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# A Note on the Partially Local Formations of Finite Soluble Groups

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**Abstract.** In this article we study the following question: If $\mathfrak{F}$ and $\mathfrak{H}$ are $\omega$-saturated formation and $\mathfrak{F}_{\mathfrak{H}}$ is the class of all soluble groups $G$ with $\mathfrak{F}$-normalizer in $\mathfrak{N}_{\omega'}\mathfrak{H}$. In general $\mathfrak{F}_{\mathfrak{H}}$ is formation but need not be an $\omega$-saturated formation. Here the smallest $\omega$-saturated formation containing $\mathfrak{F}_{\mathfrak{H}}$ is studied.

**Mathematics Subject Classification:** 20D10, 20D15

**Keywords:** Formations, $\omega$-Local formations, Screen of formation

## 1. Introduction

All groups considered in this paper are finite and soluble. The reader is assumed to be familiar with the theory of formation of finite groups. The notation and terminology is consistent with [3].

First we introduce the question we aim to analyze in this paper together with the specific notation which is used throughout.

Henceforth, $\omega$-denotes a non-empty set of primes. We denote by $\omega'$- the set $\mathbf{P}\backslash\omega$. Let $\mathfrak{H}$ be an $\omega$-saturated formation, $\mathfrak{H} = LF_{\omega}(H)$ where $H$ is the $\omega$-local screen of $\mathfrak{H}$. The class $\mathfrak{X} = \mathfrak{F}_{\mathfrak{H}}$ is a formation but in general $\mathfrak{X}$ is not $\omega$-saturated. Our purpose is to study the $\omega$-saturation $\overline{\mathfrak{X}}$ of $\mathfrak{X}$, where $\mathfrak{X} = \langle Q, R_0, E_{\Phi_\omega}\rangle\mathfrak{X}$ the smallest $\omega$-saturated formation containing $\mathfrak{X}$. The $\omega$-local satellite of $\overline{\mathfrak{X}}$ is denoted by $\overline{\mathrm{X}}$ . The analogous question for projectors, i.e. the saturation of the formation $\mathfrak{H} \downarrow \mathfrak{F} = (G : Proj_{\mathfrak{H}}(G) \subseteq \mathfrak{F})$, was fully analyzed by Doerk in [1].

**Theorem 1.1.** *(see Th. V.3.18, [3]) If $F$ is the canonical local definition of the local formation $\mathfrak{F}$, then the formation function $f(p) = \mathfrak{F}_{F(p)}$ is the largest local definition of $\mathfrak{F}$, i.e.e if $h$ is a formation function such that $\mathfrak{F} = LF(h)$ then for all primes $p$.*

**Theorem 1.2.** *(see Th.1, [6]) For every formation $\mathfrak{F}$, the following conditions are equivalent:*

*(1)* $\mathfrak{F}$ *is* $\omega$*-saturated formation.*
*(2)* $\Re_p\mathfrak{F}_{F(p)} \subseteq \mathfrak{F}$ *for all* $p \in \omega$*.*
*(3)* $\mathfrak{F} = LF_\omega(f)$*, where* $f(\omega') = \mathfrak{F}$ *and* $f(p) = \Re_p\mathfrak{F}_{F(p)}$ *for all* $p \in \omega$*.*
*(4)* $\mathfrak{F}$ *is* $\omega$*-local formation.*

## 2. A description of the class $\mathfrak{H}_\mathfrak{F}$

**Proposition 2.1.** *(a) Given a group* $G$ *the following statements are pairwise equivalent:*
*(i)* $G \in \mathfrak{H}_\mathfrak{F}$*.*
*(ii)* $Nor_\mathfrak{H}(G) = Nor_\mathfrak{F}(G)$*.*
*(iii) All* $\mathfrak{H}$*-central chief* $\omega$*-factors of* $G$ *are* $\mathfrak{F}$*-central.*
*(b)* $\mathfrak{H}_\mathfrak{F} \subseteq \mathfrak{K} = (G : G^\mathfrak{H} = G^\mathfrak{F})$*.*
*(c)* $\mathfrak{H}_\mathfrak{F} = \mathfrak{K}$ *if and only if* $\mathfrak{H}_\mathfrak{F}$ *is* $\omega$*-saturated formation.*

*Proof.* (a) By the cover and avoidance properties of the normalizer (see Th.V.3.2, [3]) that $H_\mathfrak{F}$ is class of all groups such that all $\mathfrak{H}$-central chief factors are $\mathfrak{F}$-central. Notice that in every group, $\mathfrak{H}$-critical maximal subgroups are indeed $\mathfrak{F}$-critical maximal subgroups and then every $\mathfrak{H}$-normalizer contains an $\mathfrak{F}$-normalizer.So, if $G \in \mathfrak{H}_\mathfrak{F}$ we have $Nor_\mathfrak{H}(G) = Nor_\mathfrak{F}(G)$.

