International Journal of Algebra, Vol. 3, 2009, no. 3, 125 - 135

Cyclic Pure Submodules

V. A. Hiremath[1]

Department of Mathematics, Karnatak University
Dharwad-580003, India
va_hiremath@rediffmail.com

Seema S. Gramopadhye[2]

Department of Mathematics, Karnatak University
Dharwad-580003, India
e-mail:seemaalg@yahoo.com

Abstract

P.M.Cohn [3] has introduced the notion of purity for R-modules. With respect to purity, flat, absolutely pure and regular modules are studied. In this paper we introduce and study the corresponding notions of c-flat, absolutely c-pure and c-regular modules for cyclic purity. We prove that absolutely c-pure R-modules are precisely injective modules. Also we study the relationship between c-flat and torsion-free modules over commutative integral domains and non-commutative non-integral domains. Also, we study the conditions under which c-regular R-modules are semi-simple.

Mathematics Subject Classifications: 16D40, 16D50

Keywords: Pure submodules, flat module and regular ring

Introduction

In this paper, by a ring R we mean an associative ring with unity and by an R-module we mean a unitary right R-module. $Z(M)$ denotes the singular submodule of the R-module M.

[1]Corresponding author
[2]The author is supported by University Research Scholarship.

A ring R is said to be principal projective, if every principal right ideal is projective. We denote this ring by p.p.

The notion of purity has an important role in module theory and in model theory. In model theory, the notion of pure exact sequence is more useful than split exact sequences. There are several variants of this notion. R.Wisbauer[13], generalized the notion of purity for a class $\wp$ of R-modules. He defines a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules to be $\wp$-pure, if every member of $\wp$ is projective with respect to this sequence. Cohn's purity is precisely $\wp$-purity for the class $\wp$ of all finitely presented R-modules. With respect to this purity, absolute purity, flatness and regularity are studied.

Simmons[10], considered the cyclic purity in the case of commutative integral domains. This is the $\wp$-purity for the class $\wp$ of all cyclic R-modules. Here we study the cyclic purity for general rings and also study absolute c-pure, c-flat and c-regularity.

Consider a short exact sequence, $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules. An R-module M is said to be ϵ-injective (resp. ϵ -projective) if M is injective (resp. projective) with respect to the short exact sequence ϵ.

1. Cyclic Pure Exact sequences

Definition 1.1: (i)[10] An exact sequence $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules is said to be **cyclic pure (*c*-pure in short)** if every cyclic R-module ϵ-projective.
ii) A submodule A of an R-module B is said to be **cyclic pure (c-pure in short** if the canonical short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ is c-pure.

Clearly every split exact sequence is the trivial example of c-pure exact sequence.

Simmons [10] considered for modules over commutative integral domains. He has stated in Proposition 1 (without proof) some equivalent conditions for cyclic purity. In the following proposition we prove that these conditions are true for general rings also.

Proposition 1.2: For a submodule A of an R-module B, the following conditions are equivalent.
i) A is c-pure submodule of B.
ii) For every $\overline{b} \in B/A$ there exists $b_1 \in B$ such that $\overline{b_1} = \overline{b}$ and $ann(b_1) = ann(\overline{b})$.
iii) For any $b \in B$ and any right ideal I of R with $bI \subseteq A$, there exist $a \in A$, such that $(a - b)I = 0$.

iv) Any system of equations $xr_j = a_j, j \in J$ where J is any index set, with $r_j \in R$ is solvable in A whenever it is solvable in B.

Proof: Throught the proof η denotes the canonical epimorphism from B onto B/A. **i) ⇒ ii)** Let $\overline{b} \in B/A$ and $I = ann(\overline{b})$ then R/I is a cyclic R-module. Define, a map $f : R/I \longrightarrow B/A$ by $f(\overline{r}) = \overline{b}r$ for each $\overline{r} \in R/I$. Clearly it is a well-defined homomorphism. Then by (i), there exists a homomorphism $g : R/I \longrightarrow B$ such that $\eta \circ g = f$ where $\eta : B \longrightarrow B/A$ is the natural epimorphism. Let $b_1 = g(\overline{1}) \in B$. Then, $\overline{b_1} = \eta(b_1) = \eta(g(\overline{1})) = (\eta \circ g)(\overline{1}) = f(\overline{1}) = \overline{b}$. So $ann(b_1) \subseteq ann(\overline{b})$. On the other hand, let $r \in ann(\overline{b}) = I$. Then $b_1 r = g(\overline{1})r = g(\overline{r}) = g(\overline{o}) = o$ and hence $r \in ann(b_1)$. So $ann(\overline{b}) \subseteq ann(b_1)$. This proves (ii).

