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Note on Bi-Ideals in Γ-Semigroups

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Abstract

The motivation mainly comes from the conditions of bi-ideals to be (0-)minimal or maximal that are of importance and interest in semigroups. It is well known that any semigroup can be reduced to a Γ-semigroup. The aim of this paper is to study the concept of (0-)minimal and maximal bi-ideals in Γ-semigroups, and give some characterizations of (0-)minimal and maximal bi-ideals in Γ-semigroups analogous to the characterizations of (0-)minimal and maximal bi-ideals in semigroups considered by Iampan [3].

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1 Introduction and Prerequisites

Let S be a semigroup. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$. The notion of a bi-ideal was first introduced by Good and Hughes [2] as early as 1952, and it has been widely studied. In 2008, Iampan [3] characterized the (0-)minimal and maximal bi-ideals in semigroups, and gave some characterizations of (0-)minimal and maximal bi-ideals in semigroups.

The concept of a bi-ideal is a very interesting and important thing in semigroups. Now we also characterize the (0-)minimal and maximal bi-ideals in Γ-semigroups, and give some characterizations of (0-)minimal and maximal bi-ideals in Γ-semigroups.

To present the main results we first recall some definitions which is important here.

Let M and Γ be any two nonempty sets. M is called a Γ-*semigroup* [6] if there exists a mappings $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \mapsto a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and

$\alpha, \beta \in \Gamma$. A nonempty subset K of M is called a *sub-*Γ*-semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. For nonempty subsets A, B of M, let $A\Gamma B := \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$. We also write $a\Gamma B, A\Gamma b$ and $a\Gamma b$ for $\{a\}\Gamma B, A\Gamma\{b\}$ and $\{a\}\Gamma\{b\}$, respectively.

Examples of Γ-semigroups can be seen in [4, 5] and [7] respectively.

The following example comes from Dixit and Dewan [1].

Example 1.1 *Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then M is a Γ-semigroup under the multiplication over complex number while M is not a semigroup under complex number multiplication.*

A nonempty subset I of a Γ-semigroup M is called an *ideal* of M if $M\Gamma I \subseteq I$ and $I\Gamma M \subseteq I$. A sub-Γ-semigroup Q of a Γ-semigroup M is called a *quasi-ideal* of M if $M\Gamma Q \cap Q\Gamma M \subseteq Q$. A sub-$\Gamma$-semigroup B of a Γ-semigroup M is called a *bi-ideal* of M if $B\Gamma M\Gamma B \subseteq B$. Then the notion of a quasi-ideal is a generalization of the notion of an ideal, and the notion of a bi-ideal is a generalization of the notion of a quasi-ideal. The intersection of all bi-ideals of a sub-Γ-semigroup K of a Γ-semigroup M containing a nonempty subset A of K is called the *bi-ideal of K generated by A*. For $A = \{a\}$, let $B_K(a)$ denote the bi-ideal of K generated by $\{a\}$. If $K = M$, then we also write $B_M(a)$ as $B(a)$. An element a of a Γ-semigroup M with at least two elements is called a *zero element* of M if $x\gamma a = a\gamma x = a$ for all $x \in M$ and $\gamma \in \Gamma$, and denote it by 0. If M is a Γ-semigroup with zero, then every bi-ideal of M containing a zero element. A Γ-semigroup M without zero is called *B-simple* if it has no proper bi-ideals. A Γ-semigroup M with zero is called *0-B-simple* if it has no nonzero proper bi-ideals and $M\Gamma M \neq \{0\}$. A bi-ideal B of a Γ-semigroup M without zero is called a *minimal bi-ideal* of M if there is no bi-ideal A of M such that $A \subset B$. Equivalently, if for any bi-ideal A of M such that $A \subseteq B$, we have $A = B$. A nonzero bi-ideal B of a Γ-semigroup M with zero is called a *0-minimal bi-ideal* of M if there is no nonzero bi-ideal A of M such that $A \subset B$. Equivalently, if for any nonzero bi-ideal A of M such that $A \subseteq B$, we have $A = B$. Equivalently, if for any bi-ideal A of M such that $A \subset B$, we have $A = \{0\}$. A proper bi-ideal B of a Γ-semigroup M is called a *maximal bi-ideal* of M if for any bi-ideal A of M such that $B \subset A$, we have $A = M$. Equivalently, if for any proper bi-ideal A of M such that $B \subseteq A$, we have $A = B$.

Our purpose in this paper is fourfold.