(b) Let $G \in \mathfrak{H}_\mathfrak{F}$. If $D \in Nor_\mathfrak{H}(G)$, then $D \in \mathfrak{F}$ and $G = DG^\mathfrak{F}$. Then $G/G^\mathfrak{F} \in \mathfrak{F}$ and $G^\mathfrak{H} = G^\mathfrak{F}$. So, $\mathfrak{H}_\mathfrak{F} \subseteq \mathfrak{K}$.

(c) Suppose that $\mathfrak{H}_\mathfrak{F}$ is $\omega$-local and let $G$ be a group in $\mathfrak{K}\backslash\mathfrak{H}_\mathfrak{F}$ of minimal order. Since $\mathfrak{K} = Q\mathfrak{K}$ the group G is in $b(\mathfrak{H}_\mathfrak{F})$ and then $G$ is a primitive group. Denote by $N$ a minimal normal subgroup and suppose that $N$ is an $\omega d$-group. Since $G/N \in \mathfrak{H}_\mathfrak{F}$, if $D \in Nor_\mathfrak{H}(G)$ then $D/(D \cap N) \simeq DN/N \in \mathfrak{F}$ and therefore $N \leq D$. This means that $N$ is $\mathfrak{H}$-central. Recall that $N = C_G(N)$. So $G/N \in H(p)$ and then $G \in \mathfrak{S}_pH(p) = H(p) \subseteq \mathfrak{H}$ for all $p \in \omega$. This implies that $G^\mathfrak{H} = 1$ and since $G \in \mathfrak{K}$ we have $G^\mathfrak{F} = G^\mathfrak{H} = 1$. Then $G \in \mathfrak{F}$. This implies that $G \in \mathfrak{H}_\mathfrak{F}$, a contradiction. Hence $\mathfrak{H}_\mathfrak{F} = \mathfrak{K}$.

Conversely, suppose $\mathfrak{H}_\mathfrak{F} = \mathfrak{K}$ and there exists a group $G$ of minimal order in $E_{\Phi\omega}\mathfrak{K}\backslash\mathfrak{K}$. Then $G$ is monolithic and its socle $N$ is a minimal normal subgroup of $G$ contained in $\Phi(G)$. Since $G/N \in \mathfrak{K}$, we have $G^\mathfrak{F}N = G^\mathfrak{H}N$ and since $G^\mathfrak{H} \leq G^\mathfrak{F}$ we have $G^\mathfrak{F} = G^\mathfrak{H}(G^\mathfrak{F} \cap N)$. since $G \notin \mathfrak{K}$ we have $N \leq G^\mathfrak{F}$ and $G^\mathfrak{F} = G^\mathfrak{H}N$. But $G^\mathfrak{F}/G^\mathfrak{H} = NG^\mathfrak{H}/G^\mathfrak{H} \leq \Phi(G)G^\mathfrak{H} \leq \Phi(G/G^\mathfrak{H})$. Since $G/G^\mathfrak{F} \in \mathfrak{F}$ and $\mathfrak{F}$ is $\omega$-saturated we have $G/G^\mathfrak{H} \in \mathfrak{F}$ and then $G \in \mathfrak{K}$, a contradiction. Hence $\mathfrak{K}$ is $\omega$-saturated formation. □

## 3. The $\omega$-local satellite of the $\omega$-saturated formation $\mathfrak{H}_\mathfrak{F}$

**Lemma 3.1.** *(i) For each prime* $p$ *we have* $\mathfrak{H}_{F(p)} \subseteq \overline{X}(p) \subseteq \mathfrak{S}_p\mathfrak{X}$*.*

*(ii) If moreover* $\mathfrak{F} \nsubseteq H(p)$*, for some prime* $p$ *from* $\omega$ *then* $\overline{X}(p) = \mathfrak{S}_p\mathfrak{X}$*. Hence, if the formation* $\mathfrak{X}$ *is* $\omega$*-saturated, i.e.* $\mathfrak{X} = \overline{\mathfrak{X}}$*, and* $\mathfrak{F} \nsubseteq H(p)$ *for some* $p$*, then* $\mathfrak{S}_p\mathfrak{X} = \mathfrak{X}$*.*

