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Some Remarks on the Prime Spectrum of a Noncommutative Ring

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Abstract. Let R be an associative noncommutative ring with identity. Let $Spec_l(R)$ be all prime left ideals of R. In the present study, we will give some new characterizations of $Spec_l(R)$ with the Zariski topology.

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1. Introduction

Throughout this paper, we assume that R is an associative ring (not necessarily commutative) with unity.

Let R be a ring. A proper left ideal P of R is said to be a *prime ideal* if, for any elements a and b in R such that $aRb \subseteq P$ either $a \in P$ or $b \in P$. A proper (2-sided) ideal P of R is called *prime* if, for any elements a and b in R such that $aRb \subseteq P$, either $a \in P$ or $b \in P$. Clearly, prime ideals are prime left ideals. If I is a prime left ideal, then

$$(I : R) = \{r \in R : rR \subseteq I\}$$

is a two-sided prime ideal. For this study, we need the following well known lemma in the literature.

Lemma 1.1. *Let R be a ring and I be a left ideal of R. Then $((I : R) : R) = (I : R)$.*

We write $Spec_l\,(R)$ for the set of all prime left ideals and $Spec\,(R)$ the set of all prime ideals. The map $\psi : Spec_l\,(R) \to Spec\,(R)$ is defined by $\psi\,(P) = (P : R)$ for every $P \in X_l$ will be called the *natural map* of $Spec_l\,(R)$.

In this study, we introduce a topology called *the Zariski topology* on $Spec_l\,(R)$, in which closed sets are varieties

$$V_l\,(I) = \{P \in Spec_l\,(R) : (I : R) \subseteq (P : R)\}$$

of all left ideals I of R. If I is a an ideal of R, we can say that $I \leq R$ to indicate that I is an ideal of R. If I is a left ideal of R, we can say that $I \leq_l R$ to indicate that I is a left ideal of R. Now, we give some important properties of V_l.

Lemma 1.2. *Let R be a ring. For left ideals I, J and $I_j\,(j \in \Lambda)$ of R, we have the followings:*
(1) $V_l\,(0) = Spec_l\,(R)$.
(2) $V_l\,(R)$ *is empty.*
(3) $V_l\,(I) \cup V_l\,(J) = V_l\,(I \cap J)$.
(4) $\bigcap_{j\in\Lambda} V_l\,(I_j) = V_l\left(\sum_{j\in\Lambda} (I_j : R)\right)$.

Proof. Clear from by Lemma 1.1. □

The text by Hungerford [1] is the general references for notions of rings not defined in this work.

2. The Results

As we mentioned in introduction, $V_l\,(I) = \{P \in Spec_l\,(R) : (I : R) \subseteq (P : R)\}$. By this vein, we have

$$V_l^*\,(I) = \{P \in Spec_l\,(R) : I \subseteq P\}$$

for any left ideal I of R. Hence;

Lemma 2.1. *Let R be a ring. For left ideals I, J and $I_j\,(j \in \Lambda)$ of R, we have the followings:*
(1) $V_l^*\,(0) = Spec_l\,(R)$.
(2) $V_l^*\,(R)$ *is empty.*
(3) $V_l^*\,(I) \cup V_l^*\,(J) \subseteq V_l^*\,(I \cap J)$.
(4) $\bigcap_{j\in\Lambda} V_l^*\,(I_j) = V_l^*\left(\sum_{j\in\Lambda} I_j\right)$.

Proof. By [2]. □

Proposition 2.2. *Let R be a ring. For left ideals I, J of R, we have the followings:*
(1) *Suppose that I and J are prime ideals. Then $(I : R) = (J : R)$ if and only if $V_l(I) = V_l(J)$.*
(2) *$V_l(I) = V_l((I : R)) = V_l^*((I : R))$. In particular, $V_l(JR) = V_l^*(JR)$ for every left ideal J of R.*

Proof. (1). Clear.
(2). $P \in V_l(I) \Leftrightarrow (I : R) \subseteq (P : R) \Leftrightarrow ((I : R) : R) \subseteq ((P : R) : R) = (P : R) \Leftrightarrow P \in V_l((I : R))$.
Let $P \in V_l((I : R))$. Then $((I : R) : R) \subseteq (P : R) \Leftrightarrow (I : R) \subseteq P \Leftrightarrow P \in V_l^*((I : R))$. For converse, let $P \in V_l(JR)$. Then $(JR : R) = (JR : R)R \subseteq (P : R) \subseteq P \Leftrightarrow JR \subseteq P \Leftrightarrow P \in V_l^*(JR)$. □

We consider the following sets;

$\Gamma(R) = \{V_l(I) : I \leq_l R\}$
$\Gamma^*(R) = \{V_l^*(I) : I \leq_l R\}$ and
$\Gamma^{**}(R) = \{V_l^{**}(IR) : I \leq_l R\}$.

