$.
(From Robson-McConnell [16]).

In the next theorem we use the Definition 2 of dimension.

Theorem12:
Suppose R_0 is a very nearly noetherian prime ring of level l, as in definition 1, with l minimal, and has dimension $DIM(R_0) = n$. That is there is a chain $R_0 \subseteq R_1 ...... \subseteq R_l$ where R_l is a noetherian ring, satisfying all the conditions of definition 98, and A_i is the common ideal of R_i and *R_{i-1} for $i = 1,...,l$.*
Let be $W_0 = R_0[x;\sigma,\delta]$ the Ore extension of R_0, with $\sigma : R_0 \to R_0$ an automorphism and δ a σ-derivation.
Suppose that $\mathfrak{Q} = \mathfrak{Q}(R_0) = S^{-1}R_0, S_0 = Z(R_0) - \{0\}$, and :
1) *If we extend σ and δ* to *$\mathfrak{Q}$, then for all $i = 1,...,l$ we have*:
$\sigma|_{R_i} : R_i \to R_i$, *and* $\delta|_{R_i} : R_i \to R_i$
2) $\sigma(A_i) \subseteq A_i$, and $\delta(A_i) \subseteq A_i$ *for all* $i = 1,...,l$.
3) *The same conditions as 3) and 4) hold for $\frac{R_{i-1}}{L}$, L a common ideal of* R_{i-1} and R_i, *such that $\frac{R_{i-1}}{L}$ is very nearly noetherian of level $< l$.*

Then $W_0 = R_0[x;\sigma,\delta]$ is very nearly noetherian of level l, and has dimension $\leq n$ or $\leq n+1$.

<u>Proof:</u>

In the proof we can choose the chain where R_1 has the minimal dimension of all possible chains of minimal large *l*.
We have first that all the R_i are prime (as we prove in Proposition 8 for rings having a common ideal where one of them is prime). And they all have the same ring of fractions $\mathfrak{Q}$, as we proved in Proposition 9.

We can extend σ and δ to $\mathfrak{Q}$, then for all $i = 1,...,l$ we have by hypothesis that $\sigma\big|_{R_i} : R_i \to R_i$, and $\delta\big|_{R_i} : R_i \to R_i$ are respectively an automorphism and a σ-derivation.
Then we have the chains:

$$
\begin{array}{ll}
R_l — & R_l[x;\sigma,\delta] \\
\mid \;\rangle A_l & \mid \;\rangle A_l[x] \\
R_{l-1} — & R_{l-1}[x;\sigma,\delta] \\
\mid & \mid \\
\cdot & \cdot \\
\cdot & \cdot \\
\mid & \mid \\
R_1 — & R_1[x;\sigma,\delta] \\
\mid \;\rangle A_1 & \mid \;\rangle A_1[x] \\
R_0 — & R_0[x;\sigma,\delta]
\end{array}
$$

where $A_j[x]$ is a common ideal of $R_j[x;\sigma,\delta]$ and $R_{j-1}[x;\sigma,\delta]$:
σ is onto so each element of $R_j[x;\sigma,\delta]$ can be written in the form $\sum_{i=0}^{t} x^i r_i$.
Then if $r_i \in R_j$ and $a_l \in A_j$ we have:

$$\left(\sum_{i=0}^{t} x^i r_i\right)\left(\sum_{l=0}^{t} x^l a_l\right) \underset{r_i x^l = \sum x^{l_h} r_h}{\overset{=}{\downarrow}} \sum x^m r'_m a_l \in A_j[x]$$

We want to prove that $W_0 = R_0[x;\sigma,\delta]$ is nearly noetherian of level l with dimension $\leq n$ or $\leq n+1$.
The ring $W_l = R_l[x;\sigma,\delta]$ is an Ore extension of the noetherian ring R_l, and then it is also noetherian.

Now we have to verify the other conditions we need for W_0 to be a very nearly noetherian ring.
We will prove all this by induction on the level:

$\boxed{l=1}$

$R_0 \subseteq R_1$, $A_1 \neq 0$ is a common ideal of R_0 and R_1, and $W_0 = R_0[x;\sigma,\delta] \subset W_1 = R_1[x;\sigma,\delta]$, $A_1[x]$ is a nonzero common ideal of W_0 and W_1.

We want to prove that W_0 is a very nearly noetherian ring of level l, and has dimension $\leq n$ or $\leq n+1$.
We conclude first that W_1 is noetherian (The Ore extension of a left noetherian ring is also left noetherian).
$W_1 = R_1[x;\sigma,\delta]$ is an Ore extension of R_1, then using the result of Remark 11, $\kappa(R_1) \leq \kappa(R_1[x;\sigma,\delta]) \leq \kappa(R_1)+1$.
R_0 has dimension $DIM\ (R_0) = n,$ and then $DIM\ (R_1) = \kappa(R_1) = \kappa \leq n$.