*Proof.* (i) The class $\mathfrak{N}_\omega\mathfrak{X}$ is a $\omega$-saturated formation. So $\overline{\mathfrak{X}} \subseteq \mathfrak{N}_\omega\mathfrak{X}$ . Then $\overline{X}(p) \subseteq \mathfrak{S}_p\mathfrak{X}$ for every prime $p$ of $\omega$. Suppose there exists a group $G \in \mathfrak{H}_{F(p)}\backslash\overline{X}(p)$ of minimal order. Then $G$ is monolithic and its socle is a minimal normal $\omega^{'}$-group. Let $V$ be an irreducible and faithful $G$-module over $GF(p)$ and construct the semidirect product $Y = [V]G$, (see Cor.B.10.7, [3]). If $Y \in \mathfrak{H}_{F(p)}$ we have $G \in F(p)$. Then $Y \in N_pF(p) \subseteq F(p) \subseteq \mathfrak{F}$ and $Y \in \mathfrak{X} \subseteq \overline{\mathfrak{X}}$. Now $G \simeq Y/O_{p'p}(Y) \in \overline{\mathfrak{X}}(p)$, a contradiction. So $Y \notin \mathfrak{H}$ and $Nor_{\mathfrak{H}}(G) \subseteq Nor_{\mathfrak{H}}(Y)$, by [3] Prop.V.3.8. Then $Y \in \mathfrak{X} \subseteq \overline{\mathfrak{X}}$ and again $G \in \overline{\mathfrak{X}}(p)$, a contradiction. Therefore $\mathfrak{H}_{F(p)} \subseteq \overline{X}(p)$.

(ii) Suppose that $\mathfrak{F} \nsubseteq H(p)$ for some prime $p$ of $\omega$ and $\overline{\mathfrak{X}}(p) \neq \mathfrak{N}_p\mathfrak{X}$. let $G$ be a group of minimal order in $\mathfrak{N}_p\mathfrak{X}\backslash\overline{\mathfrak{X}}(p)$. Then $G$ is monolithic group whose socle is a minimal $\omega^{'}$-subgroup of $G$. Hence, in fact, $G \in \mathfrak{X}$. Let $V$ be an irreducible and faithful $G$-model over $GF(p)$ and construct the semidirect product $Y = [V]G$. If $Y \notin \mathfrak{H}$ then $Nor_{\mathfrak{H}}(G) \subseteq Nor_{\mathfrak{H}}(Y)$. So $Y \in \mathfrak{X} \subseteq \overline{\mathfrak{X}}$ . Hence $Y/O_{p'p}(Y) \simeq G \in \overline{\mathfrak{X}}(p)$, a contradiction. Therefore $Y \in \mathfrak{H}$ and $G \in H(p) \subseteq \mathfrak{H}$. So, $G \in \mathfrak{X} \cap \mathfrak{H} = \mathfrak{F}$. Let $H$ be a group of minimal order in $\mathfrak{F}\backslash H(p)$. Again $H$ is a monolithic group whose socle is $\omega^{'}$-subgroup. Construct the direct product $D = G \times H$. Then $D \in \mathfrak{F}$ and by Cor.B.10.7, [3] there exists an irreducible and faithful $D$-module $W$ over $GF(p)$. Construct the semidirect product $K = [W]D$. If $K \in \mathfrak{H}$ then $D \in Nor_{\mathfrak{H}}(K)$ and $K \in \mathfrak{X} \subseteq \overline{\mathfrak{X}}$ . So $G \simeq K/O_{\overline{p}p}(K) \in \overline{\mathfrak{X}}(p)$, a contradiction. Hence $W \in \mathfrak{H}$ and $D \in H(p)$ for all $p \in \omega$. But this implies that $H \in H(p)$ (for all $p \in \omega$), a contradiction. So, our claim holds and $\overline{\mathfrak{X}}(p) = \mathfrak{N}_p\mathfrak{X}$. Finally, notice that if $X$ is $\omega$-saturated and $\mathfrak{F} \nsubseteq H(p)$ then $\overline{\mathfrak{X}}(p) = \mathfrak{N}_p\mathfrak{X} \subseteq \overline{\mathfrak{X}} = \mathfrak{X}$ for every $p$ from $\omega$. □

