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On the Prime Submodules of Primeful Modules

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Abstract

Let R be a commutative ring with identity. This paper deals some results concerning the radical and minimal prime submodules of a primeful R-module.

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1 Introduction

Throughout this paper R will denote a commutative ring with identity $1 \neq 0$ and all modules are unitary. Further for an ideal of R, $Min(I) = Min(V(I))$ will denote the set of all minimal prime ideals of I.

Let M be an R-module and N be a submodule of M. A proper submodule P of M is said to be prime (or p-prime) if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P : M) = p$. (Here $(N :_R M)$ (or $(N : M)$) denotes the annihilator of the R-module M/N). The set of all prime submodules of M is called the prime spectrum of M and denoted by $Spec_R(M)$ (or $Spec(M)$). Similarly, the collection of all p-prime submodules of M for any $p \in Spec(R)$ is designated by $Spec_p(M)$. Let p be a prime ideal of a ring R. For any submodule N of M, $S_p(N)$ is defined as (see [1])

$$S_p(N) = \{x \in M : tx \in N \ for\ some\ t \in R \backslash p\}.$$

Also $rad(N)$ is defined to be the intersection of all prime submodules of M containing N.

Let M be an R module. The map $\psi : Spec(M) \to Spec(R/Ann(M))$ defined by $\psi(P) = (P : M)/Ann(M)$ for every $P \in Spec(M)$, is called the natural map of $Spec(M)$. An R-module M is called primeful, if $M = 0$ or the natural map of $Spec(M)$ is surjective (see [2]).

Now let M be a primeful R-module and I be an ideal of R containing $Ann_R(M)$. In section 2 of this paper, among the other results, it is shown (see Theorem 2.1 and Proposition 2.4) that $rad(IM)$ can be specified in terms of $S_p(pM)$ (or pM when we have some further conditions), where $p \in V(I)$ (or $p \in Min(V(I))$). Moreover, it is proved that if R is an integral domain of dimension 1 and $(0) \neq p \in V(Ann(M))$, then $S_p(pM)$ is a p-prime submodule minimal over $(\mathbf{0})$ if and only if $S_{(0)}(\mathbf{0}) \nsubseteq S_p(pM)$ (see Theorem 2.6).

2 Main results

Theorem 2.1. Let M be a primeful R-module and I an ideal of R containing $Ann_R(M)$. Then $rad(IM) = \bigcap_{p\in V(I)} S_p(pM)$.

Proof. If $IM = M$, then $rad(IM) = M$. On the other hand, if $p \in V(I)$, then $p \supseteq Ann_R(M)$. This implies that $(P : M) = p$ for some $P \in Spec(M)$. Therefore $IM \subseteq pM \subseteq P \subset M$, which is a contradiction. It follows that $V(I) = \emptyset$, so

$$\bigcap_{p\in V(I)} S_p(pM) = M = rad(IM).$$

Hence we assume that $IM \neq M$. Now since M is a primeful R-module, one can see that M/IM is also primeful. By [2, Theorem 5.2], this implies that $rad(IM) = \bigcap_{p\in V(IM:M)} S_p(IM+pM)$. Now we show that $V(IM : M) = V(I+Ann_R(M))$. Clearly, $V(IM : M) \subseteq V(I + Ann_R(M))$. So we assume that, $p \in V(I+Ann_R(M))$. By [2, Proposition 3.4], $p \in V(Ann_R(M)) = Supp(M)$. But by [2, Theorem 4.1], M_p is primeful. Thus M satisfies the Nakayama's Lemma by [2, Corollary 3.2]. It follows that $p \in Supp_R(M/IM)$. Now since M/IM is primeful, $Supp_R(M/IM) = V(IM : M)$ by [2, Proposition 3.4]. Therefore $p \in V(IM : M)$, so $V(I + Ann_R(M)) \subseteq V(IM : M)$. So by the above arguments we have, $rad(IM) = \bigcap_{p\in V(I)} S_p(pM)$ as desired.

Remark 2.2. Note that in the Theorem 2.1, the condition "M be primeful" can not be omitted. To see this let $M = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ and $I = 0$. One can see that

$$Spec(M) = \{p \oplus \mathbb{Z}(p^\infty) : p \in Spec(\mathbb{Z})\}.$$

This implies that $rad(IM) = \mathbb{Z}(p^\infty)$, while $\bigcap_{p\in Min(I)} pM = 0$.

Corollary 2.3. Let M be a primeful R-module and I be an ideal of R containing $Ann_R(M)$. Then

(a) $((radIM) : M) = \sqrt{I}$ (see [2, Theorem 5.6 (4)])

(b) $radIM = \sqrt{I}M$, in each of the following cases: (see [2, Theorem 5.5])

  (1) M is a multiplication R-module;

  (2) M is a flat content module;

  (3) M is a flat module over ring R with Noetherian spectrum.

Proof. These parts follows immediately from Theorem 2.1.

Proposition 2.4. Let M be a non-zero primeful R-module and let I be an ideal of R containing $Ann(M)$. Then

$$rad(IM) = \bigcap_{p \in Min(I)} pM$$

in each of the following cases:

(a) M is a torsion-free module over a one dimensional integral domain R.

