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Energy of Zero-Divisor Graphs via Distance-Based Fuzzy Adjacency Matrices

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Abstract

In this paper, we investigate the energy of zero-divisor graphs of finite commutative rings with unity using the distance-based fuzzy adjacency matrix. In view of this, first we characterize the rings whose zero-divisor graphs have energy zero and then prove that the energy of a zero-divisor graph can never equal 1. Further we establish the sharp bounds for energy of zero-divisor graphs and characterize the rings R for which energy attains the value 2.

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Keywords: Zero-divisor graph, fuzzy adjacency matrix, distance metric, spectrum, energy

1 Introduction

Zero-divisor graphs provide a bridge between commutative algebra and graph theory by encoding the multiplicative structure of a commutative ring into a graph. The concept was first introduced by Beck [2], who associated a graph with a commutative ring by taking its elements as vertices and connecting pairs whose product is zero . This idea was later refined by Anderson and Livingston [1], who restricted the vertex set to the nonzero zero-divisors, resulting in a more concise and structurally meaningful graph. Since then, zero-divisor graphs have been extensively studied to understand how ring-theoretic properties are reflected in graph-theoretic invariants such as connectivity, diameter, chromatic number, and planarity [2, 1].

On the other hand, the notion of energy of graph, introduced by Gutman [3], has emerged as a central concept in spectral graph theory, motivated by applications in mathematical chemistry and physics. The energy of a graph is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix and serves as an important spectral invariant.

Traditional studies of graph energy rely on crisp or weighted adjacency matrices, where adjacency relations are binary or fixed-valued. However, in many contexts, particularly those involving metric considerations, it is natural to consider adjacency strengths that depend on distance rather than sharp binary relations. Motivated by this, we introduce a distance-based normalized fuzzy adjacency matrix for zero-divisor graphs and study the resulting spectral energy.

In this paper, we explore the energy of zero-divisor graphs of finite commutative rings using a distance-based fuzzy adjacency matrix. We begin by identifying exactly which rings have energy zero and show that this occurs only for $\mathbb{Z}_4$ (equivalently $\mathbb{Z}_2[x]/(x^2)$). We also prove that the energy of a zero-divisor graph can never be equal to 1. For connected zero-divisor graphs, we establish sharp bounds on the energy and completely describe the rings whose energy equals 2. Finally, we determine the smallest energy value greater than 2 and examine the behavior of Boolean rings within this framework.

2 Preliminaries

For terminology and notations in graph theory, we refer to the standard textbook by Harary [4]. Unless otherwise specified, all graphs considered here are undirected, finite, connected, and simple.

Throughout this paper, let R denote a commutative ring with unity $1 \neq 0$. The algebraic structure of R gives rise to a natural graph-theoretic representation known as the zero-divisor graph, which captures the annihilation relationships among elements of the ring as follows:

Definition 2.1. [1] The zero-divisor graph of R, denoted by $\Gamma(R)$, is a graph whose vertex set consists of all nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $x \cdot y = 0$.

Example 2.2. Let $R = \mathbb{Z}_8$. Then $Z^*(R) = \{2, 4, 6\}$. Then $\Gamma(\mathbb{Z}_8)$ is isomorphic to path with 3 vertices.

Observation 2.3. Let R is an integral domain. Then $\Gamma(R)$ is an empty graph, and hence the energy is trivially zero.

Towards this end R will be a finite commutative ring with unity which is not an integral domain. We now define the notion of distance-based fuzzy adjacency matrix as follows:

Definition 2.4. Let $\Gamma(R)$ be the zero-divisor graph of a commutative ring R with vertex set $V = Z(R)$. Let $d(u,v)$ denote the length of the shortest path between the vertices u and v in $\Gamma(R)$. Define the kernel

$$w(u,v) = \begin{cases} \dfrac{1}{d(u,v)}; & u \neq v,\ d(u,v) < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

The fuzzy adjacency matrix $A = [a_{uv}]_{u,v \in V}$ is defined by

$$a_{uv} = \begin{cases} \dfrac{w(u,v)}{\sum\limits_{t \neq u} w(u,t)}, & u \neq v, \\ 0, & u = v. \end{cases}$$

To illustrate the notion of distance-based fuzzy adjacency matrix, we have the following example:

Example 2.5. Consider the commutative ring $R = \mathbb{Z}_6$. The nonzero zero-divisors of this ring are 2, 3, and 4; hence the vertex set of the corresponding zero-divisor graph is $V(\Gamma(\mathbb{Z}_6)) = \{2, 3, 4\}$.