**Lemma 3.2.** *Assume that $\mathfrak{X} = H_{\mathfrak{F}}$ is an $\omega$-saturated formation, i.e. $\mathfrak{X} = \overline{\mathfrak{X}}$. If $\mathfrak{F} \subseteq H(p)$ then $\overline{\mathfrak{X}}(p) = \mathfrak{H}_{F(p)}$ for all $p \in \omega$. Hence if the formation $\mathfrak{X} = \mathfrak{H}_{\mathfrak{F}}$ is $\omega$-saturated, then $\overline{\mathfrak{X}}(p) = \mathfrak{H}_{F(p)}$, when $\mathfrak{F} \subseteq H(p)$ and $\overline{\mathfrak{X}}(p) = \mathfrak{N}_p\mathfrak{X}$, if $\mathfrak{F} \nsubseteq H(p)$.*

*Proof.* By lemma 3.1.(i) we have $\mathfrak{H}_{F(p)} \subseteq \overline{\mathfrak{X}}(p)$ for all $p \in \omega$. Let $G$ be a group of minimal order in $\overline{\mathfrak{X}}(p)\backslash\mathfrak{H}_{F(p)}$. Then $G$ is monolithic group. Suppose that $N = Soc(G)$ is an $\omega d$-group. Then $G \in \mathfrak{N}_p\mathfrak{H}_{F(p)}$. Let $D \in Nor_{\mathfrak{H}}(G)$. Then $DN/N \in Nor_{\mathfrak{H}}(G/N) \subseteq F(p)$ and then $D \in \mathfrak{N}_pF(p) = F(p)$. Therefore we have indeed $G \in \mathfrak{H}_{F(p)}$, a contradiction. Hence, $Soc(G)$ is minimal normal $\omega'$-subgroup and by lemma 3.1.(i) we have $G \in \mathfrak{X}$. If $G \notin \mathfrak{H}$ and $D \in Nor_{\mathfrak{H}}(G)$ then $D$ is proper subgroup of $G$ and since $D$ is a well-placed subgroup of $G$ we have $D \in \overline{\mathfrak{X}}(p)$ by [3] Cor.IV.1.15(a). The minimality of $G$ forces $G \in \mathfrak{H}_{F(p)}$ and this implies $G \in \mathfrak{H}_{F(p)}$, a contradiction. So $G \in \mathfrak{H}$. Then $G \in \mathfrak{X} \cap \mathfrak{H} = \mathfrak{F} \subseteq H(p)$. Let $V$ be an irreducible and faithful $G$-module over $GF(p)$ and construct the semidirect product $Y = [V]G$. Then $Y \in \mathfrak{N}_pH(p) = H(p) \subseteq \mathfrak{H}$. On the other hand $Y \in \mathfrak{N}_p\overline{\mathfrak{X}}(p) = \overline{\mathfrak{X}} = \mathfrak{X}$ and therefore $Y \in \mathfrak{X} \cap \mathfrak{H} = \mathfrak{F}$. So $G \simeq Y/O_{p'p}(Y) \in F(p)$ and then $G \in \mathfrak{H}_{F(p)}$, the final contradiction. Therefore our claim is true. □

**Proposition 3.3.** *The formation $\mathfrak{X} = \mathfrak{H}_{\mathfrak{F}}$ is $\omega$-saturated, i.e. $\mathfrak{X} = \overline{\mathfrak{X}}$, if and only if the following two conditions hold: (i) If $\mathfrak{F} \subseteq H(p)$ then $\overline{\mathfrak{X}}(p) = \mathfrak{H}_{F(p)}$, and (ii) if $\mathfrak{F} \nsubseteq H(p)$ then $H(p) \cap \mathfrak{F} = F(p)$ for all $p \in \omega$.*

*Proof.* Suppose that $\mathfrak{X}$ is an $\omega$-saturated formation. Then part (i) holds by Lemma 3.1. If $\mathfrak{F} \nsubseteq H(p)$ then $\overline{\mathfrak{X}}(p) = \mathfrak{N}_p\mathfrak{X}$ by (ii) of Lemma 3.1. Consider a group of minimal order in $(H(p) \cap \mathfrak{F})\backslash F(p)$. Then $G$ is a monolithic group whose socle is a minimal normal $\omega'$-subgroup. If $V$ is an irreducible and faithful $G$-module over $GF(p)$. we can construct the semidirect product $Y = [V]G$. Now $Y \in \mathfrak{N}_p\mathfrak{H}(p) = H(p) \subseteq \mathfrak{H}$, where $p \in \omega$. On the other hand $G \in \mathfrak{X}$. By Lemma 3.1, $Y \in \mathfrak{X}$ and then $Y \in \mathfrak{H} \cap \mathfrak{X} = \mathfrak{F}$. Hence $G \simeq Y/O_{p'p}(Y) \in F(p)$, a contradiction. So, (ii) holds.

