

Wreath Product Symmetric Functions

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Abstract

We systematically study wreath product Schur functions and give a combinatorial construction using colored partitions and tableaux. The Pieri rule and the Littlewood-Richardson rule are studied. We also discuss the connection with representations of generalized symmetric groups.

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1 Introduction

Since Jacobi and Frobenius, symmetric functions have played a fundamental role in representations theory and number theory. The character formula

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of the symmetric group in terms of Schur functions and power sums is undoubtedly one of the most beautiful chapters in group theory and algebraic combinatorics. Schur developed Frobenius theory further using his rational homogeneous functions, and again symmetric functions were the most important techniques in his work on generalizations of Frobenius character theory.

In Macdonald's book [4], one can see other interesting examples of applications of symmetric functions into group theory. Among them, Specht's generalization [9] of Schur functions deserves special mention. Let G be any finite group, the semidirect product $G^n \rtimes S_n$ denoted by $G \sim S_n$ is called the wreath product of the finite group G and the symmetric group S_n . The special case $1 \sim S_n$ is exactly the symmetric group S_n , while $\mathbb{Z}_2 \sim S_n$ is the hyperoctahedral group. Recently the wreath products are used in the generalized McKay correspondence [1] to realize the affine Lie algebras of simply laced types. The wreath products have also been studied in various interesting contexts [6, 2, 5]. However the basic formulation of the characters remains complicated and formidable.

It is well-known that Schur functions can also be defined in terms of Young tableaux in Stanley's work [10]. One purpose of this paper is to give a simpler and more elementary description of wreath product symmetric functions using colored tableaux. We achieve this by first considering tensor product of Schur functions and then pass the results to the case of wreath product using linear transformations. In our simplified approach one can see the structures of wreath product symmetric functions easily. In the same spirit we also obtain the description of the Littlewood-Richardson rule for the wreath product Schur functions. We deviate slightly from Macdonald's inner product by requiring that the power sum associated with irreducible characters of G to be invariant under the complex conjugation, which will facilitate us to pass the results from tensor product to wreath product easily.

The paper is organized into five parts. After introduction (section one) we give a preliminary review of the main results concerning Schur functions to set the notations and for later usage in Section two. In Section three we studied tensor products of symmetric functions and formulate several bases modeled on one variable case. The notion of colored partitions and tableaux is the fundamental combinatorial objects for us to formulate results in tensor products. In Section four wreath product Schur functions are introduced by two sets of power sum symmetric functions indexed by colored partitions, which correspond to conjugacy classes and irreducible characters of $G \sim S_n$. We show how to pass from tensor products to wreath products by carefully treating the inner product. In section five we give generalized Littlewood-Richardson rule and discuss their special cases, and finally we discuss the relationship between symmetric functions and characters of wreath products of symmetric groups.

2 Symmetric functions

In this section we review some basic materials on symmetric functions that will be used in our later discussion. Almost all statements are standard and can be found in [4] or in a slightly different notation in [10].

A *partition* λ of an non-negative integer n , denoted as $\lambda \vdash n$, is an integral decomposition of n arranged in decreasing order: $n = \lambda_1 + \dots + \lambda_l$, $\lambda_1 \geq \dots \geq \lambda_l \geq 0$. If the last part $\lambda_l \neq 0$, the partition has *length* $l(\lambda) = l$, and we also denote by $|\lambda| = n$. One sometimes uses the notation $\lambda = (1^{m_1} 2^{m_2} \dots)$ if there are m_i parts equal to i in λ .

The *Ferrer diagram* associated with the partition $\lambda \vdash n$ with $l(\lambda) = l$ consists of l layers of n boxes aligned to the left from the top to the bottom. The partition corresponding to the flip of the Ferrer diagram of λ along the northwest-to-southeast axis is called the *conjugate partition* λ' . For example the Ferrer diagram (4322) and its conjugate diagram are shown in Figure 1.

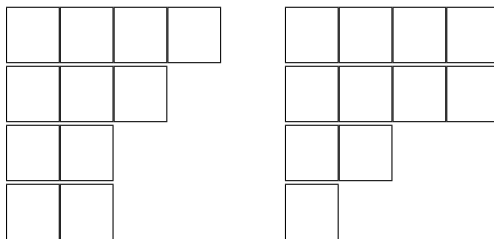


Figure 1. $\lambda = (4, 3, 2, 2)$ and $\lambda' = (4, 4, 2, 1)$

There are two partial orders in the set of partitions. The first one is the *dominance order* \geq . Let λ and μ be two partitions, $\lambda \geq \mu$ means that $\lambda_1 \geq \mu_1$, $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$, etc. The second one is the *containment order* \supset . If all parts of λ are larger than corresponding parts of μ , we say that λ contains μ and denote by $\lambda \supset \mu$.

Let $\Lambda_{\mathbb{Z}}$ be the ring of symmetric functions in the $x_i (i \in \mathbb{N})$ over \mathbb{Z} . For each non-negative integer n , the *elementary symmetric function* e_n is defined by

$$e_n = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n},$$

where the sum is taken formally, or it is in the sense of the inverse direct limit determined by the relations: $e_m(x_1, \dots, x_n, 0, \dots, 0) = e_n(x_1, \dots, x_n)$ whenever $m \geq n$.

For each partition λ with $l(\lambda) \leq n$ the *monomial symmetric function* m_λ is defined by $m_\lambda(x_1, \dots, x_n) = x_1^{\lambda_1} \cdots x_{\lambda_n}^{\lambda_n} +$ distinct permutations, where we take $\lambda_{l+1} = \dots = \lambda_n = 0$ if necessary.

For each partition λ we define

$$\begin{aligned} e_\lambda &= e_{\lambda_1} \cdots e_{\lambda_l}, \\ m_\lambda &= m_{\lambda_1} \cdots m_{\lambda_l}. \end{aligned}$$

The *homogeneous symmetric function* h_n is defined by $h_n = \sum_{|\lambda|=n} m_\lambda$. For any partition λ we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_l}$

As a graded vector space we have

$$\Lambda_{\mathbb{Z}} = \bigoplus_{\lambda} \mathbb{Z}e_\lambda = \bigoplus_{\lambda} \mathbb{Z}m_\lambda = \bigoplus_{\lambda} \mathbb{Z}h_\lambda, \quad (1)$$

where the sums are over all partitions. Moreover $\Lambda_{\mathbb{Z}} = \mathbb{Z}[e_1, e_2, \dots] = \mathbb{Z}[h_1, h_2, \dots]$ as polynomial rings, and $\{e_n\}$ and $\{h_n\}$ are algebraically independent over \mathbb{Z} . Subsequently the sets $\{e_\lambda\}$, $\{m_\lambda\}$ and $\{h_\lambda\}$ are all \mathbb{Z} -linear bases of $\Lambda_{\mathbb{Z}}$.

For each non-negative integer n , the *power sum* symmetric function $p_n = \sum_i x_i^n$ is an element in $\Lambda_{\mathbb{Q}}$, and we define for each partition λ

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}. \quad (2)$$

It is well-known that the $\{p_\lambda\}$ forms a \mathbb{Q} -linear basis in $\Lambda_{\mathbb{Q}}$, and subsequently $\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \dots]$. Moreover the p_n are algebraically independent over \mathbb{Q} .