Consequently, $\Gamma(\mathbb{Z}_6)$ has edges $(2,3)$ and $(3,4)$, and there is no edge between vertices 2 and 4. Hence, $\Gamma(\mathbb{Z}_6)$ is the path graph on 3 vertices, namely, $2 - 3 - 4$.

The distance between two vertices is either 1 or 2. Using the kernel $w(u,v) = \frac{1}{d(u,v)} \quad (u \neq v)$, we obtain the weight matrix $W = [w(u,v)]$:

$$W = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}.$$

Next, we compute the normalization factors as

$$\sum_{t \neq 2} w(2,t) = 1 + \tfrac{1}{2} = \tfrac{3}{2}, \quad \sum_{t \neq 3} w(3,t) = 1 + 1 = 2, \quad \sum_{t \neq 4} w(4,t) = \tfrac{1}{2} + 1 = \tfrac{3}{2}.$$

Therefore, the distance-based fuzzy adjacency matrix $A = [a_{uv}]$ is given by

$$A = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}.$$

Each row of A sums to one, which indicates that A is row-stochastic. This example demonstrates how the proposed distance-based fuzzy adjacency matrix incorporates both local and global distance information on the zero-divisor graph.

Remark 2.6. Consequently, by Definition 2.4, one can notice that each row of A can be viewed as a probability distribution centered at u, where nearby vertices contribute closer to u assigned higher weights. Even though the zero-divisor graph itself is undirected, the normalization process makes the matrix A row-stochastic and, in general, non-symmetric.

Definition 2.7. Let A be the distance-based fuzzy adjacency matrix of $\Gamma(R)$. The eigenvalues

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

of A are called the eigenvalues of $\Gamma(R)$. The multiset of eigenvalues is denoted by $\mathrm{Spec}(\Gamma(R))$.

Definition 2.8. *The energy of the zero-divisor graph* $\Gamma(R)$ *is defined as*

$$E(\Gamma(R)) = \sum_{i=1}^{n} |\lambda_i|.$$

Definition 2.9. Let G and H be two graphs with energies $E(G)$ and $E(H)$, respectively. The graphs G and H are said to be *equienergetic* if $E(G) = E(H)$, even if G and H are not isomorphic.

Proposition 2.10. *The trace of the distance-based fuzzy adjacency matrix of* $\Gamma(R)$ *is zero.*

Proof. By Definition 2.4, $a_{uu} = 0$ for all $u \in V$. Hence, $\mathrm{tr}(A) = \sum_{u \in V} a_{uu} = 0$. □

The distance-based fuzzy adjacency matrix has an important property: its trace is zero. Since the trace of a matrix is just the sum of all its eigenvalues, this means that all the eigenvalues together must add up to zero. So, they cannot be independent of each other. They must satisfy this basic condition. Also, because the matrix is row-stochastic, the number 1 is always one of its eigenvalues. This gives us a starting point. Once we know that 1 is an eigenvalue and that the total sum of all eigenvalues is zero, we can easily understand how the other eigenvalues are related to it. In fact, the remaining eigenvalues must balance out the dominant eigenvalue so that their total sum becomes zero.

The next corollary depict this connection between the main eigenvalue and the others.

Proposition 2.11. *If* 1 *is a simple eigenvalue of* A*, then* $\sum_{i=2}^{n} \lambda_i = -1$.

3 Main Results

In this section, we begin by determining the structure of rings whose zero-divisor graphs have the smallest possible energy. Next, we derive general bounds for the energy of zero-divisor graphs and characterize those rings for which the energy is exactly 2. Finally, we determine the least energy value that is strictly greater than 2 and discuss certain special classes of rings, including Boolean rings.