Conversely, suppose that (i) and (ii) hold and let $G$ be a group of minimal order in $\overline{\mathfrak{X}}\backslash\mathfrak{X}$. Then $G$ is a monolithic group and it is socle is minimal normal $p$-subgroup for some $p$ from $\omega$. So $O_{\omega'}(G) = 1$. Notice that $G \in \overline{\mathfrak{X}}(p)$. If $\mathfrak{F} \nsubseteq H(p)$ then, by Lemma 3.1(ii), $G \in \overline{\mathfrak{X}}(p) = \mathfrak{N}_p\mathfrak{X}$.If $G \notin \mathfrak{H}$ and $D \in Nor_{\mathfrak{H}}(G)$ then $D$ is a proper subgroup of $G$. Since $D$ is well-placed subgroup of $G$, $D \in \overline{\mathfrak{X}}$ by [3] Cor.IV.1.15(a) again. By minimality of $G$ we have $D \in \mathfrak{X}$. Thus, $D \in \mathfrak{X} \cap \mathfrak{H} = \mathfrak{F}$ and then $G \in \mathfrak{X}$, a contradiction. Therefore $G \in \mathfrak{H}$. Notice that $Soc(G) \leq O_p(G) = O_{p'p}(G)$ and so, $G/O_p(G) \in H(p)$. We have also by minimality that $G/O_p(G) \in \mathfrak{X}$ and therefore $G/O_p(G) \in \mathfrak{X} \cap \mathfrak{H} = \mathfrak{F}$. Hence $G/O_p(G) \in \mathfrak{F} \cap H(p) = F(p)$ by (ii) and this implies that $G \in \mathfrak{N}_pF(p) = F(p) \subseteq \mathfrak{F}$ and then $G \in \mathfrak{X}$, a contradiction. If $\mathfrak{F} \subseteq H(p)$ then by (i) $G \in \overline{\mathfrak{X}}(p) = \mathfrak{H}_{F(p)} \subseteq \mathfrak{H}_{\mathfrak{F}} = \mathfrak{X}$, a contradiction. Thus $\overline{\mathfrak{X}} = \mathfrak{X}$ and $\mathfrak{X}$ is $\omega$-saturated. □

## 4. The main results

The following properties will be essential in the proof of the main theorem

**Proposition 4.1.** *If the formation $\mathfrak{X} = \mathfrak{H}_{\mathfrak{F}}$ is $\omega$-saturated and $char(\mathfrak{H}) = char(\mathfrak{F}) \subseteq \omega$, then $\mathfrak{F} \subseteq H(p)$ for each prime $p \in \omega$.*