1. To introduce the concept of a B-simple and 0-B-simple Γ-semigroup.

2. To characterize the properties of bi-ideals in Γ-semigroups.

3. To characterize the relationship between (0-)minimal bi-ideals and (0-)B-simple Γ-semigroups.

4. To characterize the relationship between maximal bi-ideals and (0-)B-simple Γ-semigroups.

We shall assume throughout this paper that M stands for a Γ-semigroup. Before the characterizations of bi-ideals for the main theorems, we give some auxiliary results which are necessary in what follows. The following three lemmas are also necessary for our considerations, and easy to verify.

Lemma 1.2 *For any $a \in M$,*

$$B(a) = a\Gamma M\Gamma a \cup a\Gamma a \cup \{a\}.$$

Lemma 1.3 *The set $a\Gamma M\Gamma a$ is a bi-ideal of M for all $a \in M$.*

Lemma 1.4 *Let $\{B_\gamma \mid \gamma \in \Lambda\}$ be a collection of bi-ideals of M. Then $\bigcap_{\gamma\in\Lambda} B_\gamma$ is a bi-ideal of M if $\bigcap_{\gamma\in\Lambda} B_\gamma \neq \emptyset$.*

Lemma 1.5 *If M has no zero element, then the following statements are equivalent:*

(i) *M is B-simple.*

(ii) *$a\Gamma M\Gamma a = M$ for all $a \in M$.*

(iii) *$B(a) = M$ for all $a \in M$.*

Proof. Since M is B-simple, it follows from Lemma 1.3 that $a\Gamma M\Gamma a = M$ for all $a \in M$. Therefore (i) implies (ii). By Lemma 1.2, $B(a) = a\Gamma M\Gamma a \cup a\Gamma a \cup \{a\} = M \cup a\Gamma a \cup \{a\} = M$ for all $a \in M$. Thus (ii) implies (iii). Let B be a bi-ideal of M, and let $a \in B$. Then $M = B(a) \subseteq B \subseteq M$, so $B = M$. Hence M is B-simple, we have that (iii) implies (i). □

Lemma 1.6 *If M has a zero element, then the following statements hold:*

(i) *If M is 0-B-simple, then $B(a) = M$ for all $a \in M \setminus \{0\}$.*

(ii) *If $B(a) = M$ for all $a \in M \setminus \{0\}$, then either $M\Gamma M = \{0\}$ or M is 0-B-simple.*

Proof. (i) Assume that M is 0-B-simple. Then $B(a)$ is a nonzero bi-ideal of M for all $a \in M \setminus \{0\}$. Hence $B(a) = M$ for all $a \in M \setminus \{0\}$.

(ii) Assume that $B(a) = M$ for all $a \in M \setminus \{0\}$ and $M\Gamma M \neq \{0\}$. Let B be a nonzero bi-ideal of M, and let $a \in B \setminus \{0\}$. Then $M = B(a) \subseteq B \subseteq M$, so $B = M$. Therefore M is 0-B-simple. □

Lemma 1.7 *If B is a bi-ideal of M, and K is a sub-Γ-semigroup of M, then the following statements hold:*

(i) *If K is B-simple such that $K \cap B \neq \emptyset$, then $K \subseteq B$.*

(ii) *If K is 0-B-simple such that $K \setminus \{0\} \cap B \neq \emptyset$, then $K \subseteq B$.*

Proof. (i) Assume that K is B-simple such that $K \cap B \neq \emptyset$. Then, let $a \in K \cap B$. By Lemma 1.3, $a\Gamma K\Gamma a$ is a bi-ideal of K. It follows that $a\Gamma K\Gamma a = K$. Hence $K = a\Gamma K\Gamma a \subseteq B\Gamma M\Gamma B \subseteq B$, so $K \subseteq B$.

(ii) Assume that K is 0-B-simple such that $K \setminus \{0\} \cap B \neq \emptyset$. Then, let $a \in K \setminus \{0\} \cap B$. By Lemma 1.2 and 1.6 (i), $K = B_K(a) = a\Gamma K\Gamma a \cup a\Gamma a \cup \{a\} \subseteq a\Gamma M\Gamma a \cup a\Gamma a \cup \{a\} = B(a) \subseteq B$. Hence $K \subseteq B$.

Hence the proof is completed. □

We now give the main theorem of this paper as bellow.

2 (0-)Minimal Bi-ideals

The aim of this section is to characterize the relationship between minimal bi-ideals and B-simple Γ-semigroups, and 0-minimal bi-ideals and 0-B-simple Γ-semigroups.