Theorem 3.1. *Let R be a finite commutative ring with unity. Then $E(\Gamma(R)) = 0$ if and only if either R is isomorphic to either $\mathbb{Z}_4$ or the quotient ring $\mathbb{Z}_2[x]/(x^2)$.*

Proof. Let $\Gamma(R)$ be the zero-divisor graph of R with the distance-based fuzzy adjacency matrix A.
($\Rightarrow$) Assume that $E(\Gamma(R)) = 0$. By definition, $E(\Gamma(R)) = \sum_{i=1}^{n} |\lambda_i|$, where λ_i are the eigenvalues of A. Hence every eigenvalue of A is zero, and therefore A is the zero matrix.

This happens only when $\Gamma(R)$ has no edges. Thus assume that R is not an integral domain. Since $\Gamma(R)$ has no edges but has at least one vertex, it must consists of exactly one vertex. Hence R has exactly one nonzero zero-divisor, say a.

In a finite commutative ring, the set of non-units is the union of maximal ideals. Since there is only one nonzero non-unit, R must be a local ring with maximal ideal $\mathfrak{m} = \{0, a\}$. Thus $|\mathfrak{m}| = 2$, and the residue field $R/\mathfrak{m}$ has two elements. Therefore $R/\mathfrak{m} \cong \mathbb{F}_2$. Moreover, since a is a zero-divisor and the only one, we must have $a^2 = 0$.

Hence R is a finite local ring with residue field $\mathbb{F}_2$ and square-zero maximal ideal of order 2. Such a ring is uniquely determined up to isomorphism, and it is isomorphic to $\mathbb{Z}_4$. Note that $\Gamma(\mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_2[x]/(x^2))$. Thus the necessity follows.
($\Leftarrow$) Conversely, if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2))$, then there is only one nonzero zero-divisor. Thus $\Gamma(R)$ has exactly one vertex with no edge. The fuzzy adjacency matrix is (0), whose only eigenvalue is 0. Therefore $E(\Gamma(R)) = 0$. □

Proposition 3.2. *For any finite commutative ring R, $E(\Gamma(R)) \neq 1$.*

Proof. Let A be the distance-based fuzzy adjacency matrix associated with the zero-divisor graph $\Gamma(R)$. By construction, A is row-stochastic, and hence 1 is always an eigenvalue of A. Moreover, since the diagonal entries of A are zero, we have

$$\mathrm{tr}(A) = 0.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote the eigenvalues of A, where $\lambda_1 = 1$. Then $\lambda_1 + \lambda_2 + \dots + \lambda_n = 0$ and therefore $\lambda_2 + \lambda_3 + \dots, \lambda_n = -1$.

The energy of $\Gamma(R)$ is defined as $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_n| = 1 + |\lambda_2| + |\lambda_3| + \ldots + |\lambda_n|$. Applying the triangle inequality, we obtain,

$$\sum_{i=2}^{n} |\lambda_i| \geq \left| \sum_{i=2}^{n} \lambda_i \right| = 1.$$

Hence,

$$E(\Gamma(R)) \geq 2,$$

whenever $\Gamma(R)$ has at least one vertex.

Consequently, no finite commutative ring R can satisfy $E(\Gamma(R)) = 1$. □

The row-stochastic nature of distance-based fuzzy adjacency matrix A naturally leads to universal bounds on the energy, which depend only on the order of the graph, as follows:

Theorem 3.3. *Let $\Gamma(R)$ be a zero-divisor graph with $n \geq 2$ vertices. Then*

$$2 \leq E(\Gamma(R)) \leq n.$$

Proof. Let $V = Z^*(R)$ be the vertex set of $\Gamma(R)$ and let $|V| = n \geq 2$. Let $A = [a_{uv}]_{u,v \in V}$ be the distance-based fuzzy adjacency matrix defined in Definition 2.4. By definition, $a_{uu} = 0$ for all $u \in V$.

For each fixed $u \in V$ we have

$$\sum_{v \in V} a_{uv} = \sum_{v \neq u} \frac{w(u,v)}{\sum_{t \neq u} w(u,t)} = \frac{\sum_{v \neq u} w(u,v)}{\sum_{t \neq u} w(u,t)} = 1,$$

so every row sum of A equals 1, therefore 1 is an eigenvalue of A.