Then $\kappa(W_1) = \kappa$ or $\kappa(W_1) = \kappa+1$.

$$\frac{W_0}{A_1[x]} \underset{\substack{\sigma(A_1)\subseteq A_1 \\ \delta(A_1)\subseteq A_1}}{\overset{\cong}{\downarrow}} \frac{R_0}{A_1}[x]$$ which is an Ore extension of a noetherian ring , and thus is noetherian.

And then W_0 is nearly noetherian of level 1 and has dimension $\leq n$ or $\leq n+1$:
not all the common ideals of W_0 and W_1 (or of W_0 and an other overring $\overline{W_1}$) are of the form $L[x]$ for L a common ideal of R_0 and R_1 , then may be that for some common ideal H of W_0 and W_1 we have that $DIM(W_0) = DIM\left(\frac{W_0}{H}\right)+1$,the minimum of such dimensions , but we are taking in the list also the case of $H = L[x]$ for L a common ideal of R_0 and R_1 , then the minimum must be of dimension $\leq n$ or $\leq n+1$.

Now suppose that if for all level $\boxed{m<l}$:

If we are under the conditions of the proposition, then the chain $W_0 = R_0[x;\sigma,\delta] \subseteq W_1 = R_1[x;\sigma,\delta] \ldots\ldots \subseteq W_m = R_m[x;\sigma,\delta]$
satisfies the conditions of Def. 1.

Now if the $\boxed{\text{level is } l}$:

We have to prove that the chain $W_0 \subseteq W_1 ...... \subseteq W_l$ satisfies all the conditions of Definition 1.
We know that the chain $R_1 \subseteq R_2 ...... \subseteq R_l$ satisfies the conditions of the proposition, that is R_1 is very nearly noetherian of level $l-1$, and dimension n_1, where $n_1 \leq n$.
Then by hypothesis of induction the chain $W_1 \subseteq W_2 ...... \subseteq W_l$ satisfies the conditions of Definition 1, that is W_1 is very nearly noetherian level $l-1$, and dimension $\leq n_1$, or $\leq n_1 + 1$.
Now:

$$\frac{W_0}{A_1[x]} \underset{\substack{\sigma(A_1)\subseteq A_1 \\ \delta(A_1)\subseteq A_1}}{\cong} \frac{R_0}{A_1}[x]$$ which is an Ore extension of a ring very nearly noetherian of level $l_2 < l$.

Then by induction on the level, it is very nearly noetherian of level l_2 .

Also may be that for some common ideal H of W_0 and W_1 (or of W_0 and an overring $\overline{W_1}$) we have that $DIM(W_0) = DIM\left(\frac{W_0}{H}\right) + 1$, the minimum of such dimensions , but we are taking in the list also the case of $H = L[x]$ for L a common ideal of R_0 and R_1 , then the minimum must be of dimension $\leq n$ or $\leq n+1$.
We conclude that W_0 is very nearly noetherian of level l and dimension $\leq n$ or $\leq n+1$.

4. Deformations of very nearly noetherian rings.

<u>Remark 13</u>:

<u>Deformation of a single Algebra</u>:

Let A be a k-algebra, and let α denote its multiplication, i.e. the k-bilinear map $\alpha : A \times A \to A$ defined by $\alpha(a,b) = ab$. Taking the power series ring $k[[t]]$, we define a *formal deformation* of a k-algebra A in a category $\mathfrak{C}$ as a $k[[t]]$-algebra, again in $\mathfrak{C}$, given by a $k[[t]]$-bilinear multiplication $\alpha_t : A[[t]] \times A[[t]] \to A[[t]]$, of the form $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + ...$, where each $\alpha_i : A \times A \to A$ is a k-bilinear map, extended to be $k[[t]]$ -bilinear.
We usually write A_t for the "deformed" algebra, i.e, the $k[[t]]$ -module $A[[t]]$ with multiplication α_t .

Note that A_t may not satisfy the identities satisfied by A.
(M.Hazewinkel and M.Gerstenhaber [8]).

Proposition 14:
If R is a left noetherian ring, then R_t is left noetherian.

Proof:
Let U be a left ideal of R_t, and let $\mathfrak{a}_i$ be the set of elements $a \in R$ such that a is the coefficient of x^i in a power series ax^i + terms of higher degree , lying in U.
Then $\mathfrak{a}_i$ is a left ideal of R, and $\mathfrak{a}_i \subset \mathfrak{a}_{i+1}$. The ascending chain of left ideals stops: $\mathfrak{a}_o \subset \mathfrak{a}_1 \subset .... \subset \mathfrak{a}_{r+1} = ...$.
Let a_{ij} $(i = 0,...,r$ and $j = 1,...,n_i)$ be the generators for the left ideals $\mathfrak{a}_i$, and let f_{ij} be power series in U having a_{ij} as beginning coefficient. Given $f \in U$, starting with a term of degree d, say $d \le r$, we can find elements $c_1,...,c_{n_d} \in R$ such that
$f - c_1 f_{d1} - ... - c_{n_d} f_{dn_d}$ starts with a term with degree $\ge d+1$.
Proceeding inductively, we may assume that $d > r$.
We use a linear combination $f - c_1^{(d)} x^{d-r} f_{r1} - ... - c^{(d)}{}_{n_r} x^{d-r} f_{rn_r}$ to get a power series starting with a term of degree $\ge d+1$. In this way, if we start with a power series with degree $d > r$, then it can be expressed as a linear combination of $f_{r1},..., f_{rn_r}$ by means of the coefficients $g_1(x) = \sum_{v=d}^{\infty} c_1^v x^{v-r},..., g_{n_r}(x) = \sum_{v=d}^{\infty} c_{n_r}^v x^{v-r}$.
And we see that f_{ij} generate our ideal U, as was to be shown.