Theorem 2.1 *If M has no zero element, and B is a bi-ideal of M, then the following statements hold:*

(i) *B is a minimal bi-ideal without zero of M if and only if B is B-simple.*

(ii) *If B is a minimal bi-ideal with zero of M, then either $B\Gamma B = \{0\}$ or B is 0-B-simple.*

Proof. (i) Assume that B is a minimal bi-ideal without zero of M. Then B is a sub-Γ-semigroup of M. Now, let A be a bi-ideal of B. Then $A\Gamma B\Gamma A \subseteq A$. Define $H := \{h \in A \mid h = a_1\gamma_1 b\gamma_2 a_2$ for some $a_1, a_2 \in A, b \in B$ and $\gamma_1, \gamma_2 \in \Gamma\}$. Then $\emptyset \neq H \subseteq A \subseteq B$. To show that H is a bi-ideal of M, let $h_1, h_2 \in H, x \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. Then $h_1 = a_1\alpha_1 b_1\alpha_1' a_1'$ and $h_2 = a_2\alpha_2 b_2\alpha_2' a_2'$ for some $a_1, a_1', a_2, a_2' \in A, b_1, b_2 \in B$ and $\alpha_1, \alpha_1', \alpha_2, \alpha_2' \in \Gamma$, so $h_1\gamma_1 h_2 = a_1\alpha_1 b_1\alpha_1' a_1'\gamma_1 a_2\alpha_2 b_2\alpha_2' a_2'$ and $h_1\gamma_1 x\gamma_2 h_2 = a_1\alpha_1 b_1\alpha_1' a_1'\gamma_1 x\gamma_2 a_2\alpha_2 b_2\alpha_2' a_2'$. Since $B\Gamma M\Gamma B \subseteq B, b_1\alpha_1' a_1'\gamma_1 a_2\alpha_2 b_2 \in B$ and $b_1\alpha_1' a_1'\gamma_1 x\gamma_2 a_2\alpha_2 b_2 \in B$. Since $h_1\gamma_1 h_2 \in H\Gamma H \subseteq A\Gamma A \subseteq A$, we get $h_1\gamma_1 h_2 \in H$. Thus H is a sub-Γ-semigroup of M. Since $A\Gamma B\Gamma A \subseteq A$, we get $h_1\gamma_1 x\gamma_2 h_2 = a_1\alpha_1 b_1\alpha_1' a_1'\gamma_1 x\gamma_2 a_2\alpha_2 b_2\alpha_2' a_2' \in A$. Hence $h_1\gamma_1 x\gamma_2 h_2 \in H$, so $H\Gamma M\Gamma H \subseteq H$. Therefore H is a bi-ideal of M. Since B is a minimal bi-ideal of M, we get $H = B$. Hence $A = B$, so B is B-simple.

Conversely, assume that B is B-simple. Let A be a bi-ideal of M such that $A \subseteq B$. Then $A \cap B \neq \emptyset$, it follows from Lemma 1.7 (i) that $B \subseteq A$. Hence $A = B$, so B is a minimal bi-ideal of M.

(ii) Similar to the proof of necessary condition of statement (i).

Therefore we complete the proof of the theorem. □

Using the same proof of Theorem 2.1 (i) and Lemma 1.7 (ii), we have Theorem 2.2.

Theorem 2.2 *If M has a zero element, and B is a nonzero bi-ideal of M, then the following statements hold:*

(i) *If B is a 0-minimal bi-ideal of M, then either $A\Gamma B\Gamma A = \{0\}$ for some nonzero bi-ideal A of B or B is 0-B-simple.*

(ii) *If B is 0-B-simple, then B is a 0-minimal bi-ideal of M.*

Theorem 2.3 *If M has no zero element but it has a proper bi-ideal, then every proper bi-ideal of M is minimal if and only if the intersection of any two distinct proper bi-ideals is empty.*

Proof. Assume B_1 and B_2 are two distinct proper bi-ideals of M. Then B_1 and B_2 are minimal. If $B_1 \cap B_2 \neq \emptyset$, then $B_1 \cap B_2$ is a bi-ideal of M by Lemma 1.4. Since B_1 and B_2 are minimal, $B_1 = B_2$. It is a contradiction. Hence $B_1 \cap B_2 = \emptyset$.

The converse is obvious. □

Using the same proof of Theorem 2.3, we have Theorem 2.4.

Theorem 2.4 *If M has a zero element and a nonzero proper bi-ideal, then every nonzero proper bi-ideal of M is 0-minimal if and only if the intersection of any two distinct nonzero proper bi-ideals is $\{0\}$.*

3 Maximal Bi-ideals

The aim of this section is to characterize the relationship between maximal bi-ideals and the set $\mathcal{U}$ in Γ-semigroups.