The space $\Lambda_{\mathbb{Z}}$ has a \mathbb{Z} -valued bilinear form \langle, \rangle defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}. \quad (3)$$

Equivalently the symmetric bilinear form can also be defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\mu, \quad (4)$$

where $z_\lambda = \prod_i i^{m_i} m_i!$. The distinguished orthonormal basis—*Schur functions* are given by the following triangular relations [4]:

$$\begin{aligned} s_\lambda &= m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu, & \text{for some } c_{\lambda\mu} \in \mathbb{Z} \\ \langle s_\lambda, s_\mu \rangle &= \delta_{\lambda\mu} \end{aligned}$$

Let λ and μ be partitions such that $\mu \subset \lambda$. The *skew diagram* λ/μ is the set-theoretic difference $\lambda - \mu$. A maximal connected component of λ/μ is called a connected component. Each connected component itself is a skew diagram. For example the skew Ferrer diagram $\lambda/\mu = (6,4,2,2)/(4,2)$ has three components and is shown by Figure 2.

A *horizontal strip* is a skew Ferrer diagram λ/μ with no two boxes in the same column, i.e., $\lambda'_i - \mu'_i \leq 1$. A *vertical strip* is skew Ferrer diagram λ/μ with no two boxes in the same row, i.e., $\lambda_i - \mu_i \leq 1$

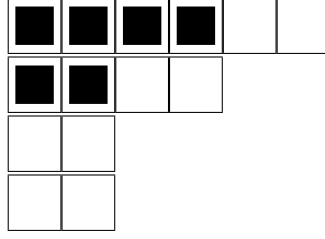


Figure 2. Skew diagram $(6422)/(42)$

The multiplication of two Schur functions is again a symmetric function and thus can be expressed as a linear combination of Schur functions:

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu}, \quad c_{\lambda\mu}^{\nu} \in \mathbb{Z}_+. \quad (5)$$

The structure constants $c_{\lambda\mu}^{\nu}$ are called the *Littlewood-Richardson coefficients*. It is known that $c_{\lambda\mu}^{\nu} \neq 0$ unless $|\nu| = |\lambda| + |\mu|$ and $\nu \supset \lambda, \mu$.

The *skew Schur function* $s_{\lambda/\mu} \in \Lambda_{\mathbb{Z}}$ is then defined by

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}. \quad (6)$$

It is clear that $\deg(s_{\lambda/\mu}) = |\lambda| - |\mu|$ and $s_{\lambda/\mu} = 0$ unless $\mu \subset \lambda$, i.e. $s_{\lambda/\mu}$ is defined for the skew Ferrer diagram λ/μ .

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two sets of variables.

Proposition 2.1 [4] (a) *If the connected components of the skew diagram λ/μ are θ_i , then $s_{\lambda/\mu}(x) = \prod_i s_{\theta_i}(x)$. In particular, $s_{\lambda/\mu}(x) = 0$ if $\mu \not\subseteq \lambda$.*

(b) *The skew Schur symmetric function $s_{\lambda}(x, y)$ satisfies*

$$s_{\lambda/\mu}(x, y) = \sum_{\nu} s_{\lambda/\nu}(x) s_{\nu/\mu}(y), \quad (7)$$

where the sum runs through all the partition ν such that $\lambda \supset \nu \supset \mu$.

(c) *In general, the skew Schur function $s_{\lambda/\mu}(x^{(1)}, \dots, x^{(n)})$ can be written as*

$$s_{\lambda/\mu}(x^{(1)}, \dots, x^{(n)}) = \sum_{(\nu)} \prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}(x^{(i)}), \quad (8)$$

where the sum runs through all sequences of partitions $(\nu) = (\nu^{(n)}, \dots, \nu^{(0)})$ such that $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \dots \subset \nu^{(n)} = \lambda$.

If there is only one variable, the Schur function can be easily computed.

Proposition 2.2 [4] (a) *We have $s_{\lambda}(x) = 0$ when $l(\lambda) > 1$. When λ/μ is a horizontal strip, $s_{\lambda/\mu}(x) = x^{|\lambda|-|\mu|}$.*

(b) *More generally $s_{\lambda/\mu}(x_1, \dots, x_n) = 0$ unless $\lambda'_i - \mu'_i \leq n$ for each i .*

In order to describe Schur functions combinatorially we introduce the concept of Young tableaux. For a diagram λ , a *semistandard tableau* T of shape λ is an insertion of natural numbers $1, 2, \dots$ into λ such that the rows weakly increase and the columns strictly increase. The *content* μ of a tableau T is the composition $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, where μ_i equals the number of i 's in T . We will write $x^T = x_1^{\mu_1} \cdots x_n^{\mu_n}$. In many situations we want to restrict the largest possible integer n inserted into the diagram λ , and the notation x^T makes it clear when one uses $x = (x_1, \dots, x_n)$.

A *semistandard Young Tableau*, denoted by $T_{\lambda\mu}$, is a tableau T having shape λ and content μ . For example, see Figure 3. The *Kostka number* $K_{\lambda\mu}$ is the number of semistandard tableaux of shape λ and content μ .

1	1	1	4
2	3	3	
3	4		
5	5		

Figure 3. $T_{\lambda\mu}$ with $\lambda = (4, 3, 2, 2)$, $\mu = (3, 1, 3, 2, 2)$

Let λ and μ be partitions such that $\lambda \supseteq \mu$ (i.e. $\lambda_i \geq \mu_i$ for all i). A *semistandard Young tableau of skew shape* λ/μ with content ν , is a diagram of shape λ/μ whose boxes have been filled with $|\nu|$ positive integers that are weakly increasing in every row and strictly increasing in every column. For example, the skew tableau of shape $\lambda/\mu = (6, 5, 4, 3)/(4, 2)$ with content $\nu = (2, 3, 3, 0, 3, 1)$ is shown below, where we will also write it as the array.

			2	2
		1	1	5
2	3	3	3	
4	5	6		

Similarly the *Kostka number* $K_{\lambda/\mu, \nu}$ is the number of semistandard tableaux of shape λ/μ and content ν .

Combining Propositions 2.2 and 2.1 one easily gets the following result

Proposition 2.3 *The Schur function $s_{\lambda/\mu}(x)$ can be expressed as a summation of all monomials x^T attached to Young tableaux of shape λ/μ :*

$$s_{\lambda/\mu}(x) = \sum_T x^T, \quad (9)$$

where the sum runs through all skew semistandard Young tableaux of shape λ/μ .