(This proof is based in the proof for the power series ring $A[[t]]$ in case A is a commutative noetherian ring, in Serge Lang [14], which is based on Hilbert's basis theorem).

Proposition 15:

1) *If R is a right noetherian ring, then $\kappa(R[[t]]) = \kappa(R) + 1$, where $R[[t]]$ is the ring of power series with coefficients in R.* (Nastasescu-Oystaeyen [17]).
2) *The same result holds for the deformation R_t : every right ideal in R_t is a right ideal in the usual ring of power series.*
3) *If R is noetherian then $G(R) = \kappa(R) + 1$. And then $G(R[[t]]) = G(R) + 1$, $G(R_t) = G(R) + 1$.*
In the next Proposition we use the definition 2 of dimension.

Theorem 16:

If R_0 *is a very nearly noetherian algebra of level* l and dimension n, *then its deformation* R_{0_t} *is also very nearly noetherian, of level* l, and dimension $\leq n+1$.

Proof:

Suppose R_0 is very nearly noetherian of level *l*, as in Definition 1 , with *l* minimal. That is there is a chain $R_0 \subseteq R_1 \subseteq ...... \subseteq R_l$ where R_l is a noetherian k-algebra, satisfying all the conditions of Definition 1, and A_i is the common ideal of R_i and R_{i-1} for $i = 1,...,l$.
Let $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + ...$, where each $\alpha_i : R_0 \times R_0 \to R_0$ is a k-bilinear map, extended to be $k[[t]]$ -bilinear.
Suppose we can extend the k-bilinear maps α_i from R_i to R_{i+1}, for all $i = 1,...,l$.
Then we have a chain of $R_0 \subseteq R_1 \subseteq ...... \subseteq R_l$ rings and the chain of their deformations $R_{0_t} \subseteq R_{1_t} \subseteq ...... \subseteq R_{l_t}$.
By Proposition 13, R_{l_t} is noetherian.
A_i is the common ideal of R_i and R_{i-1} for $i = 1,...,l$ .Then $A_i[[t]]$, the set of power series with coefficients in the ideal A_i is a common ideal of R_{i-1_t} and R_{i_t} $i = 1,...,l$.
We want to prove that R_{0_t} is very nearly noetherian of level l .
We will prove all this by induction on the level:
$\boxed{l = 0}$ by Proposition 15.
$\boxed{l = 1}$
$R_0 \subseteq R_1$, $A_1 \neq 0$ is a common ideal of R_0 and R_1, and
$R_{0_t} \subseteq R_{1_t}$, $A_1[[t]]$ is a nonzero common ideal of R_{0_t} and R_{1_t} .

We want to prove that R_{0_t} is a very nearly noetherian ring of level 1 and dimension $\leq n+1$.
We conclude first that R_{1_t} is noetherian (by Proposition 14).
$\frac{R_{0_t}}{A_1[[t]]} \cong \frac{R_{0_t}}{A_1}[[t]]$ is a deformation of a ring very nearly noetherian of level $l = 0$, then noetherian .
Then it is also noetherian, by proposition13.
Then $\frac{R_{0_t}}{A_1[[t]]}$ is very nearly noetherian of level 0.

And then R_{0_t} is very nearly noetherian of level1.
By Proposition 109 have that $\kappa(R_{1_t}) = \kappa(R_1) + 1$.
R_0 has dimension $DIM\ (R_0) = n$, and then $\kappa(R_1) = \kappa \le n$ (if we choose R_1 as in remark for def. 2).
Then $\kappa(R_{1_t}) = \kappa + 1$
The dimension of R_{0_t} is $\le n+1$ because to define it we have to consider the dimensions of R_{1_t}, and of:

$\frac{R_{0_t}}{H}$ for H a common ideal of R_{0_t} and R_{1_t}, or another ring $\overline{R_{1_t}}$.
The ideal H not necessary is of the form $L[x]$ for an ideal L of R_0.
But if $DIM\ R_{0_t} = DIM\ \left(\frac{R_{0_t}}{H}\right) + 1$, then when we take the minimum of such dimensions we consider the case $H = L[x]$ for an ideal L of R_0.

Now suppose that if for all level $\boxed{m < l}$:
If we are under the conditions of the proposition, then the chain $R_{0_t} \subseteq R_{1_t} \ldots\ldots \subseteq R_{m_t}$
satisfies the conditions of Def. 1.