ii)⇒ iii)

Let $b \in B$ and I be a right ideal of R such that $bI \subseteq A$. Then $I \subseteq ann(\overline{b})$ where $\overline{b} = b + A$. Now by (ii), there exists $b_1 \in B$ such that $ann(\overline{b}) = ann(b_1)$ and $\overline{b} = \overline{b_1}$. Hence $b - b_1 = a \in A$. Since $I \subseteq ann(\overline{b}) = ann(b_1) = ann(b - a)$, this implies $(b - a)I = o$.

iii)⇒iv)

Let $xr_j = a_j, j \in J$ (where J is any index set) $(*)$, be a system of equations with $r_j \in R$ and $a_j \in A$ have solution in B. Then there exists $b \in B$ such that $br_j = a_j$ for every $j \in J$. Let I be the right ideal of R generated by the subset $\{r_j\}_{j \in J}$ of R. Now $bI \subseteq A$. Then by (iii) there exists $a \in A$ such that, $(b - a)I$=o. This implies that $(b - a)r_j = o$ for every $j \in J$. So, $br_j = ar_j$ for every $j \in J$. Hence $(*)$ has solution in A.

iv)⇒i)

Let M be a cyclic R-module. We may assume that $M = R/I$ for some right ideal I of R. Let $f \in Hom(R/I, B/A)$ and let $\overline{b} = f(\overline{1})$. Consider the subset $J = \{r \in R/br \in A\}$ of R which we write $\{r_\alpha\}_{\alpha \in \Lambda}$. Let $br_\alpha = a_\alpha$ for every $\alpha \in \Lambda$. Hence the system of equations $xr_\alpha = a_\alpha$ has solution b in B. By (iv), there exists $a \in A$ such that $ar_\alpha = a_\alpha$ for every $\alpha \in \Lambda$. Now we define a map $g : R/I \longrightarrow B$ by, $g(\overline{r}) = (b - a)r$ for every $\overline{r} \in R/I$. If $r \in I$, $\overline{br} = \overline{b}r = f(\overline{1})r = f(\overline{r}) = f(\overline{o}) = \overline{o}$ and hence $br \in A$. So $r \in J = \{r_\alpha\}_{\alpha \in \Lambda}$. Let $r = r_{\alpha_0}$, for some $\alpha_0 \in \Lambda$. So $(b - a)r = (b - a)r_{\alpha_0} = br_{\alpha_0} - ar_{\alpha_0} = a_{\alpha_0} - a_{\alpha_0} = o$. Hence $g(\overline{r}) = o$. Hence g is a well-defined homomorphism. Also $(\eta \circ g)(\overline{r}) = \eta((b - a)r) = \overline{br} = f(\overline{r})$, since, $ar \in A = ker(\eta)$. Hence $\eta \circ g = f$. Hence the result.

Corollary 1.3: Let A be a c-pure submodule of an R-module B.

i) If B is torsion-free, so is B/A.

ii) If B is non-singular, so is B/A.

Proof: i) Let $\overline{b}r = \overline{o}$ for some $\overline{b} \in B/A$ and some $r \in R$. By Proposition 1.2 ii) there exists $b_1 \in B$ such that $ann(\overline{b}) = ann(b_1)$ and $\overline{b} = \overline{b_1}$. Since, $r \in ann(\overline{b})$, $r \in ann(b_1)$. By hypothesis, B is torsion-free, implies there is no

regular element in $ann(b_1)$. Hence, r cannot be regular. So, B/A is torsion-free
ii) Let $\overline{x} \in Z(B/A)$. Then $ann(\overline{x})$ is an essential right ideal of R. Since, A is c-pure in B, by definition, there exists, $y \in B$ such that, $ann(\overline{x}) = ann(y)$ and $\overline{x} = \overline{y}$. Since, $\overline{x} \in Z(B/A)$, $y \in Z(B)$. Since, B is non-singular, $Z(B) = 0$ and hence $y = 0$. So, $\overline{x} = \overline{y} = \overline{0}$. Hence, $Z(B/A) = 0$, which implies B/A is non-singular.