Proof. In Proposition 2.2(c) we take $x^{(i)} = x_i$ and using Proposition 2.1 it follows that

$$s_{\lambda/\mu}(x_1, \dots, x_n) = \sum_{(\nu)} \prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}(x_i) = \sum_{(\nu)} \prod_{i=1}^n x_i^{|\nu^{(i)}| - |\nu^{(i-1)}|},$$

where the sum runs through all sequences of partitions $(\nu) = (\nu^{(n)}, \dots, \nu^{(0)})$ such that $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \dots \subset \nu^{(n)} = \lambda$ and $\nu^{(i)}/\nu^{(i-1)}$ are horizontal strips. For each horizontal strip $\nu^{(i)}/\nu^{(i-1)}$, we assign i in the diagram λ/μ where the horizontal strip occupies and thus obtain a semistandard Young tableau of shape λ/μ with content $T = (|\nu^{(1)}| - |\nu^{(0)}|, |\nu^{(2)}| - |\nu^{(1)}|, \dots, |\nu^{(n)}| - |\nu^{(n-1)}|)$. In this way the sequence of such partitions gives rise a semistandard Young tableau of shape λ/μ with content T . Conversely given a semistandard Young tableau of shape λ/μ with content T ($|T| = |\lambda| - |\mu|$), let $\nu^{(i)}$ be the subdiagram consisting of the diagram μ together with entries numbered $1, \dots, i$. Then the skew diagram $\nu^{(i)}/\nu^{(i-1)}$ is horizontal and gives rise a sequence of partitions $(\nu) = (\nu^{(n)}, \dots, \nu^{(0)})$ such that $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \dots \subset \nu^{(n)} = \lambda$ and $\nu^{(i)}/\nu^{(i-1)}$ are horizontal strips. Therefore we have

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

where T runs through Young tableaux of shape λ/μ . \square

Corollary 2.4 *We have $s_{\lambda/\mu} = \sum_{\nu} K_{\lambda/\mu, \nu} m_{\nu}$.*

3 Tensor product of symmetric functions

Fix a natural number r we consider the tensor product $\Lambda_{\mathbb{Z}}^{\otimes r}$. Clearly the tensor products of basis elements from a given basis of $\Lambda_{\mathbb{Z}}$ will form a basis for the tensor product space $\Lambda_{\mathbb{Z}}^{\otimes r}$.

To parametrize the basis elements, we introduce the notion of *colored partitions*. A colored partition $\underline{\lambda}$ is a partition-valued function: $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$, where $\lambda^{(i)}$ are partitions. Here we intuitively color the i th diagram $\lambda^{(i)}$ by the color i . Also we set $I = \{0, 1, \dots, r-1\}$, the set of the indices or *colors*. If we want to specify the number of colors or partitions inside $\underline{\lambda}$, we also say that $\underline{\lambda}$ is a r -colored partition. We denote by $|\underline{\lambda}|$ the sum of all weights: $|\underline{\lambda}| = |\lambda^{(0)}| + \dots + |\lambda^{(r-1)}|$. Thus we will also use the terms such as the *colored Ferrer diagrams*, *colored skew Ferrer diagrams* etc.

The dominance order \leq and containment order \subset can be extended to colored partitions as follows. For two colored partitions $\underline{\lambda}$ and $\underline{\mu}$, $\underline{\lambda} \leq \underline{\mu}$ means that $\lambda^{(i)} \leq \mu^{(i)}$ for $i \in I$. Similarly $\underline{\lambda} \subset \underline{\mu}$ means that $\lambda^{(i)} \subset \mu^{(i)}$ for $i \in I$.

Let u_λ be any element from our bases $\{e_\lambda\}$, $\{m_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$ or $\{s_\lambda\}$ of $\Lambda_{\mathbb{Z}}$ in Section 2. Let $x^{(i)} = (x_{i,1}, x_{i,2}, \dots)$ be the variables in the i th ring $\Lambda_{\mathbb{Z}}$ ($i \in I$), and for any r -colored partition $\underline{\lambda}$ we define

$$u_{\underline{\lambda}} = u_{\lambda^{(0)}}(x^{(0)}) \cdots u_{\lambda^{(r-1)}}(x^{(r-1)}). \quad (10)$$

Then the $u_{\underline{\lambda}}$ forms a basis for the tensor product space $\Lambda_{\mathbb{Q}}^{\otimes r}$. We also define similarly

$$z_{\underline{\lambda}} = z_{\lambda^{(0)}} \cdots z_{\lambda^{(r-1)}}.$$

We canonically extend the bilinear scalar product of $\Lambda_{\mathbb{Z}}$ to $\Lambda_{\mathbb{Z}}^{\otimes r}$. Namely we define on $\Lambda_{\mathbb{Z}}^{\otimes r}$ by

$$\langle u_1 \otimes \cdots \otimes u_r, v_1 \otimes \cdots \otimes v_r \rangle = \langle u_1, v_1 \rangle \cdots \langle u_r, v_r \rangle$$

for any two sets of elements $\{u_i\}, \{v_i\} \subset \Lambda_{\mathbb{Z}}$ and extend bilinearly. Then we have the following result.

Theorem 3.1 (a) *The tensor product ring $\Lambda_{\mathbb{Z}}^{\otimes r}$ of symmetric functions has the following four sets of linear bases: $\{e_{\underline{\lambda}}\}$, $\{h_{\underline{\lambda}}\}$, $\{m_{\underline{\lambda}}\}$ and $\{s_{\underline{\lambda}}\}$. The set $\{p_{\underline{\lambda}}\}$ is a \mathbb{Q} -linear basis for $\Lambda_{\mathbb{Q}}^{\otimes r}$. Namely we have*

$$\begin{aligned} \Lambda_{\mathbb{Z}}^{\otimes r} &= \bigoplus_{\underline{\lambda}} \mathbb{Z}e_{\underline{\lambda}} = \bigoplus_{\underline{\lambda}} \mathbb{Z}m_{\underline{\lambda}} = \bigoplus_{\underline{\lambda}} \mathbb{Z}h_{\underline{\lambda}} = \bigoplus_{\underline{\lambda}} \mathbb{Z}s_{\underline{\lambda}}, \\ \Lambda_{\mathbb{Q}}^{\otimes r} &= \bigoplus_{\underline{\lambda}} \mathbb{Q}p_{\underline{\lambda}}, \end{aligned}$$

where the sums are over all colored partitions. Moreover

$$\begin{aligned} \Lambda_{\mathbb{Z}} &= \bigotimes_{i=0}^{r-1} \mathbb{Z}[e_{i,1}, e_{i,2}, \dots] = \bigotimes_{i=0}^{r-1} \mathbb{Z}[h_{i,1}, h_{i,2}, \dots] \\ &\simeq \mathbb{Z}[e_1, e_2, \dots]^{\otimes r} \simeq \mathbb{Z}[h_1, h_2, \dots]^{\otimes r} \end{aligned}$$

as polynomial rings, and $\{e_n\}$ and $\{h_n\}$ are algebraically independent over \mathbb{Z} . Subsequently the sets $\{e_{\underline{\lambda}}\}$, $\{m_{\underline{\lambda}}\}$ and $\{h_{\underline{\lambda}}\}$ are all \mathbb{Z} -linear bases of $\Lambda_{\mathbb{Z}}$.