(b) M is a flat R-module.

Proof. (a) By Theorem 2.1, we have $rad(IM) = \bigcap_{p \in V(I)} S_p(pM)$. If $Ann(M) \neq 0$, then for every $p \in V(I)$, $S_p(pM) = pM$. It turns out that $rad(IM) = \bigcap_{p \in Min(I)} pM$ as required. So we assume that M is a faithful R-module and $I = 0$. Then $rad(IM) = 0$. Also $\bigcap_{p \in Min(I)} S_p(pM) \subseteq S_{(0)}(\mathbf{0}) = (\mathbf{0})$.

(b) Let $p \in V(I)$. Then $p \in V(Ann_R(M))$, so $pM \neq M$ by [2, Result 2]. This implies that $pM \in Spec(M)$ by [3, Theorem 3]. Therefore $S_p(pM) = pM$ for every $p \in V(I)$. It follows that $rad(IM) = \bigcap_{p \in Min(I)} pM$. This completes the proof.

Proposition 2.5. Let M be an R-module and $N \leq M$ such that M/N is a primeful R-module. If p is a minimal ideal of $(N : M)$, then $S_p(pM + N)$ is a p-prime submodule of M which is minimal over N.

Proof. Set $\bar{M} = M/N$. Since $\bar{M}$ is primeful and $p \in V(N : M) = V(Ann(\bar{M}))$, $S_p(p\bar{M})$ is a p-prime submodule of $\bar{M}$ by [2, Theorem 2.1]. But $S_p(p\bar{M}) = S_p(\overline{N + pM})$. Hence $S_p(N + pM)$ is a p-prime submodule of M by [1, Result 1]. Now let $K \in Spec(M)$ with $N \subseteq K \subseteq S_p(N + pM)$. Then $(N : M) \subseteq (K : M) \subseteq (S_p(N + pM) : M) = p$. This implies that $(K : M) = p$, so $pM \subseteq K$. Hence $N + pM \subseteq K \subseteq S_p(N + pM)$. By [1, Result 3], it follows that $S_p(N + pM) = K$. This completes the proof.

Theorem 2.6. Let R be an integral domain of dimension 1, M any non-zero primeful R-module, and $(0) \neq p \in V(Ann(M))$. Then $S_p(pM)$ is a p-prime submodule minimal over $(\mathbf{0})$ if and only if $S_{(0)}(\mathbf{0}) \nsubseteq S_p(pM)$.

Proof. By [2, Theorem 2.1], $S_p(pM) \in Spec_p(M)$. Now let $S_p(pM)$ be a p-prime submodule minimal over $(\mathbf{0})$. If $S_{(0)}(\mathbf{0}) = M$, then $S_{(0)}(\mathbf{0}) \nsubseteq S_p(pM)$. Hence we may assume that $S_{(0)}(\mathbf{0}) \neq M$. This implies that $S_{(0)}(\mathbf{0}) \in Spec_{(0)}(M)$ by [1, Lemma 4.5]. Now $S_{(0)}(\mathbf{0}) \subseteq S_p(pM)$ implies that $S_{(0)}(\mathbf{0}) = S_p(pM)$, so $p = (0)$, a contradiction. To see the revers implication let $S_{(0)}(\mathbf{0}) \nsubseteq S_p(pM)$ and K be a prime submodule of M with $(\mathbf{0}) \subseteq K \subseteq S_p(pM)$. Then we have $(0) \subseteq ((\mathbf{0}) : M) \subseteq (K : M) \subseteq (S_p(pM) : M) = p$. Since $dimR = 1$, $(K : M) = (0)$ or p. If $(K : M) = (0)$, then $K \in Spec_{(0)}M$. By assumptions $S_{(0)}(\mathbf{0}) \neq (\mathbf{0})$, so $(\mathbf{0})$ is not a (0)-prime submodule. This implies that $S_{(0)}(\mathbf{0}) \subseteq K$ by [1, Result 3(2)]. Thus $(\mathbf{0}) \neq S_{(0)}(\mathbf{0}) \subseteq K \subseteq S_p(pM)$, a contradiction. Hence we may assume that $(K : M) = p$, so $K \in Spec_pM$ and $pM \subseteq K$. This implies that $k = S_p(pM)$ and the proof is completed.

Corollary 2.7. Let R be an integral domain of dimension 1. If M is a non-zero primeful torsion R-module. Then for every $(0) \neq p \in V(Ann(M))$, p is a minimal prime ideal of $Ann(M)$ and $S_p(pM)$ is a p-prime submodule minimal over $(\mathbf{0})$.

Proof. By [2, Proposition 2.6], $Ann(M) \neq (0)$. Also $S_p(pM) \in Spec_p(M)$ by [2, Theorem 2.1]. It follows that $(0) \subsetneq Ann(M) \subseteq p$. Since $dimR = 1$, p is a minimal prime ideal of $Ann(M)$. But $S_{(0)}(\mathbf{0}) = T(M) = M \neq S_p(pM)$, where $T(M)$ is the torsion submodule of M. By Theorem 2.6, it turns out that $S_p(pM)$ is minimal over $(\mathbf{0})$ as desired.

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