Now we give some simple properties of cyclic pure submodules. The proofs are straightforward and hence omitted.
Proposition 1.4: Let A, B, C be R-modules such that A is a submodule of B and B is a submodule of C.
i) If A is c-pure in B and B is c-pure in C then A is c-pure in C.
ii) If A is c-pure in C then A is c-pure in B.
iii) If B is c-pure in C then B/A is c-pure in C/A.
iv)If A is c-pure in C and B/A is c-pure in C/A then B is c-pure in C.

Now we will give one more characterization for c-pure submodules, which will be used later.
Proposition 1.5: A submodule A of an R-module B is c-pure submodule of B if and only if for every submodule C of B containing A, with C/A cyclic, A is a direct summand of C.
Proof:Only if: Let A be a c-pure submodule of an R-module B and let C be a submodule of B containing A such that C/A cyclic. Then by Proposition 1.4(ii) above, A is a c-pure submodule of C. Then C/A, being cyclic, is projective with respect to the canonical short exact sequence, $0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0$ and hence A is direct summand of C.
If: Consider the canonical short exact sequence $\epsilon : 0 \longrightarrow A \longrightarrow B \stackrel{\eta}{\longrightarrow} B/A \longrightarrow 0$ of R-modules. Let M be a cyclic R-module and let $f \in Hom_R(M, B/A)$. Now $f(M)$ is a submodule of B/A and being homomorphic image of a cyclic R-module, it is also cyclic. Let $f(M) = C/A$ for some submodule C of B containing A. Then, by the hypothesis, A is a direct summand of C. Then there exists a homomorphism, say, $g : C \longrightarrow A$ which is identity on A. We now define a map $h : M \longrightarrow B$ as follows. Let $m \in M$. Then $f(m) = c + A$ for some $c \in C$. Define $h(m) = c - g(c)$. We prove that h is a well-defined map. Suppose $f(m) = c_1 + A = c_2 + A$, for some $c_1, c_2 \in C$. Then $c_1 - c_2 \in A$ and hence, $g(c_1 - c_2) = c_1 - c_2$. Then $g(c_1) - g(c_2) = c_1 - c_2$. Hence, $c_1 - g(c_1) = c_2 - g(c_2)$. Thus $h(m)$ is a well-defined element of B and is independent of the choice of c such that $f(m) = c + A$. Clearly h is a homomorphism. Now $(\eta \circ h)(m) = \eta(c - g(c)) = \eta(c) - \eta(g(c)) = \eta(c) = c + A = f(m)$. Hence $\eta \circ h = f$. Hence M is projective with respect to the short exact sequence ϵ. Hence A is a c-pure submodule of B.

Definition 1.6: We recall the definition of **Proper Class** in the sense of MacLane [9].

Let ϕ be a class of short exact sequences of R-modules. Let ϕ_m denote the class of all monomorphisms $f : A \longrightarrow B$ such that, $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is in ϕ for some R-module C and an epimorphism g. Let ϕ_e denote the class of all epimorphisms $g : B \longrightarrow C$ such that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is in ϕ for some R-module A and a monomorphism f. The members of ϕ_m and ϕ_e are called ϕ-monomorphisms and ϕ-epimorphisms, respectively. ϕ is said to be a **Proper Class** in the sense of MacLane[9], if it satisfies the following conditions:

1)ϕ contains all short exact sequences isomorphic to its members.

2)For any two R-modules A,B the canonical short exact sequence,
$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$ is in ϕ.

3)If $\alpha, \beta \in \phi_m$ and $\alpha \circ \beta$ is defined, then $\alpha \circ \beta \in \phi_m$.

4)If α, β are monomorphisms such that $\alpha \circ \beta$ is defined and $\alpha \circ \beta \in \phi_m$ then $\beta \in \phi_m$.