Let the eigenvalues of A be $\lambda_1, \lambda_2, \ldots, \lambda_n$, counted with algebraic multiplicity, and choose the indexing so that $\lambda_1 = 1$. Since $\operatorname{tr}(A) = \sum_{u \in V} a_{uu} = 0$, we have

$$\sum_{i=1}^{n} \lambda_i = \operatorname{tr}(A) = 0.$$

Thus

$$\sum_{i=2}^{n} \lambda_i = -\lambda_1 = -1.$$

By the triangle inequality,

$$\sum_{i=2}^{n} |\lambda_i| \geq \left| \sum_{i=2}^{n} \lambda_i \right| = |-1| = 1.$$

Therefore

$$E(\Gamma(R)) = \sum_{i=1}^{n} |\lambda_i| = |\lambda_1| + \sum_{i=2}^{n} |\lambda_i| \geq 1 + 1 = 2,$$

which proves the lower bound.

For the upper bound, since each row of A sums to 1 and all entries of A are nonnegative, A is a row-stochastic matrix. Hence its spectral radius is 1, and every eigenvalue satisfies $|\lambda_i| \leq 1$. Consequently,

$$E(\Gamma(R)) = \sum_{i=1}^{n} |\lambda_i| \leq \sum_{i=1}^{n} 1 = n.$$

This proves $2 \leq E(\Gamma(R)) \leq n$. □

Theorem 3.4. *Let R be a finite commutative ring with unity and let $\Gamma(R)$ be its zero-divisor graph. Then $E(\Gamma(R)) = 2$ if and only if one of the following holds:*

(i) R is a finite local ring with $Z(R)^2 = 0$ and $|Z^*(R)| \geq 2$;

(ii) $R \cong \mathbb{F}_q \times \mathbb{F}_r$ for some finite fields $\mathbb{F}_q, \mathbb{F}_r$ with $q, r \geq 2$.

Proof. Let A denote the distance-based fuzzy adjacency matrix of $\Gamma(R)$ and let

$$\mathrm{Spec}(\Gamma(R)) = \{\lambda_1, \ldots, \lambda_n\}$$

be its spectrum. Since A is row-stochastic, 1 is an eigenvalue of A. Hence

$$E(\Gamma(R)) = 1 + \sum_{i=2}^{n} |\lambda_i|.$$

($\Rightarrow$) Suppose $E(\Gamma(R)) = 2$. Then

$$\sum_{i=2}^{n} |\lambda_i| = 1.$$

Now our claim is $\Gamma(R)$ is either complete or complete bipartite. To do this, suppose to the contrary, $\Gamma(R)$ is neither complete nor complete bipartite, then in first possibility there exist $u, v \in V$ with $d(u, v) \geq 2$, so $w(u, v) \leq \frac{1}{2}$ and hence $a_{uv} \leq \frac{1}{2}$ after normalization. Since the diagonal is 0, every row has at least two positive entries when $|V| \geq 3$, and the normalization is strictly affected by having both distance-1 and distance-≥ 2 vertices.

Next, if the graph is not complete bipartite, then there exists an induced path of length 3, say $x - y - z - t$ with $x \not\sim z$ and $y \not\sim t$ and with x, t in the same part of any attempted bipartition. A direct computation of the 4×4 principal submatrix of A on $\{x, y, z, t\}$ (using the facts that $d(x, y) = d(y, z) = d(z, t) = 1$ and $d(x, z) = d(y, t) = 2$, $d(x, t) = 3$) shows that this principal submatrix has energy strictly greater than 2. Since the energy of a matrix dominates the energy of any principal submatrix (by eigenvalue interlacing for real symmetric matrices, applied to the symmetrized matrix $(A+A^T)/2$ which has the same absolute-eigenvalue sum lower bounded by that of A), it follows that $E(\Gamma(R)) > 2$, a contradiction. Hence $\Gamma(R)$ must be complete or complete bipartite.