(b) *The bases $\{h_{\underline{\lambda}}\}$ and $\{m_{\underline{\lambda}}\}$ are dual under the scalar product:*

$$\langle h_{\underline{\lambda}}, m_{\underline{\mu}} \rangle = \delta_{\underline{\lambda}\underline{\mu}}.$$

(c) *The power sum symmetric functions are orthogonal:*

$$\langle p_{\underline{\lambda}}, p_{\underline{\mu}} \rangle = \delta_{\underline{\lambda}\underline{\mu}} z_{\underline{\lambda}}.$$

(d) *The Schur functions $s_{\underline{\lambda}}$ are orthonormal and uniquely determined by the triangular relations:*

$$\begin{aligned} s_{\underline{\lambda}} &= m_{\underline{\lambda}} + \sum_{\underline{\mu} < \underline{\lambda}} c_{\underline{\lambda}\underline{\mu}} m_{\underline{\mu}}, \quad \text{for some } c_{\underline{\lambda}\underline{\mu}} \in \mathbb{Z} \\ \langle s_{\underline{\lambda}}, s_{\underline{\mu}} \rangle &= \delta_{\underline{\lambda}\underline{\mu}}. \end{aligned}$$

Proof. All statements are trivial when one uses the tensor product structure $\Lambda_{\mathbb{Z}}^{\otimes r}$ or $\Lambda_{\mathbb{Q}}^{\otimes r}$. For example, one has

$$\begin{aligned} s_{\underline{\lambda}} &= s_{\lambda^{(0)}} \cdots s_{\lambda^{(r-1)}} \\ &= \left(\sum_{\mu^{(0)} \leq \lambda^{(0)}} c_{\lambda^{(0)}\mu^{(0)}} m_{\mu^{(0)}} \right) \cdots \left(\sum_{\mu^{(r-1)} \leq \lambda^{(r-1)}} c_{\lambda^{(r-1)}\mu^{(r-1)}} m_{\mu^{(r-1)}} \right) \\ &= m_{\underline{\lambda}} + \sum_{\underline{\mu} < \underline{\lambda}} c_{\underline{\lambda}\underline{\mu}} m_{\underline{\mu}}, \quad \text{for some } c_{\underline{\lambda}\underline{\mu}} \in \mathbb{Z} \end{aligned}$$

□

Let $\underline{\lambda}$ and $\underline{\mu}$ be colored partitions such that $\underline{\mu} \subset \underline{\lambda}$. A *colored skew diagram* $\underline{\lambda}/\underline{\mu}$ is the sequence of set-theoretic differences $\{\lambda^{(i)} - \mu^{(i)}\}_{i \in I}$. A colored maximal connected component of $\underline{\lambda}/\underline{\mu}$ is a sequence consisting of maximal connected components of $\lambda^{(i)}/\mu^{(i)}$. Each colored connected component itself is a colored skew diagram. For example the colored skew Ferrer diagram $\underline{\lambda}/\underline{\mu} = (6,4,2,2)/(4,2)$ has three components and is shown by

A *colored horizontal strip* is a sequence of horizontal strips, i.e., $\{\lambda^{(i)}/\mu^{(i)}\}_{i \in I}$ (with no two boxes in the same column) such that $(\lambda^{(i)})'_k - (\mu^{(i)})'_k \leq 1$. A *colored vertical strip* is a sequence of vertical strips $\{\lambda^{(i)}/\mu^{(i)}\}_{i \in I}$, which has no two boxes in the same row, i.e., $\lambda_k^{(i)} - \mu_k^{(i)} \leq 1$.

A *colored semistandard tableau* \underline{T} of shape $\underline{\lambda}/\underline{\mu}$ is a sequence of semistandard tableau $\{\lambda^{(i)}/\mu^{(i)}\}_{i \in I}$. The *content* $\underline{\nu}$ of a colored tableau $\underline{T} = (T^{(0)}, \dots, T^{(r-1)})$ is the sequence consisting of the contents $\nu^{(i)}$ of $\{\lambda^{(i)}/\mu^{(i)}\}$. For a sequence of variables $x = (x^{(0)}; \dots; x^{(r-1)})$ we will write $x^{\underline{T}} = (x^{(0)})^{T^{(0)}} \cdots (x^{(r-1)})^{T^{(r-1)}}$ associated with the colored tableau \underline{T} . Here $x^{(i)} = (x_{i1}, x_{i2}, \dots)$ for each $i \in I$.

We use $T_{\underline{\lambda}/\underline{\mu}, \underline{\nu}}$ to denote the colored skew tableau of shape $\underline{\lambda}/\underline{\mu}$ and content $\underline{\nu}$. The *Kostka number* $K_{\underline{\lambda}/\underline{\mu}, \underline{\nu}}$ is the number of semistandard tableaux of shape $\underline{\lambda}/\underline{\mu}$ and content $\underline{\nu}$. For example, in Figure 4 the colored tableau $((6, 5, 4, 3), (4, 3, 2)/((4, 2), (2, 1))$ with content $((2, 3, 3, 0, 3, 1), (3, 2, 1))$ is

				2	2			1'	1'
		1	1	5			1'	2'	
2	3	3	3			2'	3'		
5	5	6							

Figure 4. Colored tableau of shape $((6, 5, 4, 3), (4, 3, 2)/((4, 2), (2, 1))$

which is also shown as an array

$$\begin{array}{cccc} & & 2 & 2 \\ & 1 & 1 & 5 \\ 2 & 3 & 3 & 3 \\ 5 & 5 & 6 & \end{array}, \quad \begin{array}{cc} & 1' & 1' \\ & 1' & 2' \\ 2' & 3' & \end{array}$$

The multiplication of two colored Schur functions is expressed as a linear combination of colored Schur functions:

$$s_{\underline{\lambda}}s_{\underline{\mu}} = \sum_{\underline{\nu}} c_{\underline{\lambda}\underline{\mu}}^{\underline{\nu}} s_{\underline{\nu}}, \quad c_{\underline{\lambda}\underline{\mu}}^{\underline{\nu}} \in \mathbb{Z}_+. \quad (11)$$

The structure constants $c_{\underline{\lambda}\underline{\mu}}^{\underline{\nu}}$ are called the colored *Littlewood-Richardson coefficients*. It is known that $c_{\underline{\lambda}\underline{\mu}}^{\underline{\nu}} \neq 0$ unless $|\underline{\nu}| = |\underline{\lambda}| + |\underline{\mu}|$ and $\underline{\nu} \supset \underline{\lambda}, \underline{\mu}$.

We define *skew Schur function* $s_{\underline{\lambda}/\underline{\mu}} \in \Lambda_{\mathbb{Z}}$ by

$$s_{\underline{\lambda}/\underline{\mu}} = \sum_{\underline{\nu}} c_{\underline{\mu}\underline{\nu}}^{\underline{\lambda}} s_{\underline{\nu}}. \quad (12)$$

It is clear that $\deg(s_{\underline{\lambda}/\underline{\mu}}) = |\underline{\lambda}| - |\underline{\mu}|$ and $s_{\underline{\lambda}/\underline{\mu}} = 0$ unless $\underline{\mu} \subset \underline{\lambda}$, i.e. $s_{\underline{\lambda}/\underline{\mu}}$ is defined for the skew Ferrer diagram $\underline{\lambda}/\underline{\mu}$.