5)If $\alpha, \beta \in \phi_e$ and $\alpha \circ \beta$ is defined, then $\alpha \circ \beta \in \phi_e$.

6)If α, β are epimorphisms such that $\alpha \circ \beta$ is defined and $\alpha \circ \beta \in \phi_e$, then $\alpha \in \phi_e$.

PROPOSITION 1.7: The family of all c-pure short exact sequences of R-modules forms a **Proper Class** in the sense of MacLane [9].
Proof: We prove that the family of all c-pure short exact sequence of R-modules satisfies the conditions of Definition 1.6. Conditions 1) and 2) are obvious. 3),4),5) and 6) follow from i) , ii), iv) and iii) respectively of Proposition 1.4.

Simmons [10,Remark(C)], mentioned the following result. For completeness we give the proof of this result.
Proposition 1.8: If R is a commutative integral domain then every short exact sequence, $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$ of R-modules with M torsion-free, is c-pure.
Proof: Let N be a cyclic R-module. We may assume that $N = R/I$ for some ideal I of R. If $I = 0$ then $N = R$ is projective, and hence, ϵ-projective. If $I \neq 0$. Since R is commutative integral domain, this implies R/I is torsion module. Since by hypothesis, M is torsion-free, $Hom(R/I, M) = 0$. So R/I is trivially ϵ-projective. Thus ϵ is c-pure.

We generalize the above result to noncommutative non-integral domains. Prior to this, we prove the following proposition. The idea of the proof of the following proposition is from the proof of the Theorem 5.2 of Levy [8]. Here our proof is much simpler compare to Levy's.

First we recall the definition of Principal Projective Ring [7, p.249]. A ring R is said to be Principal Projective Ring (in short p.p. ring), if every principal ideal is
projective.

Proposition 1.9: Let R be a semi-prime, two-sided Goldie, right pp.ring. Then every torsion-free, cyclic R-module is projective.

Proof: Since, by hypothesis, R is semiprime, right Goldie ring, by [5,Theorem 5.1], R has a right quotient ring, say, Q. Since, by hypothesis, R is also left Goldie, R has a left quotient ring. So, by [5,p.81],we may assume that Q is also a left quotient ring of R. Let M be a torsion-free, cyclic R-module. Since M is cyclic, we may assume that $M = R/I$ for some right ideal I of R. Since, by [5,Theorem 5.4], Q is semisimple, Q/IQ is projective as a Q-module and hence the canonical short exact sequence $0 \longrightarrow IQ \longrightarrow Q \longrightarrow Q/IQ \longrightarrow 0$ of Q-modules splits. So, there exists a Q-monomorphism, say, $\phi : Q/IQ \longrightarrow Q$. Since Q is a left quotient ring of R, $\phi(1 + IQ) = b^{-1}a$ for some $a, b \in R$ with b regular. We now define a map $\varphi : R/I \longrightarrow R$ by, $\varphi(x + I) = ax$ for each $x + I \in R/I$. We prove that φ is a well defined map. Suppose $x \in I$. Then $x \in IQ$ and hence $x + IQ = \overline{o}$. So, $o = \phi(x + IQ) = \phi(1 + IQ)x = b^{-1}ax$ which implies that $ax = o$. Thus φ is a well-defined map. Clearly it is an R-homomorphism. We claim that φ is a monomorphism. Let $x + I \in R/I$ be such that $\varphi(x + I) = ax = o$. Then $\phi(x + IQ) = \phi(1 + IQ)x = b^{-1}ax = o$ which implies that $x \in IQ$ as ϕ is a monomorphism. Then, by [5, Lemma 5.2], $x = cd^{-1}$ for some $c \in I$, $d \in R$ regular. Then $xd = c \in I$ which implies that $(x + I)d = xd + I = c + I = I$. Since R/I is torsion-free, this implies that $x + I = \overline{o}$. Thus φ is a monomorphism. Since, by hypothesis, R is a right p.p. ring, it follows that R/I, and hence M, is projective.

Proposition 1.10: Let R be a semiprime, two sided Goldie and right pp ring. If A is a submodule of an R-module B such that the factor module B/A is torsion-free, then A is c-pure in B.