It is well known from [1, Corollary 2.7 and Theorem 2.8] that the zero-divisor graph of a finite commutative ring is complete if and only if R is a finite local ring with $Z(R)^2 = 0$, and from [1, Example 2.1(d)] shows that "the zero-divisor graph of a finite commutative ring is complete bipartite if and only if R is isomorphic to a direct product of two fields." Thus (i) or (ii) holds.
($\Leftarrow$) Conversely, assume that (i) holds. Then every pair of distinct nonzero zero-divisors annihilates each other, so $\Gamma(R)$ is complete. Hence its spectrum is

$$\left\{1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}\right\},$$

and therefore

$$E(\Gamma(R)) = 1 + (n-1)\frac{1}{n-1} = 2.$$

Assume that condition (ii) is satisfied and $R \cong \mathbb{F}_q \times \mathbb{F}_r$. In this setting, the nonzero zero-divisors naturally divide into two distinct subsets, each associated with one of the field components. Any element belonging to one subset multiplies with every element of the other subset to give zero, so elements from opposite parts annihilate one another. Therefore, the graph $\Gamma(R)$ is complete bipartite. Under the distance-based fuzzy normalization, its spectrum again consists of 1 and equal negative eigenvalues whose absolute values sum to 1. Consequently, $E(\Gamma(R)) = 2$. Hence the result. □

Corollary 3.5. *Let R be a commutative ring with unity. If every nonzero zero-divisor of R is nilpotent, then either $E(\Gamma(R)) = 0$ or $E(\Gamma(R)) = 2$.*

Proof. Assume that every nonzero zero-divisor of the commutative ring R is nilpotent. We consider two cases.
(i) If $\Gamma(R)$ has at most one vertex, the following cases arises: If R contains no nonzero zero-divisors, then R is an integral domain and the graph $\Gamma(R)$ has no vertices or edges. Consequently, the graph is empty. In this case, the distance-based fuzzy adjacency matrix does not contain any entries, and therefore the energy satisfies $E(\Gamma(R)) = 0 < 2.$.

If R has exactly one nonzero zero-divisor, then $\Gamma(R)$ consists of a single isolated vertex. The corresponding fuzzy adjacency matrix is the 1×1 zero matrix, whose only eigenvalue is 0. Thus, $E(\Gamma(R)) = 0 < 2$.
(ii) Suppose that $\Gamma(R)$ has at least two vertices. Then the ring R contains at least two distinct nonzero zero-divisors. Let x and y be two distinct nonzero zero-divisors of R. Since both x and y are nilpotent, there exist positive

integers m, n such that

$$x^m = 0 \quad \text{and} \quad y^n = 0.$$

Then

$$(xy)^{m+n} = x^m y^n = 0,$$

which implies that xy is nilpotent. Since x and y are nonzero zero-divisors, it follows that $xy = 0$. Hence, every pair of distinct vertices in $\Gamma(R)$ is adjacent, and therefore $\Gamma(R)$ is a complete graph.

Consequently, $\Gamma(R)$ has diameter 1. By the characterization of the distance-based fuzzy energy for diameter 1 graphs, the fuzzy adjacency matrix A of $\Gamma(R)$ has eigenvalues

$$\lambda_1 = 1, \qquad \lambda_2 = \cdots = \lambda_n = -\frac{1}{n-1},$$

where $n = |V(\Gamma(R))| \geq 2$. Hence,

$$E(\Gamma(R)) = |1| + (n-1)\left|-\frac{1}{n-1}\right| = 2.$$

Thus $E(\Gamma(R))$ is 2 in this case.

Combining both cases, one have $E(\Gamma(R))$ is either 0 or 2. □

At this stage one may ask the following problem:

Problem 3.6. Determine finite commutative rings R for which $E(\Gamma(R))$ attains the next minimal value greater than 2.