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two sets of variables.

Theorem 3.2 (a) *If the connected components of the colored skew diagram $\underline{\lambda}/\underline{\mu}$ are θ_j , then $s_{\underline{\lambda}/\underline{\mu}}(x) = \prod_j s_{\theta_j}(x)$. In particular, $s_{\underline{\lambda}/\underline{\mu}}(x) = 0$ if $\underline{\mu} \not\subset \underline{\lambda}$.*

(b) *The skew Schur symmetric function $s_{\underline{\lambda}}(x, y)$ satisfies*

$$s_{\underline{\lambda}/\underline{\mu}}(x, y) = \sum_{\underline{\nu}} s_{\underline{\lambda}/\underline{\nu}}(x) s_{\underline{\nu}/\underline{\mu}}(y), \quad (13)$$

where the sum runs through all the colored partitions $\underline{\nu}$ such that $\underline{\lambda} \supset \underline{\nu} \supset \underline{\mu}$.

(c) *In general, the skew Schur function $s_{\underline{\lambda}/\underline{\mu}}(x^{(1)}, \dots, x^{(n)})$ can be written as*

$$s_{\underline{\lambda}/\underline{\mu}}(x^{(1)}, \dots, x^{(n)}) = \sum_{(\underline{\nu})} \prod_{j=1}^n s_{\underline{\nu}^{(j)}/\underline{\nu}^{(j-1)}}(x^{(j)}), \quad (14)$$

where the sum runs through all sequences of colored partitions $(\underline{\nu}) = (\underline{\nu}^{(n)}, \dots, \underline{\nu}^{(0)})$ such that $\underline{\mu} = \underline{\nu}^{(0)} \subset \underline{\nu}^{(1)} \subset \dots \subset \underline{\nu}^{(n)} = \underline{\lambda}$.

Proof. (a) Suppose the skew diagram $\lambda^{(i)}/\mu^{(i)}$ has m_i connected components $\theta_j^{(i)}$, $j = 1, \dots, m_i$. From Proposition 2.1 (a) it follows that

$$\begin{aligned} s_{\underline{\lambda}/\underline{\mu}}(x) &= \prod_{i \in I} s_{\lambda^{(i)}/\mu^{(i)}}(x^{(i)}) \\ &= \prod_{i \in I} \prod_{j=1}^{m_i} s_{\theta_j^{(i)}}(x^{(i)}) = \prod_j s_{\theta_j}(x), \end{aligned}$$

where m is the number of connected components of the colored skew diagram $\underline{\lambda}/\underline{\mu}$.

(b) It follows from Proposition 2.1 (b) that

$$\begin{aligned} s_{\underline{\lambda}/\underline{\mu}}(x, y) &= \prod_{i \in I} s_{\lambda^{(i)}/\mu^{(i)}}(x^{(i)}, y^{(i)}) \\ &= \prod_{i \in I} \left(\sum_{\nu^{(i)}} s_{\lambda^{(i)}/\nu^{(i)}}(x^{(i)}) s_{\nu^{(i)}/\mu^{(i)}}(y^{(i)}) \right) \\ &= \sum_{\underline{\nu}} \prod_{i \in I} s_{\lambda^{(i)}/\nu^{(i)}}(x^{(i)}) s_{\nu^{(i)}/\mu^{(i)}}(y^{(i)}) \\ &= \sum_{\underline{\nu}} s_{\underline{\lambda}/\underline{\nu}}(x) s_{\underline{\nu}/\underline{\mu}}(y) \end{aligned}$$

where the sum runs through all colored partitions $\underline{\nu}$ such that $\underline{\lambda} \supset \underline{\nu} \supset \underline{\mu}$.

Statement (c) follows from (b). \square

If there is only one variable in each $x^{(i)}$, and $x = (x_0; x_1; \dots; x_{r-1})$, then the Schur function can be easily computed.

Theorem 3.3 (a) We have $s_{\underline{\lambda}}(x) = 0$ when $l(\lambda^{(i)}) > 1$ for some $i \in I$. When $\underline{\lambda}/\underline{\mu}$ is a colored horizontal strip with $|\lambda^{(r)}| - |\mu^{(r)}| = a_i$, $i \in I$ and $x = (x_0; x_1; \dots; x_{r-1})$, then $s_{\underline{\lambda}/\underline{\mu}}(x) = x_0^{a_0} \cdots x_{r-1}^{a_{r-1}}$.

(b) More generally $s_{\underline{\lambda}/\underline{\mu}}(x_1, \dots, x_n) = 0$ unless $\lambda'_i - \mu'_i \leq n$ for each i .

Proof. The first identity in Part (a) and Part (b) are a direct consequence of Proposition 2.2. Now assume $\underline{\lambda}/\underline{\mu}$ is a colored horizontal strip with $|\lambda^{(r)}| - |\mu^{(r)}| = a_i$, $i \in I$, then using Proposition 2.2 (a) we have

$$\begin{aligned} s_{\underline{\lambda}/\underline{\mu}}(x) &= \prod_{i \in I} s_{\lambda^{(i)}/\mu^{(i)}}(x_i) \\ &= \prod_{i \in I} x_i^{|\lambda^{(r)}| - |\mu^{(r)}|} = \prod_{i \in I} x_i^{a_i}, \end{aligned}$$

as required. \square

Combining Propositions 4.4 and 3.2 one easily gets the following result.

Proposition 3.4 The Schur function $s_{\underline{\lambda}/\underline{\mu}}(x)$ can be expressed as a summation of all monomials x^T attached to Young tableaux of shape $\underline{\lambda}/\underline{\mu}$:

$$s_{\underline{\lambda}/\underline{\mu}}(x) = \sum_{\underline{T}} x^T, \quad (15)$$

where the sum runs through all colored semistandard skew tableaux of shape $\underline{\lambda}/\underline{\mu}$.

Proof. In Proposition 2.2(c) we take $x^{(i)} = x_i$ and using Proposition 2.1 it follows that

$$\begin{aligned} s_{\lambda/\mu}(x_1, \dots, x_n) &= \sum_{(\nu)} \prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}(x_i) \\ &= \sum_{(\nu)} \prod_{i=1}^n x_i^{|\nu^{(i)}| - |\nu^{(i-1)}|}, \end{aligned}$$

where the sum runs through all sequences of partitions $(\nu) = (\nu^{(n)}, \dots, \nu^{(0)})$ such that $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \dots \subset \nu^{(n)} = \lambda$ and $\nu^{(i)}/\nu^{(i-1)}$ are horizontal strips. For each horizontal strip $\nu^{(i)}/\nu^{(i-1)}$, we assign i in the diagram λ/μ where the horizontal strip occupies and thus obtain a semistandard Young tableau of shape λ/μ with content $T = (|\nu^{(1)}| - |\nu^{(0)}|, |\nu^{(2)}| - |\nu^{(1)}|, \dots, |\nu^{(n)}| - |\nu^{(n-1)}|)$. In this way the sequence of such partitions gives rise a colored semistandard Young tableau of shape λ/μ with content \underline{T} . Conversely given a colored semistandard Young tableau of shape λ/μ with content \underline{T} ($|\underline{T}| = |\lambda| - |\mu|$), let $\nu^{(i)}$ be the subdiagram consisting of the diagram μ together with entries numbered $1, \dots, i$. Then the skew diagram $\nu^{(i)}/\nu^{(i-1)}$ is horizontal and gives rise a sequence of partitions $(\nu) = (\nu^{(n)}, \dots, \nu^{(0)})$ such that $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \dots \subset \nu^{(n)} = \lambda$ and $\nu^{(i)}/\nu^{(i-1)}$ are horizontal strips. Therefore we have

$$s_{\lambda/\mu}(x) = \sum_{\underline{T}} x^{\underline{T}},$$

where \underline{T} runs through Young tableaux of shape λ/μ . \square

Corollary 3.5 *We have $s_{\lambda/\mu} = \sum_{\nu} K_{\lambda/\mu, \nu} m_{\nu}$.*