Proof: By Proposition 1.5, we need only prove the following. If C is a submodule of B containing A such that C/A is cyclic then A is a direct summand of C. So let C be a submodule of B containing A such that C/A is cyclic. C/A, being a submodule of the torsion-free R-module B/A, is torsion-free. Hence, by above result C/A is projective. This implies A is a direct summand of C. Hence A is c-pure in B.

Since every commutative integral domain is clearly a semi prime, two sided Goldie and right p.p. ring, the above proposition generalizes Simmons' observation mentioned above.

Now, we study the rings, in which every c-pure exact sequence splits.

Proposition 1.11: Let R be a commutative integral domain. Then every c-pure exact sequence of R-modules splits if and only if R is a field.
Proof: We need only prove the 'Only if' part. First we prove that every torsion-free R-module is projective. Let M be any torsion-free R-module and let $M = F/K$ for some free module F and a submodule K of F. Then, by Proposition 1.8, the canonical short exact sequence $\epsilon : 0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ is c-pure. Hence, by hypothesis, ϵ splits. So, M is projective. Thus we have proved that every torsion-free R-module is projective and hence Q, the quotient field of R is projective. Now, Q, being projective, $Hom(Q, R) \neq 0$. Let $0 \neq f \in Hom(Q, R)$. Now, $A = f(Q)$ is a nonzero torsion-free, divisible ideal of R. Since, by hypothesis, R is a commutative integral domain A will be injective and hence direct summand of R. Then $A = R$, because a commutative integral domain will not have proper direct summands. This implies that R is divisible as an R-module and hence R is a field.

We now generalize the above result to the non-commutative case.
Proposition 1.12: Let R be a prime, two-sided Goldie, right p.p. ring. Then every c-pure exact sequence splits if and only if R is a simple, Artinian ring.
Proof: We need only prove the 'Only if' part. First we prove that every torsion-free R-module is projective. Let M be any torsion-free R-module . Then,by Proposition 1.10, every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$ is c-pure and hence by hypothesis, splits. So, M is projective. By hypothesis, R has a two-sided quotient ring Q. Now Q, being torsion-free R-module, it is projective. Hence $Hom(Q, R) \neq 0$. Let $0 \neq f \in Hom(Q, R)$. Then $I = f(Q)$ is a non-zero, divisible right ideal of R. So, $K = RI$ is a non-zero ideal of R and is divisible as a right R-module. K, being an ideal in the ring R, which is prime by hypothesis, it is essential as a right ideal of R, by [8, Lemma 5.5]. Hence, by [5, Theorem 4.8], K contains a regular element, say, s. Then by the divisibility of K, there exists $x \in K$ such that, $xs = s$. So, $x = 1$ and hence $K = R$, proving that R is divisible. Then every regular element of R is invertible in R and hence $R = Q$. So, by [5, Theorem 5.4], R is a simple, Artinian ring.

2. Purity versus cyclic purity

In [10] Simmons studied the relationship between purity and cyclic purity. He has not discussed the inter-implications between these two concepts. We give below examples to show that neither implies the other.
Remark 2.1. In general, purity does not imply cyclic purity.
Example. Let $R = \prod_{\alpha \in \Lambda} R_\alpha$ and $S = \oplus_{\alpha \in \Lambda} R_\alpha$ where $\{R_\alpha\}_{\alpha \in \Lambda}$ is any family of fields. Clearly S in an essential ideal of R. If S is cyclic pure in R then R/S is

projective with respect to the exact sequence $0 \to S \to R \to R/S \to 0$. Then S is a direct summand of R. Since S is essential ideal of R, this implies $R = S$ which is impossible. So S is not cyclic pure in R. Since R is a von-Neumann regular ring, by[4, Theorem 11.24] it follows that S is pure in R.
Remark 2.2. In general, cyclic purity does not imply purity.
Example. Let $R = k[X, Y]$ be a polynomial ring in two variables X, Y over a field k. Here the ideal (X, Y) of R is torsion-free but not flat as R-module. (cf. [1,Chapter I, Exercise 2.3]) Let $(X, Y) = F/K$ for a free module F and a submodule K. Since, R is an integral domain, by Proposition 1.8 above, K is cyclic pure in F. Since $(X, Y) = F/K$ is not flat, K is not pure in F.