To answer the result we have the following result:

Proposition 3.7. *The smallest value of* $E(\Gamma(R))$ *strictly greater than* 2 *is* $2.042389169\ldots$, *attained when* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

Proof. The nonzero zero-divisors of $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ are

$$Z^*(R) = \{(0,1), (0,2), (0,3), (1,0), (1,2)\},$$

so $|Z^*(R)| = 5$. Denote these vertices by

$$v_1 = (0,1), \quad v_2 = (0,2), \quad v_3 = (0,3), \quad v_4 = (1,0), \quad v_5 = (1,2).$$

Two distinct vertices u, v are adjacent in $\Gamma(R)$ if and only if $uv = (0,0)$ in R. The zero-divisor graph is shown in the Figure 1

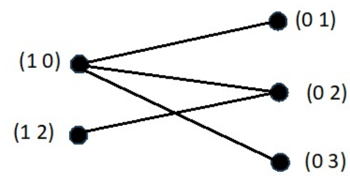

Figure 1: The zero-divisor graph $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$

From the zero-divisor graph we obtains the distance table as follows:

$d(\cdot,\cdot)$	v_1	v_2	v_3	v_4	v_5
v_1	0	2	2	1	3
v_2	2	0	2	1	1
v_3	2	2	0	1	3
v_4	1	1	1	0	2
v_5	3	1	3	2	0

Using Definition 2.4, $w(u,v) = 1/d(u,v)$ for $u \neq v$ and $w(u,u) = 0$. For convenience, set $S(u) = \sum_{t\neq u} w(u,t)$. From the distance table we get:

$$S(v_1) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{7}{3}, \quad S(v_2) = 1 + 1 + \frac{1}{2} + \frac{1}{2} = 3,$$

$$S(v_3) = S(v_1) = \frac{7}{3}, \quad S(v_4) = 1+1+1+\frac{1}{2} = \frac{7}{2}, \quad S(v_5) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{3} = \frac{13}{6}.$$

Therefore, for $i \neq j$,

$$a_{ij} = \frac{w(v_i,v_j)}{S(v_i)},$$

and with the vertex order $(v_1, v_2, v_3, v_4, v_5)$ this yields

$$A = \begin{pmatrix} 0 & \frac{3}{14} & \frac{3}{14} & \frac{3}{7} & \frac{1}{7} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{14} & \frac{3}{14} & 0 & \frac{3}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{2}{7} & \frac{2}{7} & 0 & \frac{1}{7} \\ \frac{2}{13} & \frac{6}{13} & \frac{2}{13} & \frac{3}{13} & 0 \end{pmatrix}.$$

A direct computation of the characteristic polynomial gives

$$\chi_A(\lambda) = \det(\lambda I - A) = \frac{(\lambda-1)(14\lambda+3)\big(3822\lambda^3 + 3003\lambda^2 + 548\lambda - 13\big)}{53508}.$$

Hence the eigenvalues of A are

$$1, \quad -\frac{3}{14}, \quad \lambda_1, \lambda_2, \lambda_3,$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of $3822\lambda^3 + 3003\lambda^2 + 548\lambda - 13 = 0$. Numerically,

$$\lambda_1 \approx 0.0211945848, \qquad \lambda_2 \approx -0.3555696387, \qquad \lambda_3 \approx -0.4513392318.$$

Therefore,

$$E(\Gamma(R)) = \sum_{i=1}^{5} |\lambda_i| = 1 + \frac{3}{14} + |\lambda_1| + |\lambda_2| + |\lambda_3| \approx 2.0423891696513943.$$

$\square$

Let $x = (x_1, x_2, \ldots, x_k) \in \mathbb{Z}_2^k$. The *support* of x, denoted by $\operatorname{supp}(x)$, is defined as

$$\operatorname{supp}(x) = \{\, i \in \{1, \ldots, k\} \mid x_i = 1 \,\}.$$

That is, the support of x is the set of indices at which the coordinates of x are equal to 1.

For $x, y \in \mathbb{Z}_2^k$, the condition $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \varnothing$ means that there is no index i such that $x_i = 1$ and $y_i = 1$.

Theorem 3.8. *Let $R \cong \mathbb{Z}_2^k$, $(k > 1)$ be a finite Boolean ring. Then*

$$E(\Gamma(R)) = \begin{cases} 2, & k = 2, \\ > 2, & k \geq 3. \end{cases}$$

Proof. Since $R \cong \mathbb{Z}_2^k$, every element of R is a binary vector of length k and satisfies $x^2 = x$. The only unit of R is $(1, 1, \ldots, 1)$; hence the set of nonzero zero-divisors is

$$Z^*(R) = R \setminus \{0, 1\}.$$

Thus $|V(\Gamma(R))| = 2^k - 2$.