*Proof.* Suppose there exists a prime $p \in \omega$ such that $\mathfrak{F} \nsubseteq H(p)$. By Lemma 3.1(ii) we have $\mathfrak{X}(p) = \mathfrak{N}_p\mathfrak{X}$ and by proposition 3.3, $H(p) \cap \mathfrak{F} = F(p)$. Let $G$ be a group of minimal order in $H(p)\backslash\mathfrak{F}$; then $G$ is primitive and it can be factorized as $G = MN$ where $N$ is self-centralizing minimal normal $q$-subgroup of $G$ for some prime $q$, and $M$ is a core-free maximal subgroup of $G$ complementing $N$. Clearly $M \in \mathfrak{F}$ and $M \in H(q)\backslash F(q)$. First notice that $q \neq p$ since otherwise we could have $G \in \mathfrak{N}_p\mathfrak{X} = \mathfrak{X}$ and $G \in \mathfrak{H} \cap \mathfrak{X} = \mathfrak{F}$, a contradiction. Denote by $I$ a set composed by one and only one representative of each isomorphism class of irreducible $GF(p)[M]$-modules. If $V \in I$ denote by $P(V)$ its projective cover. Construct the direct sum $P = \oplus_{V \in I} P(V)$. Since the regular module is faithful we have $1 = ker(M \text{ on } GF(p)[M]) = ker(M \text{ on } P) = C_M(P)$ and $P$ is faithful for $M$. Notice that $Soc(P) = \oplus_{V \in I} V$ and therefore it is the direct sum of pairwise non-isomorphic irreducible $M$-modules. Construct the semidirect product $K = [P]M$. Since $P$ is faithful, we have $Soc(K) = Soc(P)$. If $V$ is an $\mathfrak{H}$-central chief $\omega$-factor of $K$ under $P$, then $M/C_M(V)$ is in $H(p)$ and therefore $M/C_M(V) \in H(p) \cap \mathfrak{F} = F(p)$, i.e. $V$ is $\mathfrak{F}$-central. This means that $K \in \mathfrak{X}$. Since $q \neq p$ we apply Cor. B.10.7 of [3] and there exists an irreducible and faithful $K$-module $W$ over $GF(q)$. Construct the semidirect product $Y = [W]K$. Assume that $W$ is $\mathfrak{H}$-central in $Y$. Then $K \in H(q) \cap \mathfrak{X} \subseteq \mathfrak{F}$. Since $q \in \omega$ we can say that $K/O_{q'}(K) \in F(p)$. But $O_{q'}(K) \leq C_K(P) = P$, that is $O_{q'}(K) = 1$, and $K, M \in F(q)$, a contradiction. Hence $W$ is $\mathfrak{H}$-eccentric in $Y$ and $Y \in \mathfrak{X}$. We can consider $N$ as an irreducible $K$-module with $ker(K$ on $N) = P$; since $W$ is a faithful $K$-module we can apply Steinberg's theorem (see Th.B.10.3, [3]) to conclude there exists a natural number $n$ such that $N$ is an irreducible $K$-submodule of $W^n = W \oplus ... \oplus W$. Consider the Hartley group $H = H(W_1, ...W_n)$ where $W_i \simeq W$ for $i = 1, ..., n$ (see [3] pp. 197-203). Then $H/\Phi(H) \simeq W_1 \oplus ... \oplus W_n$ and $W^n$ is isomorphic to a subgroup in $Z(H) \cap \Phi(H)$. Construct the semidirect product $L = [H]K$. Then $L/\Phi(L) \in QR_0(Y) \subseteq \mathfrak{X}$. But in $L$ there exists a minimal normal subgroup which is $L$-isomorphic to $N$ and it is $\mathfrak{H}$-central and $\mathfrak{F}$-eccentric. So $L \notin \mathfrak{X}$. Therefore $\mathfrak{X}$ is not an $\omega$-saturated formation, a contradiction. Hence our claim is true: $\mathfrak{F} \subseteq H(p)$ for every prime number $p \in \omega$. □

**Proposition 4.2.** *If the formation $\mathfrak{X} = \mathfrak{H}_{\mathfrak{F}}$ is $\omega$-saturated then:*

*(i) $\mathfrak{H}$ is full characteristic, and*
*(ii) $char(\mathfrak{F}) = char(\mathfrak{X}) = \omega$.*

*Proof.* (i) Suppose that $char(\mathfrak{H}) = \pi$ and consider a prime number $p \in \pi'$. Let $G$ be a group of minimal order in $\mathfrak{H} \backslash \mathfrak{F}$. Following the notation of proposition 4.1 we can say $q \neq p$ since $q \in \pi$. Clearly $M$ is $p'$-group. So, now $GF(p)[M]$ is completely reducible and $P$ is the direct sum. Construct the semidirect product. Notice that $K \in \mathfrak{H}$ since $p$ divides $|K|$. Since $P$ is faithful for $M$ we have $Soc(K) = P$. Clearly $K \in \mathfrak{X}$ and since $q \neq p$ there exists an irreducible and faithful $K$-module $W$ over $GF(q)$. Construct the semidirect product $Y = [W]K$. Since $Y/C_Y(W) \simeq K \notin H(q)$ we have that $W$ is $\mathfrak{H}$-eccentric in $Y$ and $Y \in \mathfrak{X}$. Arguing as in proposition 4.1 we obtain that $\mathfrak{X}$ is not an $\omega$-saturated formation. This is again a contradiction. So, $\mathfrak{H}$ is of full characteristic.