3. Absolutely cyclic-pure modules

Definition 3.1: An R-module A is said to be absolutely c-pure if every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules is c-pure.

It is easy to verify that A is absolutely c-pure if and only if it is c-pure in its injective hull.

We prove below some properties of absolutely c-pure modules.
Proposition 3.2: Every c-pure submodule of an injective module is injective.
Proof: Let A be a c-pure submodule of an injective module B. Let I be any right ideal of R. We have the following exact sequence $Hom_R(R/I, B) \xrightarrow{f} Hom_R(R/I, B/A) \longrightarrow Ext^1_R(R/I, A) \longrightarrow Ext^1_R(R/I, B) = 0$ of abelian groups. Here the last group is 0 since, by hypothesis, B is injective . Since A is c-pure in B, f is surjective, and hence, $Ext^1_R(R/I, A) = 0$. Now consider the exact sequence $Hom_R(R, A) \longrightarrow Hom_R(R, A) \longrightarrow Hom_R(I, A) \longrightarrow Ext^1_R(R/I, A) = 0$ of abelian groups. The last group is 0 from above. Hence $Hom_R(R/I, A) \longrightarrow Hom(I, A)$ is surjective. It follows by the Baer's criterion of injectivity that A is injective.

Theorem 3.3: An R-module is absolutely c-pure if and only if it is injective.
Proof: We need only prove the 'only if' part. Let A be an absolutely c-pure R-module then it is c-pure in its injective hull and hence by above proposition it is injective.

4. Cyclic-flat R-modules

Definition 4.1: An R-module M is said to be cyclic flat (in short, c-flat) if every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$ of R-modules is c-pure.
Remark 4.2. i) Clearly every projective module is c-flat.
ii) If R is a commutative integral domain, every torsioin-free R-module is c-flat

(cf. Proposition 1.8).
iii) If R is a semiprime, two sided Goldie and right p.p. ring, every torsion-free R-module is c-flat. (cf. Proposition 1.10).

Proposition 4.3: If an R-module M is c-flat then M is torsion-free.
Proof: Let M be a c-flat R-module and $M = F/K$ for some free module F and a submodule K of F. Then K is c-pure in F. Since, F is torsion-free, by Corollary 1.3(i), M is torsion-free.
Proposition 4.4: Let R be a commutative integral domain. Then an R-module M is c-flat if and only if M is torsion-free.
Proof: Follows from above Proposition 4.3 and Proposition 1.8.
Proposition 4.5: Let R be a semiprime, two sided Goldie and right p.p. ring. An R-module M is c-flat if and only if M is torsion-free.
Proof: Follows from above Proposition 4.3 and Proposition 1.10.

Remark 4.6: A flat module need not be c-flat.
Example: In the example, given after Remark 2.1, R is a von Neumann regular ring and hence R/S is flat. Since, S cannot be c-pure in R, R/S is not c-flat.
Remark 4.7: A c-flat module need not be flat.
Example: In the example, given after Remark 2.2, we have a torsion-free, but not flat R-module. Since, R is commutative integral domain and hence, being torsion-free R-module, (X, Y) is c-flat (cf. Remark 4.2(ii)).

Next we find the rings in which every module is c-flat.
Proposition 4.8: For any ring R, every R-module is c-flat if and only if R is semi-simple.
Proof: We need only prove the 'Only if' part. Let I be any right ideal of R. By hypothesis, the R-module R/I is c-flat. Since R/I is cyclic, it is projective with respect to the canonical cyclic pure short exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ and hence I is a direct summand of R. Hence R is semi-simple.

Remark 4.9: By the Remark 4.7, a c-flat module need not be flat. But, by the above proposition, it follows that if for a ring R every R-module is c-flat then every R-module is flat since every semisimple ring is a von-Neumann regular ring and since over a von-Neumann regular ring R, every R-module is flat [4, Theorem 11.24]. But the converse need not be true. (cf. Example given after Remark 2.1).

5. Cyclic-Regular R-modules

Definition 5.1: An R-module B is called **Cyclic-Regular (in short c-regular)** if every short exact sequence, $\epsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules is c-pure.