Now if the $\boxed{\text{level is } l}$:

We have to prove that the chain $R_{0_t} \subseteq R_{1_t} \ldots\ldots \subseteq R_{l_t}$ satisfies all the conditions of Definition 1.
We know that the chain $R_1 \subseteq R_2 \ldots\ldots \subseteq R_l$ satisfies the conditions of the proposition, that is R_1 is very nearly noetherian of level l-1.
Then by hypothesis of induction the chain $R_{1_t} \subseteq \ldots\ldots \subseteq R_{l_t}$ satisfies the conditions of Definition 1, that is R_{1_t} is very nearly noetherian of level $l-1$.
Now:
$\frac{R_{0_t}}{A_1[[t]]} \cong \frac{R_{0_t}}{A_1}[[t]]$ is a deformation of a ring very nearly noetherian of level l_2, with $l_2 < l$.
Then by induction on the level, it is very nearly noetherian of level $l_2 < l$.

We conclude that R_{0_t} is very nearly noetherian of level l.

The dimension of R_{0_t} is $\leq n+1$ because to define it we have to consider the dimensions of R_{1_t}, which is by induction $\geq$ of the dimension of R_1 and of : $\frac{R_{0_t}}{H}$ for H a common ideal of R_{0_t} and R_{1_t}, or another overring $\overline{R_{1_t}}$.

The ideal H not necessary is of the form $L[x]$ for an ideal L of R_0.

But if $DIM\ (R_{0_t}) = DIM\left(\frac{R_{0_t}}{H}\right)+1$, then when we take the minimum of such dimensions we consider the case $H = L[x]$ for an ideal L of R_0.

5. Finite extensions of a very nearly noetherian ring

Remark 17:

If R is a prime PI ring with centre C, and $S = C - \{0\}$, then the quotient ring $\Omega = S^{-1}R$ has centre $Z = S^{-1}C$ (the quotient field of C) ,and it is a central simple algebra , satisfying $\Omega = RZ$.

(Robson-MacConnell [16] .This result is due to Kaplansky).

Remark 18:

Let $S = \sum_{i=1}^{t} a_i R$ with each a_i normalizing R. *Then S is right noetherian if and only if R is right noetherian, and then* $\kappa(S) = \kappa(R)$.

(Robson-MacConnell [16]).

Remark 19: Some theorems about noetherian PI-rings:

Proposition: *Suppose R is a finitely spanned extension of W (that is $R = \sum_{i=1}^{u} W r_i$ with each $r_i \in C_R(W)$). If W satisfies ACC (ideals, resp.left ideals, resp. right ideals), then so does R.*

Theorem: *(Cauchon's theorem)*: *If R is a prime PI-ring satisfying ACC (ideals), then R is noetherian.*

(From L.Rowen [19])

Remark 20:

Theorem: *Let $R \subseteq S$ be PI-rings such that:*

(1) S is prime and integral over R.
(2) $\frac{R}{N(R)}$ is noetherian.

Then $\mathfrak{C}_R(N(R)) \subseteq \mathfrak{C}_S(0)$, i.e. the elements of R which are regular in $\frac{R}{N(R)}$ are actually regular in S. As a consequence, if $\mathfrak{Q}(R)$ exists, it embeds into $\mathfrak{Q}(S)$.

($N(R)$ is the prime radical of *R*, that is the intersection of all prime ideals, and $\mathfrak{C}_R(I) = \left\{ r \in R \,\middle|\, r + I \text{ is regular in } \frac{\mathrm{R}}{\mathrm{I}} \right\}$)
(A.Braun, N.Vonessen [1]).

Proposition 21:

Suppose R_0 is a very nearly noetherian PI ring of level l, and dimension n, as in definition 1, with l minimal. That is there is a chain $R_0 \subseteq R_1 \subseteq ...... \subseteq R_l$ where R_l is a noetherian ring, satisfying all the conditions of definition 1, and $0 \neq A_i$ is the common ideal of R_i and *R_{i-1} for $i = 1,...,l$.*
If we have elements $q_1,...,q_t \in Z(\mathfrak{Q}(R_0))$, such that the set $\{q_1,...,q_t\}$ is closed under the multiplication, we can define the chain
$$W_0 = \sum_{i=1}^{t} q_i R_0 \subseteq W_1 = \sum_{i=1}^{t} q_i R_1 \subseteq .... \subseteq W_l = \sum_{i=1}^{t} q_i R_l \quad .$$
Suppose $(A_i W_{i-1}) \cap R_{i-1} = A_i$, for i=1,…,l (and an analogous condition holds for $\frac{R_{i-1}}{L}$, L a common ideal of R_{i-1} and *R_i, such that $\frac{R_{i-1}}{L}$ is nearly noetherian of level $< l$).*

Then:

1) $W_0 = \sum_{i=1}^{t} q_i R_0$ *is very nearly noetherian ring of level l, and dimension $\leq n$.*
2) *If W_0 is a prime ring, then, $\mathfrak{Q}(R_0) = \mathfrak{Q}(R_1) = ... = \mathfrak{Q}(R_l) \subseteq \mathfrak{Q}(W_l)$.*