Theorem 3.1 *Let B be a bi-ideal of M. If either $M \setminus B = \{a\}$ for some $a \in M$ or $M \setminus B \subseteq b\Gamma M\Gamma b$ for all $b \in M \setminus B$, then B is a maximal bi-ideal of M.*

Proof. Let A be a bi-ideal of M such that $B \subset A$. Then we consider the following two cases:

Case 1: $M \setminus B = \{a\}$ for some $a \in M$.

Then $M = B \cup \{a\}$. Since $B \subset A, \emptyset \neq A \setminus B \subseteq M \setminus B = \{a\}$. Thus $A \setminus B = \{a\}$ and $A = B \cup \{a\} = M$.

Case 2: $M \setminus B \subseteq b\Gamma M\Gamma b$ for all $b \in M \setminus B$.

If $b \in A \setminus B \subseteq M \setminus B$, then $M \setminus B \subseteq b\Gamma M\Gamma b \subseteq A\Gamma M\Gamma A \subseteq A$. Hence $M = B \cup M \setminus B \subseteq B \cup A = A \subseteq M$, so $A = M$.

Therefore B is a maximal bi-ideal of M. □

Theorem 3.2 *If B is a maximal bi-ideal of M, and $B \cup B(a)$ is a bi-ideal of M for all $a \in M \setminus B$, then either*

(i) *$M \setminus B \subseteq a\Gamma a \cup \{a\}$ and $a\Gamma a\Gamma a \subseteq B$ for some $a \in M \setminus B$, and $b\Gamma M\Gamma b \subseteq B$ for all $b \in M \setminus B$ or*

(ii) *$M \setminus B \subseteq B(a)$ for all $a \in M \setminus B$.*

Proof. Assume that B is a maximal bi-ideal of M and $B \cup B(a)$ is a bi-ideal of M for all $a \in M \setminus B$. Then we have the following two cases:

Case 1: $a\Gamma M\Gamma a \subseteq B$ for some $a \in M \setminus B$.

Then $a\Gamma a\Gamma a \subseteq a\Gamma M\Gamma a \subseteq B$, so $a\Gamma a\Gamma a \subseteq B$. Since $B \cup a\Gamma a \cup \{a\} = (B \cup a\Gamma M\Gamma a) \cup a\Gamma a \cup \{a\} = B \cup (a\Gamma M\Gamma a \cup a\Gamma a \cup \{a\}) = B \cup B(a)$, so $B \cup a\Gamma a \cup \{a\}$ is a bi-ideal of M. Since $a \in M \setminus B$, we have $B \subset B \cup a\Gamma a \cup \{a\}$. Thus $B \cup a\Gamma a \cup \{a\} = M$ because B is a maximal bi-ideal of M, so $M \setminus B \subseteq a\Gamma a \cup \{a\}$. If $b \in M \setminus B$, then $b \in a\Gamma a \cup \{a\}$. If $b = a$, then $b\Gamma M\Gamma b = a\Gamma M\Gamma a \subseteq B$. If $b = a\gamma a$ for some $\gamma \in \Gamma$, then $b\Gamma M\Gamma b = a\gamma a\Gamma M\Gamma a\gamma a \subseteq a\Gamma M\Gamma a \subseteq B$. Hence $b\Gamma M\Gamma b \subseteq B$ for all $b \in M \setminus B$.

Case 2: $a\Gamma M\Gamma a \nsubseteq B$ for all $a \in M \setminus B$.

If $a \in M \setminus B$, then $B \subset B \cup a\Gamma M\Gamma a \subseteq B \cup B(a)$ by Lemma 1.2. Since $B \cup B(a)$ is a bi-ideal of M, and B is a maximal bi-ideal of M, we have $B \cup B(a) = M$. Hence $M \setminus B \subseteq B(a)$ for all $a \in M \setminus B$.

Hence the proof is completed. □

For a Γ-semigroup M, let $\mathcal{U}$ denote the union of all nonzero proper bi-ideals of M if M has a zero element, and let $\mathcal{U}$ denote the union of all proper bi-ideals of M if M has no zero element. Then it is easy to verify Lemma 3.3.

Lemma 3.3 *$M = \mathcal{U}$ if and only if $B(a) \neq M$ for all $a \in M$.*

As a consequence of Theorem 3.2 and Lemma 3.3, we obtain Theorem 3.4.