4 Wreath Product of symmetric functions

Let G be a finite group. We denote by G^* its set of complex irreducible characters and by $G_* = \{c\}$ conjugacy classes and index them as follows.

$$G^* = \{\gamma^{(0)}, \dots, \gamma^{(r-1)}\}, \quad G_* = \{c_0, \dots, c_{r-1}\} \quad (16)$$

We also let $\gamma_j^{(i)} = \gamma^{(i)}(c_j)$ and $\gamma_{j^*}^{(i)} = \overline{\gamma^{(i)}(c_j)} = \gamma^{(i)}(c_j^{-1})$.

Let ζ_c be the order of the centralizer of the class c , then $\zeta_c = |G|/|c|$. We also write $\zeta_s = \zeta_{c_s}$. The orthogonal relations of the irreducible characters $\gamma^{(i)}$

read

$$\sum_{s=0}^{r-1} \zeta_s^{-1} \gamma_s^{(i)} \gamma_{s^*}^{(j)} = \delta_{ij}, \quad (17)$$

$$\sum_{i=0}^{r-1} \gamma_s^{(i)} \gamma_{t^*}^{(i)} = \delta_{st} \zeta_s. \quad (18)$$

Let $p_n(\gamma)$ ($\gamma \in G^*, n \in \mathbb{Z}_+$) be the independent indeterminate over \mathbb{C} associated with the irreducible character γ and let

$$\Lambda_{\mathbb{Q}}(G) = \mathbb{Q}[p_n(\gamma) : n \geq 1, \gamma \in G^*] \simeq \Lambda_{\mathbb{Q}}^{\otimes r}.$$

For each partition $\lambda = (\lambda_1, \dots, \lambda_l)$, $\lambda_1 \geq \dots \geq \lambda_l \geq 0$, we denote $p_{\lambda}(\gamma) = p_{\lambda_1}(\gamma) \cdots p_{\lambda_l}(\gamma)$. The \mathbb{Q} -algebra $\Lambda_{\mathbb{Q}}(G)$ is a graded algebra with degree given by $\deg(p_n(\gamma)) = n$. For each colored partition $\underline{\rho} = (\rho(\gamma))_{\gamma \in G^*}$ we define

$$p_{\underline{\rho}} = \prod_{i=0}^{r-1} p_{\rho(\gamma^i)}(\gamma^i) \quad (19)$$

We define the inner product of $\Lambda_{\mathbb{Q}}(G)$ to be the tensor product of that of $\Lambda_{\mathbb{Q}}$. Thus the inner product is given by

$$\langle p_{\underline{\lambda}}, p_{\underline{\rho}} \rangle = \delta_{\underline{\lambda}, \underline{\rho}} z_{\underline{\lambda}}, \quad (20)$$

where $z_{\underline{\lambda}} = \prod_{\gamma \in G^*} z_{\lambda(\gamma)}$.

We will extend the inner product to the hermitian space $\Lambda_{\mathbb{C}}(G)$ naturally (or sesque-linearly) as follows. For $f = \sum_{\underline{\lambda}} c_{\underline{\lambda}} p_{\underline{\lambda}}$ and $g = \sum_{\underline{\lambda}} d_{\underline{\lambda}} p_{\underline{\lambda}}$ we define

$$\langle f, g \rangle = \sum_{\underline{\lambda}} c_{\underline{\lambda}} \overline{d_{\underline{\lambda}}} z_{\underline{\lambda}}. \quad (21)$$

In other words, we require that the power sum $p_n(\gamma)$ be invariant under complex conjugation: $\overline{p_{\underline{\lambda}}} = p_{\underline{\lambda}}$. We remark that this differs from the inner product in [4] where another power sum basis is invariant.

We now introduce the second power sum symmetric functions indexed by colored partitions associated with conjugacy classes of G . For each $c \in G_*$ we define

$$p_n(c) = \sum_{\gamma \in G^*} \gamma(c^{-1}) p_n(\gamma). \quad (22)$$

Using the orthogonality of the characters of G we have

$$p_n(\gamma) = \sum_{c \in G_*} \zeta_c^{-1} \gamma(c) p_n(c) \quad (23)$$

For each partition $\lambda = (\lambda_1, \dots, \lambda_l)$, $\lambda_1 \geq \dots \geq \lambda_l \geq 0$, we denote $p_\lambda(c) = p_{\lambda_1}(c) \cdots p_{\lambda_l}(c)$. For a colored partition ρ we define the power sum

$$P_\rho = \prod_{i=0}^{r-1} p_{\rho^{(i)}}(c) \quad (24)$$

Then P_ρ is of degree n if ρ is a colored partition of n . Clearly the power sum symmetric function P_ρ forms a \mathbb{Q} -linear basis for the space $\Lambda_{\mathbb{Q}}^{\otimes r} \simeq \Lambda_{\mathbb{Q}}(G)$. However the second power sum basis is not invariant under the complex conjugation.

Then we arrive to the following equivalent inner product.

Proposition 4.1 *The induced inner product on the wreath product of the ring Λ is given by*

$$\langle P_\lambda, P_\rho \rangle = \delta_{\lambda, \rho} Z_\lambda, \quad (25)$$

where $Z_\lambda = \prod_{c_i \in G_*} z_{\lambda^{(i)}} \zeta_{c_i}^{l(\lambda^{(i)})}$ for λ with $\lambda^{(i)} = \lambda(c_i)$.

Proof. Using definition (22) and the orthogonal relations (17-18) we compute that

$$\begin{aligned} \langle p_m(c), p_n(c') \rangle &= \sum_{\gamma, \gamma'} \gamma(c^{-1}) \gamma'(c') \langle p_m(\gamma), p_n(\gamma') \rangle \\ &= m \delta_{m,n} \sum_{\gamma} \gamma(c^{-1}) \gamma(c') = m \zeta_c \delta_{m,n} \delta_{c,c'}. \end{aligned}$$

The general case then follows easily by induction. \square

For each colored partition $\underline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(r-1)})$ we define the wreath product Schur symmetric function by

$$S_{\underline{\lambda}} = s_{\lambda^{(0)}}(P_{\lambda^{(0)}}) \cdots s_{\lambda^{(r-1)}}(P_{\lambda^{(r-1)}}), \quad (26)$$

where $s_{\lambda^{(i)}}(P_{\lambda^{(i)}})$ is the Schur function spanned by the power sum $p_\mu(\gamma_i)$ with $|\mu| = |\lambda^{(i)}|$.