<u>Proof:</u>
Given the chain of rings $R_0 \subseteq R_1 \subseteq ...... \subseteq R_l$, we can build the chain:
$$W_0 = \sum_{i=1}^{t} q_i R_0 \subseteq W_1 = \sum_{i=1}^{t} q_i R_1 \subseteq .... \subseteq W_l = \sum_{i=1}^{t} q_i R_l .$$
First we check that the product of two elements of W_i is in W_i:

$$\left(\sum_{1}^{t} q_i h_i\right)\left(\sum_{1}^{t} q_k m_k\right) = \sum q_i q_k h_i m_k \underset{q_i q_k = q_j}{\underset{\downarrow}{=}} \sum q_j r_j \in \sum q_i R_i .$$

That is the chain we built is a chain of rings.

Using Proposition 8-2, we conclude that all R_i are prime rings.

Now, using Proposition 9 we conclude that all R_i have the same ring of quotients that is $\mathfrak{Q}(R_0) = \mathfrak{Q}(R_1) = ... = \mathfrak{Q}(R_l)$

$B_i = A_i W_i = \sum_{1}^{t} q_j A_i$ is a common ideal of W_i and W_{i-1}.

Using Remark 19 we have that W_l is a noetherian ring.
We want to prove that W_0 is very nearly noetherian of level l, and has dimension $\leq n$.
We will prove all this by induction on the level:

$\boxed{l=1}$

$R_0 \subseteq R_1$, $A_1 \neq 0$ is a common ideal of R_0 and R_1, and
$W_0 \subseteq W_1$, B_1 is a nonzero common ideal of W_0 and W_1.
We want to prove that W_0 is a nearly noetherian ring of level *l*.
We conclude first that W_1 is noetherian (by Remark19).
By Remark 18 we have that $\kappa(W_1) = \kappa(R_1)$.
R_0 has dimension $DIM(R_0)=n$, and then $\kappa(R_1) = \kappa \leq n$ (by remark for definition 2). Then $\kappa(W_1) = \kappa \leq n$.
The dimension of W_0 is $\leq n$ because to define it we have to consider the dimensions of W_1, and of :
$\frac{W_0}{H}$ for H a common ideal of W_0 and W_1, (or another overring $\overline{W_1}$) .
We see that $H \cap R_0 = L$ which is not necessary a common ideal ideal of R_0 and R_1.
If $H \cap R_0 = L$ is a common ideal of R_0 and R_1, and $\frac{C}{L}$ is very nearly noetherian of lower level than C, then $\frac{W_0}{H} = \sum_{1}^{t} (q_i + L)\frac{C}{L}$, and it is noetherian with $\kappa(\frac{W_0}{H}) = \kappa(\frac{C}{L})$.
Now

$$\frac{W_0}{B_1}=\frac{\sum_{i=1}^{t} q_i R_0}{B_1}=\sum_{i=1}^{t}(q_i+B_1)\frac{R_0+B_1}{B_1}\cong \sum_{i=1}^{t}(q_i+B_1\cap R_0)\frac{R_0}{B_1\cap R_0},$$ and we have that

$B_1\cap R_0=A_1$. Then $\frac{W_0}{B_1}$ is a finite centralizing extension of the ring $\frac{R_0}{B_1\cap R_0}$, with generators $(q_i+B_1)\in \mathfrak{Q}\left(\frac{R_0}{B_1\cap R_0}\right)$.

But $\frac{R_0}{B_1\cap R_0}$ is very nearly noetherian of level $l=0$.

Then $\frac{W_0}{B_1}$ it is also noetherian , and then very nearly noetherian of level 0.

And then W_0 is very nearly noetherian of level 1.

Now suppose that if for all level $\boxed{m<l}$:

If we are under the conditions of the proposition, then the chain $W_0\subseteq W_1......\subseteq W_m$
satisfies the conditions of Def. 1.

Now if the $\boxed{\text{level is } l}$:

We have to prove that the chain $W_0\subseteq W_1......\subseteq W_l$ satisfies all the conditions of Definition 1.

We know that the chain $R_1\subseteq R_2......\subseteq R_l$ satisfies the conditions of the proposition, that is R_1 is very nearly noetherian of level $l-1$, with dimension $n_1\leq n$.

Then by hypothesis of induction the chain $W_1\subseteq......\subseteq W_l$ satisfies the conditions of Definition1, that is W_1 is very nearly noetherian of level l-1, and has dimension $n_1\leq n$.

Now:

$\frac{W_0}{B_1}$ s a finite centralizing extension of the ring $\frac{R_0}{B_1\cap R_0}=\frac{R_0}{A_1}$,with generators $(q_i+B_1)\in Q\left(\frac{R_0}{B_1\cap R_0}\right)$, and $\frac{R_0}{A_1}$ is very nearly noetherian of level and $l_2<l$.

Then by induction on the level, $\frac{W_0}{B_1}$ it is very nearly noetherian of level $l_2<l$,

with dimension $\leq$ of the dimension of $\frac{R_0}{A_1}$.