Theorem 3.4 *If M has no zero element, then one of the following four conditions is satisfied:*

(i) *$\mathcal{U}$ is not bi-ideal of M.*

(ii) *$B(a) \neq M$ for all $a \in M$.*

(iii) *There exists $a \in M$ such that $B(a) = M, a\Gamma a \cup \{a\} \nsubseteq a\Gamma M\Gamma a$ and $a\Gamma a\Gamma a \subseteq \mathcal{U}$, M is not B-simple, $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$ and $\mathcal{U}$ is the unique maximal bi-ideal of M.*

(iv) *$M \setminus \mathcal{U} \subseteq B(a)$ for all $a \in M \setminus \mathcal{U}$, M is not B-simple, $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$ and $\mathcal{U}$ is the unique maximal bi-ideal of M.*

Proof. Assume that $\mathcal{U}$ is a bi-ideal of M. Then $\mathcal{U} \neq \emptyset$. Now, we consider the following two cases:

Case 1: $\mathcal{U} = M$.

By Lemma 3.3, $B(a) \neq M$ for all $a \in M$. In this case, the condition (ii) is satisfied.

Case 2: $\mathcal{U} \neq M$.

Then M is not B-simple. To show that $\mathcal{U}$ is the unique maximal bi-ideal of M, let A be a bi-ideal of M such that $\mathcal{U} \subset A$. If $A \neq M$, then A is a proper bi-ideal of M. Thus $A \subseteq \mathcal{U}$, so it is a contradiction. Hence $\mathcal{U}$ is a maximal bi-ideal of M. Next, assume that B is a maximal bi-ideal of M. Then $B \subseteq \mathcal{U} \subset M$ because B is a proper bi-ideal of M. Since B is a maximal bi-ideal of M, we have $B = \mathcal{U}$. Hence $\mathcal{U}$ is the unique maximal bi-ideal of M. Since $\mathcal{U} \neq M$, it follows from Lemma 3.3 that $B(a) = M$ for some $a \in M$. Clearly, $B(a) = M$ for all $a \in M \setminus \mathcal{U}$. Thus $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$, so $\mathcal{U} \cup B(a) = M$ is a bi-ideal of M for all $a \in M \setminus \mathcal{U}$. By Theorem 3.2, we have the following two cases:

(i) $M \setminus \mathcal{U} \subseteq a\Gamma a \cup \{a\}$ and $a\Gamma a\Gamma a \subseteq \mathcal{U}$ for some $a \in M \setminus \mathcal{U}$, and $b\Gamma M\Gamma b \subseteq \mathcal{U}$ for all $b \in M \setminus \mathcal{U}$ or

(ii) $M \setminus \mathcal{U} \subseteq B(a)$ for all $a \in M \setminus \mathcal{U}$.

Assume $M \setminus \mathcal{U} \subseteq a\Gamma a \cup \{a\}$ and $a\Gamma a\Gamma a \subseteq \mathcal{U}$ for some $a \in M \setminus \mathcal{U}$, and $b\Gamma M\Gamma b \subseteq \mathcal{U}$ for all $b \in M \setminus \mathcal{U}$. If $a\Gamma a \cup \{a\} \subseteq a\Gamma M\Gamma a$, then $M = B(a) = a\Gamma M\Gamma a \cup a\Gamma a \cup \{a\} = a\Gamma M\Gamma a$ by Lemma 1.2. By hypothesis, $M = a\Gamma M\Gamma a \subseteq \mathcal{U}$ and so $\mathcal{U} = M$. This is a contradiction. Hence $a\Gamma a \cup \{a\} \nsubseteq a\Gamma M\Gamma a$. In this case, the condition (iii) is satisfied. Now, assume $M \setminus \mathcal{U} \subseteq B(a)$ for all $a \in M \setminus \mathcal{U}$. In this case, the condition (iv) is satisfied.

Hence the theorem is now completed. □

Using the same proof of Theorem 3.4, we have Theorem 3.5.

Theorem 3.5 *If M has a zero element and $M\Gamma M \neq \{0\}$, then one of the following five conditions is satisfied:*

(i) *$\mathcal{U}$ is not bi-ideal of M.*

(ii) *$B(a) \neq M$ for all $a \in M$.*

(iii) *$\mathcal{U} = \{0\}, M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$ and $\mathcal{U}$ is the unique maximal bi-ideal of M.*

(iv) *There exists $a \in M$ such that $B(a) = M, a\Gamma a \cup \{a\} \nsubseteq a\Gamma M\Gamma a$ and $a\Gamma a\Gamma a \subseteq \mathcal{U}$, M is not 0-B-simple, $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$ and $\mathcal{U}$ is the unique maximal bi-ideal of M.*

(v) *$M \setminus \mathcal{U} \subseteq B(a)$ for all $a \in M \setminus \mathcal{U}$, M is not 0-B-simple, $M \setminus \mathcal{U} = \{x \in M \mid B(x) = M\}$ and $\mathcal{U}$ is the unique maximal bi-ideal of M.*

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