Clearly, every semi-simple R-module is regular. We don't know about the converse. However, we have following results, in which we have proved that a c-regular R-module M is semi-simple with some extra condition on M.

Proposition 5.2: Let M be a cyclic R-module. If M is c-regular then M is semi-simple.
Proof: Let N be any submodule of M. Since, M is c-regular, the canonical short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ is c-pure. Since, M is cyclic, the quotient module M/N is also cyclic. Hence M/N is ϵ-projective. So, ϵ splits. Hence, M is semi-simple.

Theorem 5.3: For the ring R the following conditions are equivalent.
i)R is c-regular.
ii)R is a semisimple ring.
iii)Every R-module is c-regular.
Proof: i)$\Longrightarrow$ii) follows from the above Proposition.
i)$\Longrightarrow$iii) and iii)$\Longrightarrow$i) are obvious.

Proposition 5.4: Let M be a finitely generated R-module. If M is c-regular then M is semi-simple.
Proof: Let N be a submodule of M and let K be a complement submodule of M. Now, $N \oplus K$ is an essential submodule of M. Suppose, $N \oplus K$ is a proper submodule of M. Since, M is finitely generated, by [1,Theorem 2.8], there exists a maximal submodule M_o of M containing $N \oplus K$. Since, $N \oplus K$ is essential in M, so is M_o. Since M is c-regular, M is c-pure submodule of M. M/M_o being simple, it is cyclic. So, M/M_o is projective with respect to the canonical c-pure short exact sequence, $0 \longrightarrow M_o \longrightarrow M \longrightarrow M/M_o \longrightarrow 0$. Hence, M_o is a direct summand of M. Since, M is essential, this implies $M_o = M$ which is a contradiction. Hence, $N \oplus K = M$, proving that, N is a direct summand of M. So, M is semi-simple.

Proposition 5.5: Let M be an injective R-module. If M is c-regular then M is semi-simple.
Proof: Let N be any submodule of M. Since, M is c-regular and injective, by above Proposition 3.2, N also injective. Hence, N is a direct summand of M. So, M is semi-simple.

Proposition 5.6: Every c-regular, non-singular R-module is semi-simple.
Proof: Let M be any c-regular, non-singular R-module. By [5, Proposition 1.15], we need only prove that M has no proper essential submodule. Let N be any essential submodule of M. Since, M is c-regular, N is c-pure submodule of M. So, by above Lemma, M/N is non-singular. Also, by [5, Proposition1.21], M/N is singular. This implies M/N is 0, and hence $M = N$. So M is semi-simple.

Corollary 5.7: Let R be a semiprime, right Goldie ring. Then every c-regular torsion-free R-module is semi-simple.
Proof: Follows from the fact that, in a semiprime, right Goldie ring every torsion-free R-module is non-singular [5,Theorem 4.8]

References

1) F.W.Anderson and K.R.Fuller, Rings and Categories of Modules, Springer-Verlag New York. Heidelberg. Berlin, 1991.
2) N.Bourbaki, Commutative Algebra, Springer, 1991.
3) P.M.Cohn, On the free product of associative rings,I,Math.Z., 71 (1959), 380-398.
4) C.Faith, Algebra I:Rings,Modules and Categories, Springer publications 1973.
5) A.W.Goldie, Rings with Maximum Condition, (Lecture Notes). Yale University Math. Dept. (1961).
6) K.R.Goodearl, Ring Theory Nonsingular Rings and Modules, Marcel Dekker Inc.1976.
7) T.Y.Lam, Lectures on Modules and Rings, Springer, 1999.
8) L.Levy, Torsion-free and divisible modules over non-integral-domains, Canad. J. Math., 15 (1963), 132-151.
9) S.S.MacLane, Homology, Springer(1967).
10) J.Simmons, Cyclic purity, a generalization of purity for modules, Houston J.Math. 13, No.1 (1987), 135-150.
11) B.Stenstrom, Pure Submodules, Arkiv for Mat. 7(1967), 159-171.
12) R.B.Warfield Jr., Purity and algebraic compactness for modules, Pacific J.Math. 28(3) (1969), 699-719.
13) R.Wisbauer, Foundations of Modules and Ring Theory, Gordon and Breach Science Publishers 1991.

Received: September 15, 2008