As our ring $\Lambda_{\mathbb{C}}(G)$ is isomorphic to the r -fold tensor product $\Lambda_{\mathbb{C}}$, we can carry the results from the previous section to the case of wreath product symmetric functions.

Likewise we can define wreath product Schur functions for colored skew partitions. Let $\underline{\lambda}$ and $\underline{\mu}$ be colored partitions such that $\underline{\mu} \subset \underline{\lambda}$. We define *skew wreath product Schur function* $S_{\underline{\lambda}/\underline{\mu}} \in \Lambda_{\mathbb{Z}}$ by

$$S_{\underline{\lambda}/\underline{\mu}} = \sum_{\underline{\nu}} c_{\underline{\mu}\underline{\nu}}^{\underline{\lambda}} S_{\underline{\nu}}, \quad (27)$$

where $c_{\underline{\mu}\underline{\nu}}^{\underline{\lambda}}$ is the Littelwood-Richardson coefficient.

Proposition 4.2 *The wreath product Schur functions form an orthonormal basis in the ring $\Lambda_{\mathbb{Q}}(G)$.*

Proof. As the wreath product Schur function is the product of the usual Schur functions, and moreover the variables of each Schur functions are perpendicular to those of the other Schur functions by (25), the result follows immediately. \square

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two sets of variables.

Theorem 4.3 (a) *If the connected components of the colored skew diagram $\underline{\lambda}/\underline{\mu}$ are $\underline{\theta}_j$, then $S_{\underline{\lambda}/\underline{\mu}}(x) = \prod_j S_{\underline{\theta}_j}(x)$. In particular, $S_{\underline{\lambda}/\underline{\mu}}(x) = 0$ if $\mu \not\subseteq \lambda$.*

(b) *The skew wreath product Schur symmetric function $S_{\underline{\lambda}}(x, y)$ satisfies*

$$S_{\underline{\lambda}/\underline{\mu}}(x, y) = \sum_{\underline{\nu}} S_{\underline{\lambda}/\underline{\nu}}(x) S_{\underline{\nu}/\underline{\mu}}(y), \quad (28)$$

where the sum runs through all the colored partitions $\underline{\nu}$ such that $\underline{\lambda} \supset \underline{\nu} \supset \underline{\mu}$.

(c) *In general, the skew wreath product Schur function $S_{\underline{\lambda}/\underline{\mu}}(x^{(1)}, \dots, x^{(n)})$ can be written as*

$$S_{\underline{\lambda}/\underline{\mu}}(x^{(1)}, \dots, x^{(n)}) = \sum_{(\underline{\nu})} \prod_{j=1}^n S_{\underline{\nu}^{(j)}/\underline{\nu}^{(j-1)}}(x^{(j)}), \quad (29)$$

where the sum runs through all sequences of colored partitions $(\underline{\nu}) = (\underline{\nu}^{(n)}, \dots, \underline{\nu}^{(0)})$ such that $\underline{\mu} = \underline{\nu}^{(0)} \subset \underline{\nu}^{(1)} \subset \dots \subset \underline{\nu}^{(n)} = \underline{\lambda}$.

If there is only one variable in each $x^{(i)}$, and $x = (x_0; x_1; \dots; x_{r-1})$, then the wreath product Schur function can be easily computed.

Theorem 4.4 (a) *We have $S_{\underline{\lambda}}(x) = 0$ when $l(\lambda^{(i)}) > 1$ for some $i \in I$. When $\underline{\lambda}/\underline{\mu}$ is a colored horizontal strip with $|\lambda^{(r)}| - |\mu^{(r)}| = a_i$, $i \in I$ and $x = (x_0; x_1; \dots; x_{r-1})$, then $s_{\underline{\lambda}/\underline{\mu}}(x) = x_0^{a_0} \dots x_{r-1}^{a_{r-1}}$.*

(b) *More generally $S_{\underline{\lambda}/\underline{\mu}}(x_1, \dots, x_n) = 0$ unless $\lambda'_i - \mu'_i \leq n$ for each i .*

Proposition 4.5 *The wreath product Schur function $S_{\underline{\lambda}/\underline{\mu}}(x)$ can be expressed as a summation of all monomials $x^{\underline{T}}$ attached to Young tableaux of shape $\underline{\lambda}/\underline{\mu}$:*

$$S_{\underline{\lambda}/\underline{\mu}}(x) = \sum_{\underline{T}} x^{\underline{T}}, \quad (30)$$

where the sum runs through all colored semistandard skew tableaux of shape $\underline{\lambda}/\underline{\mu}$.

Corollary 4.6 *We have $S_{\underline{\lambda}/\underline{\mu}} = \sum_{\underline{\nu}} K_{\underline{\lambda}/\underline{\mu}, \underline{\nu}} m_{\underline{\nu}}$.*

Let T be a tableau. We derive the *word* of T or $w(T)$ by reading the symbols in T from right to left in successive rows, starting with the top row. For instance, if T is the tableau

$$\begin{array}{cccccc} & & & & 5 & 5 \\ & & & 1 & 1 & 6 & 7 \\ 2 & 3 & 3 & 3 & 7 & 8 \\ 4 & 4 & 6 & 7 & 8 & 9 \end{array}$$

$w(T)$ is the word 557611873332987644.

If \underline{T} is a colored tableau, then we define the colored word $\underline{w}(\underline{T})$ to the composition consists of the words $w(T^{(i)})$. We also say that the colored word $\underline{w}(\underline{T})$ is a lattice permutation if each word $w(T^{(i)})$ is a lattice permutation.

Theorem 4.7 . *Let $\underline{\lambda}, \underline{\mu}, \underline{\nu}$ be colored partitions. Then $c_{\underline{\mu}\underline{\nu}}^{\underline{\lambda}}$ is equal to the number of colored Young tableau \underline{T} of shape $\underline{\lambda}/\underline{\mu}$ and content $\underline{\nu}$ such that $\underline{w}(\underline{T})$ is a lattice permutation. The product $s_{\underline{\mu}}s_{\underline{\nu}}$ is an integral linear combination of Schur functions:*

$$S_{\underline{\mu}}S_{\underline{\nu}} = \sum_{\underline{\lambda}} c_{\underline{\mu}\underline{\nu}}^{\underline{\lambda}} S_{\underline{\lambda}} \quad (31)$$

or equivalently

$$S_{\underline{\lambda}/\underline{\mu}} = \sum_{\underline{\nu}} c_{\underline{\mu}\underline{\nu}}^{\underline{\lambda}} S_{\underline{\nu}} \quad (32)$$

Proof. It follows from the construction of $S_{\underline{\lambda}/\underline{\mu}}$ that

$$c_{\underline{\mu}\underline{\nu}}^{\underline{\lambda}} = c_{\mu^{(0)}\nu^{(0)}}^{\lambda^{(0)}} \cdots c_{\mu^{(r-1)}\nu^{(r-1)}}^{\lambda^{(r-1)}}, \quad (33)$$

and each $c_{\mu^{(i)}\nu^{(i)}}^{\lambda^{(i)}}$ is equal to the number of Young tableau T of shape $\lambda^{(i)}/\mu^{(i)}$ and content $\nu^{(i)}$ such that $w(T)$ is a lattice permutation. Taking product and noticing that they are independent we get the result in the statement. \square

A special case of the Littlewood-Richardson rule is the Pieri rule.