Theorem 2.3. *For a ring R, we have the followings.*
(1) *$\Gamma(R) = \Gamma^{**}(R) \subseteq \Gamma^*(R)$.*
(2) *The Zariski topology Γ on $Spec(R)$ is identical with Γ^{**}.*

Proof. Clear from definitions and Proposition 2.2. □

For each subset E of R, let $V^R(E) = \{P \in Spec(R) : E \subseteq P\}$.

Proposition 2.4. *Let R be a ring. Then the surjective map $f : Spec_l(R) \to Spec(R)$, defined by $f(P) = (P : R)$, is continuous.*

Proof. Let X be any closed set in $Spec(R)$. Then $X = V^R(I)$ for some ideal I of R. For any $P \in Spec_l(R)$, we have $f(P) = (P : R) \in V^R(I) \Leftrightarrow I = (I : R) \subseteq ((P : R) : R) = (P : R) \Leftrightarrow P \in V_l(I)$. This implies that $f^{-1}(X) = f^{-1}\left(V^R(I)\right) = V_l(I)$, and so f is continuous. □

Let $\Lambda^R = \left\{V^R(E) : E \subseteq R\right\}$. Then Λ^R is a closed set of a unique topology on $Spec(R)$. The topology Λ^R on $Spec(R)$ is called Zariski topology. Any open subset of $Spec(R)$ is of the form $Spec(R) - V^R(E)$ for some $E \subseteq R$. It is clear that $V^R(f) = V^R(RrR)$ for any $r \in R$. For $r \in R$, if we define $Spec(R)^r = Spec(R) - V^R(RrR)$. Clearly, $\{Spec(R)^r : r \in R\}$ is a basis for the Zariski topology.

Theorem 2.5. *For a ring R, let $f : Spec_l(R) \to Spec(R)$ be the surjective map of $Spec_l(R)$. Then the following statements hold.*
(1) *f is closed and open.*
(2) *$Spec_l(R)$ is connected if and only if $Spec(R)$ is connected.*

Proof. (1) By Proposition 2.4, for every ideal I of R, f is a continuous map and $f^{-1}\left(V^R(I)\right) = V_l(I)$. By Proposition 2.2, $f^{-1}\left(V^R((I:R))\right) = V_l((I:R)) = V_l(I)$ for every left ideal I of R. But $f(V_l(I)) = V^R((I:R))$ because of the surjectivity of f. By the similar technic, we can see that

$$\begin{aligned} f\left(Spec(R)_l - V_l(I)\right) &= f\left(f^{-1}(Spec(R)) - f^{-1}\left(V^R((I:R))\right)\right) \\ &= f\left(f^{-1}\left(Spec(R) - V^R((I:R))\right)\right) \\ &= Spec(R) - V^R((I:R)) \end{aligned} .$$

(2) We assume that $Spec(R)$ is connected. For contradiction, we suppose that $Spec_l(R)$ is not connected. Then $Spec_l(R)$ must contain a non-empty proper subset X that is open and closed. By Proposition 2.4, $f(X)$ is non-empty subset of $Spec(R)$ that is both open and closed . Since X is open, $X = Spec_l(R) - V_l(I)$ for some $I \leq_l R$ whence $f(X) = Spec(R) - V^R((I:R))$ by Proposition 2.4. This implies that , if $f(X) = Spec(R)$ then $V^R((I:R))$ is empty. Hence $(I:R) = R$,i.e, $I = R$. Therefore $X = Spec_l(R) - V_l(I) = Spec_l(R) - V_l(R) = Spec_l(R)$. It is a contradiction. That is (X) is a proper subset of $Spec(R)$.

Converse is clear by (1). □

Let R be a division ring and S a ring of $n \times n$ matrices over R. Then $Spec(S) = \{0\}$ is connected. By Theorem 2.5, $Spec_l(S)$ is also connected.

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