(ii) If $p \in char(\mathfrak{F})$ then $C_p \in \mathfrak{F} \subseteq \mathfrak{X}$ and then $p \in char(\mathfrak{X})$. If $p \notin char(\mathfrak{X})$ then $C_p \in \mathfrak{H} \backslash \mathfrak{F}$ . So $C_p \notin \mathfrak{X}$ and $p \notin char(\mathfrak{X})$. □

With this, if the formation $\mathfrak{H}_{\mathfrak{F}}$ is $\omega$-saturated we distinguish two possibilities:
(A) $char(\mathfrak{F}) = \pi$ is a proper subset of prime numbers, or
(B) $\mathfrak{F}$ is of full characteristic.
We analyze first case (A). Denote by $\mathfrak{N}$ the class of all nilpotent groups.

**Lemma 4.3.** *If $X = \mathfrak{H}_F$ is $\omega$-saturated, $\mathfrak{H}$ is of full characteristic and $char(\mathfrak{F}) = \pi \subseteq \omega$ is a proper subset of prime numbers, then $\mathfrak{N}_p\mathfrak{F} \subseteq \mathfrak{H}$ for every prime number $p$ from $\omega$. Hence, $\mathfrak{N}_{\mathfrak{F}} \subseteq \mathfrak{H}$.*

*Proof.* Suppose that $\mathfrak{N}_p\mathfrak{F} \nsubseteq \mathfrak{H}$ and consider a group $G$ of minimal order in $\mathfrak{N}_p\mathfrak{F}\backslash\mathfrak{H}$; then $G$ is a primitive group, $G = MN$ where $N = Soc(G)$ is a self-normalizing minimal normal $p$-subgroup of $G$ and $M$ is a core-free maximal subgroup such that $M \in \mathfrak{F}$. Since $G \notin \mathfrak{H}$ it follows that $M \in Nor_{\mathfrak{H}}(G)$ and then $G \in \mathfrak{X}$. Since $p$ divides $|G|$ and $\mathfrak{X}$ is $\omega$-saturated we have $p \in char(\mathfrak{X}) = char(\mathfrak{F}) = \pi = \omega$. Take a prime $q \in \omega^{'}$ and consider an irreducible and faithful $G$-module over $GF(p)$. Construct the semidirect product $Y = [V]K$. Now $Y \notin \mathfrak{H}$ and therefore $Nor_{\mathfrak{H}}(G) \subseteq Nor_{\mathfrak{H}}(Y)$ and $Y \in \mathfrak{X}$. Since $q$ divides $|Y|$ and $\mathfrak{X}$ is $\omega$-saturated we have that $q \in \omega$, a contradiction. Hence $\mathfrak{N}_p\mathfrak{F} \subseteq \mathfrak{H}$ for every prime number $p \in \omega$. Consider a group $G$ of minimal order in $\mathfrak{N}\mathfrak{F}\backslash\mathfrak{H}$. Then $G = MN$ where $N = Soc(G)$ is a self-centralizing minimal normal $p$-subgroup of $G$ and $M$ is a core-free maximal subgroup such that $M \in \mathfrak{F}$. But this means that $G \in \mathfrak{N}_p\mathfrak{F} \subseteq \mathfrak{H}$, a contradiction. Thus $\mathfrak{N}_p\mathfrak{F} \subseteq \mathfrak{H}$, for all $p$ for all $p$ from $\omega$. □

Let us examine now case (B).

**Lemma 4.4.** *If the formation $\mathfrak{N}_{\mathfrak{F}}$ is $\omega$-saturated and $\mathfrak{H}$ and $\mathfrak{F}$ are both of full characteristic then $\mathfrak{N}_p\mathfrak{F} \subseteq \mathfrak{H}$ for every prime $p$ of $\omega$.*

*Proof.* In this case $char(\mathfrak{F}) = char(\mathfrak{H})$ and we can apply proposition 4.1 to obtain that for every $p \in \omega$ we have $\mathfrak{F} \subseteq H(p)$. Then we have $\mathfrak{N}_p\mathfrak{F} \subseteq \mathfrak{H}$ for all $p$ from $\omega$. □

Now we can state the main theorem.