Corollary 4.8 (a) *Let $\mathbf{m} = (m_0, m_1, \dots, m_{r-1})$ then we have*

$$S_{\underline{\lambda}}(P) S_{\mathbf{m}}(P) = \sum_{\underline{\mu}} S_{\underline{\mu}}(P)$$

summed over all colored partitions $\underline{\mu}$ such that $\mu^{(i)}/\lambda^{(i)}$ is a horizontal m_i strip.

(b) *Let $\mathbf{1}^{\mathbf{m}} = (1^{m_0}, 1^{m_1}, \dots, 1^{m_{r-1}})$ then we have $S_{\underline{\lambda}}(P) S_{\mathbf{1}^{\mathbf{m}}}(P) = \sum_{\underline{\mu}} S_{\underline{\mu}}(P)$*

summed over all partitions $\underline{\mu}$ such that $\mu^{(i)}/\lambda^{(i)}$ is a vertical m_i strip.

5 Characters of wreath products

For any partition λ of n , let χ^λ be the irreducible character of S_n corresponding to λ and V^λ be an irreducible $\mathbb{C}S_n$ module affording χ^λ . Then the degree $\chi^\lambda(1) = \dim V^\lambda$ is equal to $n!/h(\lambda)$. Let $\{W_0, \dots, W_{r-1}\}$ be the complete set of representatives of isomorphism classes of irreducible $\mathbb{C}\Gamma$ -modules. We fix the numbering of W_0, \dots, W_{r-1} and put the $d_i = \dim W_i$. Then the irreducible $\mathbb{C}(\Gamma \sim S_n)$ -modules can be constructed as follows. For $\underline{\lambda} = (\lambda^0, \dots, \lambda^{(r-1)}) \in (Y^{(r-1)})_n$, we put

$$\begin{aligned} K^\lambda &= \Gamma^n \left(S_{n_0} \times \dots \times S_{n_{(r-1)}} \right), \\ T^\lambda &= T^{n_0}(W_0) \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} T^{n_{(r-1)}}(W_{r-1}), \\ &\quad \text{and} \\ V^\lambda &= V^{\lambda^0} \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} V^{\lambda^{(r-1)}} \end{aligned}$$

where $n_i = |\lambda^i|$ and $T^i(V)$ denotes the i -fold tensor product of a vector space V . We regard T^λ and V^λ as $\mathbb{C}K^\lambda$ -modules by defining

$$\begin{aligned} (g_1, \dots, g_n; \sigma)(t_1 \otimes \dots \otimes t_n) &= g_1 t_{\sigma^{-1}(1)} \otimes \dots \otimes g_n t_{\sigma^{-1}(n)}, \\ (g_1, \dots, g_n; \sigma)(v_0 \otimes \dots \otimes v_{(r-1)}) &= \sigma_0 v_0 \otimes \dots \otimes \sigma_{(r-1)} v_{(r-1)}, \end{aligned}$$

where $g = (g_1, \dots, g_n) \in \Gamma^n$, $\sigma = \sigma_1 \dots \sigma_{(r-1)} \in S_{n_0} \times \dots \times S_{n_{(r-1)}}$, $t_0 \otimes \dots \otimes t_{(r-1)} \in T^\lambda$, and $v_0 \otimes \dots \otimes v_{(r-1)} \in V^\lambda$:

$$W^\lambda = \mathbb{C}(\Gamma \sim S_n) \otimes_{\mathbb{C}K^\lambda} (T^\lambda \otimes_{\mathbb{C}} V^\lambda).$$

Then

$$\dim W^\lambda = n! \prod_{i=0}^{r-1} \frac{d_i^{|\lambda^{(i)}|}}{h(\lambda^{(i)})}.$$

Theorem 5.1 [7] *For each colored partition $\underline{\lambda}$ the $\mathbb{C}(\Gamma \sim S_n)$ -module W^λ constructed above is irreducible. These W^λ are pairwise non-isomorphic and exhaust the isomorphism classes of irreducible $\mathbb{C}(\Gamma \sim S_n)$ modules.*

Let f be a class function of a finite group G , the Frobenius characteristic $ch : F(G) \rightarrow \Lambda(G)$, a ring of symmetric functions associated to G , is defined by

$$ch(f) = \sum_c \frac{f(c)}{z_c} p(c) \quad (34)$$

where the power sum symmetric function $p(c)$ is associated with the conjugacy class c , and z_c is the order of the conjugacy class c . In all examples of G considered in this paper, the conjugacy class c is indexed by partitions or

colored partitions, so the meaning of $p(c)$ is self-clear from the context. For example, if $G = S_n$, then $p(c)$ is the usual power sum symmetric function corresponding to the cycle-type of the class c and $z_c = z_\rho$ with $\rho = \rho(c)$ the cycle type of the class c ; if $G = \Gamma \sim S_n$, the conjugacy classes are indexed by colored partitions $\underline{\rho}$, and $p(c) = p_{\underline{\rho}}(c)$, the second type of power sum symmetric function introduced in Eq. (22) and $z_c = Z_{\underline{\rho}}$ (cf. Eq. (25)).

The characteristic map enjoys the nice property being an isometric isomorphism from the ring of class functions to that of symmetric functions (cf. [4, 1]). When $G = S_n$, Frobenius' formula says that $ch(\chi_\lambda) = s_\lambda$, the Schur symmetric function associated with the partition λ . In the case of the wreath product group $G \sim S_n$ the generalized Frobenius-type formula and the character table was obtained by Specht [9] and recounted by Macdonald [4]. The idea is to compute the special case of one-part partition and then extended to general partition using the isomorphism of ch .

Theorem 5.2 [9, 4] *For a family of partitions $\underline{\rho} = (\rho^{(0)}, \rho^{(1)}, \dots)$ we have*

$$S_\lambda = \sum_{\underline{\rho}} Z_{\underline{\rho}}^{-1} \chi_{\underline{\rho}} P_{\underline{\rho}}$$

or equivalently

$$\chi_{\underline{\rho}}^\lambda = \langle S_\lambda, P_{\underline{\rho}} \rangle$$

The simplest example of this theory is when $r = 2$. We then have the hyperoctahedral group, which is the semi-direct product $H_n = \mathbb{Z}_2^n \rtimes S_n$, where $\mathbb{Z}_2 = \{-1, 1\}$ is considered as the multiplicative group of two elements.

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