We conclude that W_0 is very nearly noetherian of level l.
The dimension of W_0 is $\leq n$ because to define it we have to consider the dimensions of W_1, and of :
$\frac{W_0}{H}$ for H a common ideal of W_0 and W_1, (or another overring $\overline{W_1}$).
We see that $H \cap R_0 = L$ which is not necessary a common ideal ideal of R_0 and R_1.
If $H \cap R_0 = L$ is a common ideal of R_0 and R_1, and $\frac{C}{L}$ is very nearly noetherian of lower level than C, then $\frac{W_0}{H} = \sum_1^t (q_i + L)\frac{C}{L}$, and has dimension $DIM(\frac{W_0}{H}) \leq DIM(\frac{C}{L})$.
Now we want to prove that $\mathfrak{Q}(R_0) = \mathfrak{Q}(R_1) = ... = \mathfrak{Q}(R_l) \subseteq \mathfrak{Q}(W_l)$.
By Remark 20, we are under the same conditions: $R_l \subset W_l = \sum_1^t q_i R_l$, and they are both PI rings.
W_l is a finite centralizing PI extension of R_l, and then is an integral extension. R_l is noetherian. Then $\mathfrak{Q}(R_l)$ embeds into $\mathfrak{Q}(W_l)$.

6. Gabriel dimension. Does a very nearly noetherian ring have Gabriel dimension?

Remark 22:
Affine PI algebras:

a) *The classical Krull dimension of an affine PI ring is finite.* (C.Procesi [18]).
b) *Affine PI algebras need not have Krull dimension.*
c) *Affine PI rings satisfy the ascending chain condition on semiprime ideals.* (C. Procesi [18]).

d) *Gordon-Small* :***Affine PI rings have finite Gabriel dimension.****(And then all modules over an affine PI algebra have Gabriel dimension)*
This is a Corollary of the next theorem:

Theorem **:**(*Gordon-Small*) *: If R is a PI ring with the ascending chain condition on semiprime ideals and such that the prime radical of every factor ring is nilpotent , then G(R) exists and it satisfies G(R)=KdimR+1.*

The assumptions of the theorem are equivalent to:
If R is a PI ring such that:
R satisfies the ascending chain condition on prime ideals
For any proper ideal J of R there is a finite set of prime ideals containing J such that their product is contained in J.

Then G(R) exists and it satisfies G(R)=KdimR+1.
(Nastasescu-Oystaeyen [16]).

e) *If C is a subalgebra of a PI-algebra R, and R is either affine or else prime. Then C has a finite number of minimal prime ideals and its prime radical is nilpotent.*
(N. Vonessen [22]).

Proposition 23:
Suppose the ring C is very nearly noetherian.
Then :
i) The prime radical of C is nilpotent.
ii) If for some ideal I, $\frac{C}{I}$ is also nearly noetherian, then the prime radical $\sqrt{\frac{C}{I}} = \frac{\sqrt{I}}{I}$ is nilpotent.

Proof:
i) If C is very nearly noetherian, then it is a subring of a noetherian ring, and using remark 5 -3 we conclude that the prime radical of C is nilpotent.
ii) using i).

Theorem 24
Suppose the ring C is very nearly noetherian.
If I is a proper ideal of C, there are a finite number of prime ideals containing I, such that their product is contained in I.

Proof:
For level 0 we are in the noetherian case so the proposition is true in both cases.

For level *t>0*
In Proposition 10 we proved that if I is an ideal of a very nearly noetherian ring, there is a finite number of prime ideals minimal over I.

No we have to prove more than this: that there is a finite number of prime ideals containing I such that their product is contained in I.

In the proof we take A and R as in the definition 1, that is $C \subset R$, and A is the common ideal of them.

Case 1: $A \subseteq I$: Then $\frac{C}{I} \cong \frac{\frac{C}{A}}{\frac{I}{A}}$, and because that $\frac{C}{A}$ is nearly is nearly noetherian of lower level than C we conclude that there are a finite number of prime ideals containing I such that $\frac{P_1}{I} \cdot \ldots \cdot \frac{P_t}{I} = 0 \Rightarrow P_1 \ldots P_t \subseteq I$.

Case 2 : $A \not\subseteq I$

The ideal AIA is a common ideal of R and C. By induction of the level, there is a finite number of prime ideals of R containing AIA, such their product is contained in it:

$\left(Q_1 \ldots Q_k\right).\left(Q_{k+1} \ldots Q_t\right) \subseteq L$.

Suppose the k first factors don't contain A, and the *t-k* last factors contain A.

Then $Q_i \cap C$ is prime for i=1,...,k , and

$\left(Q_1 \cap C\right) \ldots (Q_k \cap C).A^{t-k} \subseteq AIA \cap C = AIA \Rightarrow \left(Q_1 \cap C\right) \ldots (Q_k \cap C).\left(A^{t-k} + I\right) \subseteq I$

But, $\left(A^{t-k} + I\right)$ is an ideal of C containing A so by case one we conclude there are a finite number of prime ideals of C $H_1, \ldots, H_l$ containing $\left(A^{t-k} + I\right)$ such that their product is contained in it.