**Theorem 4.5.** *Let $\omega$ be a set of primes. Suppose that $\mathfrak{H}$ is an $\omega$-saturated formation such that $charH \subseteq \omega$ and $\mathfrak{F}$ is a proper $\omega$-saturated subformation of $\mathfrak{H}$. Then the following statements are pairwise equivalent:*

*(i) The formation $\mathfrak{N}_{\mathfrak{F}}$ is $\omega$-saturated.*
*(ii) $\mathfrak{N}_p\mathfrak{F} \subseteq \mathfrak{H}$.*
*(iii) $\mathfrak{N}_{\mathfrak{F}} = \mathfrak{F}$.*

*Proof.* (i) $\Rightarrow$ (ii). If $char(\mathfrak{F})$ is a proper subset of $\omega$ we apply lemma 4.2 to deduce that $\mathfrak{N}_p\mathfrak{F} \subseteq \mathfrak{H}$ for every $p \in \omega$. If $\mathfrak{F}$ is of full characteristic the conclusion follows from lemma 4.3.

(ii) $\Rightarrow$ (iii). Suppose there exists a group $G$ of minimal order in $\mathfrak{H}_{\mathfrak{F}}\backslash\mathfrak{F}$. Clearly $G \in \mathfrak{H}_{\mathfrak{F}} \cap \mathfrak{N}_p\mathfrak{F}$, that is $G \in \mathfrak{H}_{\mathfrak{F}} \cap \mathfrak{H} = \mathfrak{F}$, a contradiction. Therefore $\mathfrak{H}_{\mathfrak{F}} = \mathfrak{F}$. (iii) $\Rightarrow$ (i).

It is trivial. □

*Example* 4.6. If $\mathfrak{H} = \mathfrak{N}_p$, the class of $p$-nilpotent groups, and $\omega = P$ the set of all prime numbers, then $\mathfrak{H}_{\mathfrak{F}}$ is saturated if and only if either

1. $\mathfrak{F} = (1)$, and then $\mathfrak{H}_{\mathfrak{F}} = (1)$, or
2. $\mathfrak{F} = \mathfrak{S}_p$, and then $\mathfrak{H}_{\mathfrak{F}} = \mathfrak{S}_p$, or
3. $\mathfrak{F} = \mathfrak{H}$, and then $\mathfrak{H}_{\mathfrak{F}} = \mathfrak{S}$.

Obviously when $\mathfrak{F} = \mathfrak{H}$ we have $\mathfrak{H}_{\mathfrak{F}} = \mathfrak{S}$ and if $\mathfrak{F} = (1)$ then $\mathfrak{H}_{\mathfrak{F}} = (1)$. So, suppose $(1) \subset \mathfrak{F} \subset \mathfrak{H}$. Let $p$ a prime in $\omega$. If there exists another prime $q \in \omega$, $q \neq p$, then the group $E(q/p) \in \mathfrak{N}\mathfrak{F} \backslash \mathfrak{H}$ (where $E(q/p)$ is the group of B.12.5, [3]). Therefore $\mathfrak{H}_{\mathfrak{F}}$ is not saturated by theorem 4.5.

Our purpose now is to characterize the $\omega$-saturated formations $\mathfrak{H}$ such that for every $\omega$-saturated formation $\mathfrak{F}$, the class $\mathfrak{H}_{\mathfrak{F}}$ is a $\omega$-saturated formation. It is clear that for all $\omega$-saturated formation $\mathfrak{F}$, we have $(1)_{\mathfrak{F}} = \mathfrak{S}_\omega$ and $(\mathfrak{S}_\omega)_{\mathfrak{F}} = \mathfrak{F}$. Next we prove that $\mathfrak{S}_\omega$ and $(1)$ are the only $\omega$-saturated formations with this property.

**Corollary 4.7.** *Let $\mathfrak{H}$ be an $\omega$-saturated formation. The following statements are equivalent:*

1. *for every $\omega$-saturated formation $\mathfrak{F}$, the formation $\mathfrak{H}_{\mathfrak{F}}$ is $\omega$-saturated.*
2. *$\mathfrak{H}$ is either $(1)$ or $\mathfrak{S}_\omega$.*

*Proof.* Since $\mathfrak{H}$ must be of full characteristic, arguing by induction, we have that the class $\mathfrak{N}_\omega^k$ of all $\omega$-nilpotent groups of length at most $k$ is contained in $\mathfrak{H}$ for every $k$. So, in fact, $\mathfrak{S}_\omega \subseteq \mathfrak{H}$. Therefore $\mathfrak{H} = \mathfrak{S}_\omega$. □

## Final remarks

• In [2], N.Muller presents an approach to these questions from the point of view of the pronormal maximal subgroups.

• In [4], A.Ballester-Bolinches, K.Doerk and L.M.Ezquerro considered a similar question from the point of view of the local(saturated) formations.

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