Two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. In coordinate form this is equivalent to

$$\operatorname{supp}(x) \cap \operatorname{supp}(y) = \varnothing,$$

Thus, the graph $\Gamma(R)$ can be described as the graph whose vertices represent the nonempty proper subsets of $\{1, \ldots, k\}$, where two vertices are adjacent precisely when the corresponding subsets are disjoint.

Case $k = 2$. Then R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and

$$Z(R)^* = \{(1, 0), (0, 1)\}.$$

Hence $\Gamma(R) \cong K_2$. The distance-based fuzzy adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose eigenvalues are $\{1, -1\}$. Therefore

$$E(\Gamma(R)) = |1| + |-1| = 2.$$

Case $k \geq 3$. In this case, the number of vertices in $\Gamma(R)$ is $|V(\Gamma(R))| = 2^k - 2 \geq 6$. Moreover, the graph is not complete. For instance, consider the elements $(1, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$ have nonempty intersection of supports and hence are not adjacent. Thus the diameter D of $\Gamma(R)$ is at least 2.

For a connected graph, the distance-based fuzzy adjacency matrix is row-stochastic, so 1 is an eigenvalue. If $\Gamma(R)$ were complete, the spectrum would be

$$\left\{1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1}\right\}$$

and the energy would equal 2. However, since $\Gamma(R)$ is not complete for $k \geq 3$, its spectrum cannot have this form. In particular, at least one nonprincipal eigenvalue has absolute value strictly greater than $\frac{1}{n-1}$. Consequently,

$$\sum_{i=2}^{n} |\lambda_i| > 1,$$

and hence

$$E(\Gamma(R)) = 1 + \sum_{i=2}^{n} |\lambda_i| > 2.$$

Therefore $E(\Gamma(R)) = 2$ when $k = 2$, and $E(\Gamma(R)) > 2$ for all $k \geq 3$. □

From the forgoing analysis one have the following problem:

Problem 3.9. Characterize all equienergetic zero-divisor graphs arising from finite commutative rings with unity. In particular, determine all pairs of finite commutative rings R and S such that

$$E(\Gamma(R)) = E(\Gamma(S))$$

while $\Gamma(R)$ is not isomorphic to $\Gamma(S)$, and describe the corresponding classes of rings that produce such equienergetic graphs.

The Problem 3.9 has been settled in established results partially as we obtain a complete description of equienergetic zero-divisor graphs in the minimal energy cases. When $E(\Gamma(R)) = 0$, this occurs precisely when R is an integral domain or $R \cong \mathbb{Z}_4$. In particular, all finite fields are equienergetic, since their zero-divisor graphs are empty and hence have energy zero. Thus, there exist infinitely many non-isomorphic rings whose zero-divisor graphs share the same energy value 0.

Similarly, when $E(\Gamma(R)) = 2$, we obtain a full structural classification. In this case, the ring R must either be a finite local ring satisfying $Z(R)^2 = 0$ with at least two nonzero zero-divisors, or a direct product of two finite fields. Consequently, all rings belonging to these two families are equienergetic. Although these rings may be algebraically non-isomorphic, their zero-divisor graphs have identical energy equal to 2. Therefore, the values 0 and 2 give rise to large and structurally distinct classes of equienergetic zero-divisor graphs among finite commutative rings with unity. For energies strictly greater than 2, the problem is still open.

4 Conclusion

In this paper, we characterize finite commutative rings in terms of the energy of their zero-divisor graphs under the distance-based fuzzy adjacency matrix. We showed that energy zero occurs precisely for the unique finite ring with exactly one nonzero zero-divisor, namely $\mathbb{Z}_4$ (equivalently $\mathbb{Z}_2[x]/(x^2)$). We proved that energy one is impossible and derived sharp bounds for connected zero-divisor graphs. A characterization of rings satisfying $E(\Gamma(R)) = 2$ was obtained, revealing that minimal positive energy arises exactly in two situations: finite local rings with square-zero zero-divisor ideals and direct products of two finite fields. We further determined the smallest energy strictly greater than two and showed that it is derived by $\mathbb{Z}_2 \times \mathbb{Z}_4$. The behavior of Boolean rings was also analyzed.

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