Then

$\left(Q_1 \cap C\right) \ldots (Q_k \cap C) H_1 \ldots H_l \subseteq \left(Q_1 \cap C\right) \ldots (Q_k \cap C)\left(A^{t-k} + I\right) \subseteq I$

And for i=1,...,k $A \not\subseteq Q_i \cap C$, then :

$AIA \subseteq Q_i \cap C$ implies that $I \subseteq Q_i \cap C$.

An then we found a finite set of prime ideals of C containing I which product is contained in I.

Theorem 25:

Suppose the ring C is a PI ring very nearly noetherian.
Then:
i) *The classical Krull dimension, $K\dim C$, exists.*
ii) *C has Gabriel dimension, and G(C) =KdimC+1.*

Proof:

i) If C satisfies ACC on prime ideals then $K\dim C$ (the Classical Krull dimension) exists.

ii) As we saw in Remark 22, the theorem of Gordon Small is valid for PI rings with ACC on prime ideals such that for any proper ideal I of the ring there is a finite set of prime ideals containing I such that their product is contained in I.

By Proposition 10 and theorem 24 we conclude that the requirements of Gordon Small are satisfied in our case, then C has *Gabriel dimension,* and :
G(C)=KdimC+1.

7. Do very nearly affine PI rings have Gabriel dimension?

Remark 26 :

We have seen that affine PI algebras have finite Gabriel dimension. (Theorem of Gordon Small).
Also we know that they satisfy the ascending chain condition on semiprime ideals.
(C.Procesi, [18]).
A prime (or affine) PI ring has a finite number of minimal prime ideals whose product is zero.
We also know that for a subring of a prime PI ring the prime radical is nilpotent and there are a finite number of minimal prime ideals.(Braun-Vonhessen).
Now we will check if a very nearly affine ring, PI, has Krull and Gabriel dimension.

Theorem 27 :

Suppose the ring C is very nearly affine, and it is a prime PI ring.
Then:
i) *C has only a finite number of minimal prime ideals whose product is zero.*
ii) *C satisfies ACC on prime ideals.*
iii) *The classical KRULL dimension $K\dim C$ exists.*
iv) *If I is a proper ideal of C, there are a finite number of prime ideals containing I, such that their product is contained in I.*
v) *C has Gabriel dimension, and G(C) =KdimC+1.*

Proof:
In the proof we take A and R as in the definition 1, that is $C \subset R$, and A is the common ideal of them.
i) We know that a prime PI ring has a finite number of minimal prime ideals which product is zero.
ii) In Remark 22-c we view that affine PI rings satisfy the ascending chain condition on semiprime ideals.
Suppose we have an infinite ascending chain of prime ideals of C.
If one of them contains A, then passing to $\frac{C}{A}$ we get a contradiction (by induction on the level). (For level one, we have that $\frac{C}{A}$ is an affine PI ring, and then satisfies the ascending chain condition on semiprime ideals).
So we may assume each does not contain A.

But then applying LO and GU, yields an infinite ascending chain of prime ideals in R, which is a contradiction by induction on the level.

iii) We know that if C is a ring with ACC on prime ideals, then $K\dim C$ (Classical Krull dimension) exists.

iv) We prove this by induction on the level:

<u>For level 0</u> we are in the case C is an affine PI ring, and then $\frac{C}{I}$ is also affine PI, so the proposition is true by Remark 22 e) : There are is a finite number of minimal prime ideals in $\frac{C}{I}$, and its prime radical is nilpotent, that is

If $P_1,...,P_t$ are the minimal primes, then:

$$0=\left(\bigcap_i \frac{P_i}{I}\right)^k \supseteq \left(\frac{P_1}{I}.....\frac{P_t}{I}\right)^k \Rightarrow \left(\frac{P_1}{I}.....\frac{P_t}{I}\right)^k = 0 \Rightarrow \left(P_1....P_t\right)^k \subseteq I.$$

<u>For level *t>0*</u>

<u>Case 1</u> : $A\subseteq I$: Then $\frac{C}{I}\cong\frac{\frac{C}{A}}{\frac{I}{A}}$, and because that $\frac{C}{A}$ is nearly is nearly affine of lower level than C we conclude that there are a finite number of prime ideals containing I such that $\frac{P_1}{I}\cdot...\cdot\frac{P_t}{I}=0\Rightarrow P_1....P_t\subseteq I.$

<u>Case 2 :</u> $A\not\subseteq I$

The ideal AIA is a common ideal of R and C. By induction of the level, there is a finite number of prime ideals of R containing AIA, such their product is contained in it:

$(Q_1.....Q_k).(Q_{k+1}.....Q_t)\subseteq L$.

Suppose the *k* first factors don't contain A, and the *t-k* last factors contain A.

Then $Q_i\cap C$ is prime for i=1,...,k , and

$$(Q_1\cap C)....(Q_k\cap C).A^{t-k}\subseteq AIA\cap C=AIA \Rightarrow (Q_1\cap C)....(Q_k\cap C).\left(A^{t-k}+I\right)\subseteq I$$

But:

$\left(A^{t-k}+I\right)$ is an ideal of C containing A so by case one we conclude there are a finite number of prime ideals of C $H_1,...,H_l$ containing $\left(A^{t-k}+I\right)$ such that their product is contained in it.

Then

$$(Q_1\cap C)....(Q_k\cap C)H_1....H_l\subseteq(Q_1\cap C)....(Q_k\cap C)\left(A^{t-k}+I\right)\subseteq I$$

And for i=1,...,k $A\not\subseteq Q_i\cap C$, then :

$AIA\subseteq Q_i\cap C$ implies that $I\subseteq Q_i\cap C$.

An then we found a finite set of prime ideals of C containing I which product is contained in I.

v) It is a consequence of the Theorem of Gordon-Small, which says that if *R* is a PI ring with the ascending chain condition on semiprime ideals and such that the prime radical of every factor ring is nilpotent , then *G*(*R*) exists and it satisfies *G(R)=KdimR*+1.

We have seen at Remark 22 that the hypothesis of the theorem of Gordon –Small are equivalent to:
R is a PI ring such that satisfies the ascending chain condition on prime ideals and for any proper ideal *J* of *R* there is a finite set of prime ideals containing *J* such that their product is contained in *J*.
So, after proving ii) and iv), we conclude that *G*(*R*) exists and it satisfies :
G(*R*) =*KdimR*+1.

BIBLIOGRAPHY

[1]. A. Braun and N. Vonessen, *Integrality of PI-Rings,* Journal of Algebra 151, (1992), 39-79.

[2]. K. A. Brown and K. R. Godearl, *Lectures on Algebraic Quantum groups,* Advanced Curses in Mathematics, CRM, Barcelona. (2002).

[3]. R. Gilmer**,** *Multiplicative Ideal Theory,* Pure and Applied Mathematics 12, New York (1972).

[4]. B.J.Gardner and R. Wiegandt, *Radical Theory of Rings,* Pure and Applied Mathematics 261, (2004).

[5]. A. Goldie and G. Michler, *Ore Extensions and Polycyclic Group Rings,* J.London Math.Soc.(2), 9. (1974), 337-345.

[6]. K.R.Goodearl & R.B.Warfield, Jr, *An Introduction to Noncommutative Noetherian Rings*, London Math.Soc.Student Texts 16. (Cambridge Univ. Press, 1989).

[7]. K.R.Goodearl and E.S.Letzter, *Prime Ideals in Skew and q-Skew Polynomial Rings,* Memories of the American Mathematical Society, Vol. 109, Number 521.

[8]. M. Hazewinkel and Murray Gerstenhaber, *Deformation Theory of Algebras and Structures and Applications,* NATO ASI Series, Series C: Mathematical and Physical Sciences –Vol. 247, Kluer Academic Publishers.

[9]. A.G.Heinicke and J.C.Robson: *Normalizing extensions: Prime Ideals and Incomparability.* Journal of Algebra 91, (1984), 142-165.

[10]. R..S. Irving**,** *Prime Ideals of Ore Extensions over commutative rings,* I, II, Journal of Algebra 56 and 58 (1979), pages 315-342, 399-423 .

[11]. S. Jandrup , *Representations of Skew Polynomial Algebras,* Proceedings of the American Mathematical Society, Vol. 128, Number 5, (1999), 1301-1305.

[12]. I. Kantor and L. Rowen, *Ring-Theoretic Properties of Commutative Algebras of Invariants.* Journal of Algebra, 266, (2003), 239-260.

[13]. T.Y.Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Math.Vol. 131, Springer Verlag, Berlin-Heidelberg-New York (2001).

[14]. S. Lang, *Algebra*, Graduate Texts in Math. 211, Springer Verlag, Berlin-Heidelberg-New York

[15]. A. Leroy and J. Matczuk, *Ore Extensions Satisfying a Polynomial Identity*. Polish KBN grant No. 1 PO3A 032 27.

[16]. J.C.McConnell and J.C.Robson, *Noncommutative Noetherian Rings*, New York Wiley-Interscience (1987).

[17]. C. Nastasescu and F. van Oystaeyen, *Dimensions of Ring Theory,* Mathematics and Its Applications, Reidel Publishing Company, Dordrecht, Holland (1987) .

[18]. C. Procesi ,*Rings with polynomial identities ,* Pure and Applied Mathematics , 17 (1973) .

[19]. L. H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York –London.

[20]. L. H. Rowen, *Ring Theory*, Vols. I, II, Academic Press, New York (1988).

[21]. W. Schelter, *Integral Extensions of Rings satisfying a Polynomial Identity*, Journal of Algebra 40 (1976).

[22]. N. Vonessen, *Actions of lineary reductive groups on affine PI-algebras*, Memories of the American Mathematical Society, 414 (1989).

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