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Same Rank in Curve $y^2 = x^3 - 2px$

Shin-Wook Kim

Deokjin-gu, Songcheon 54823
I-Park Apt
Jeonju, Jeonbuk, Korea

Abstract

We denote E_{-2p} as an elliptic curve $y^2 = x^3 - 2px$ with prime $p = Hu^4 \pm Iu^2v^2 + Kv^4$ then, we research the rank and submit examples.

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1 Introduction

Prime of the form $p = Hu^4 + Iu^2v^2 + Kv^4 \cdots\cdots$ (자) is important in treating the ranks of elliptic curves $y^2 = x^3 \pm Apx$. If the curve is gotten as generalized rank 1 then, it is appeared often. In $y^2 = x^3 - px$ and $y^2 = x^3 - 2px$ and $y^2 = x^3 \pm 3px$ and $y^2 = x^3 - 4px$ and $y^2 = x^3 - 5px$ we can obtain this form [4], [6], [8], [10], [7]. We also can attain rank 1 in form $p = t^4 + 324$ in curve $y^2 = x^3 - 4px$([2]) but form (자) is significant since from this we can obtain more terms of primes as $p = 163t^4 - 4t^3u - 8tu^3 + 8t^2u^2 + 4u^4$ and $p = 163t^4 + 4t^3u - 8tu^3 + 4u^4$ and $p = 3t^4 + 4t^3u - 8tu^3 + 4u^4$ and $p = 1251t^4 + 4t^3u - 8tu^3 + 4u^4$ in $y^2 = x^3 - 2px$ ([13]) and $p = 400s^4 + 4t^4 - 11u^4 + 80s^2t^2 - 40s^2u^2 - 4t^2u^2$ and $p = 400s^4 + 4t^4 - 11u^4 - 80s^2t^2 + 40s^2u^2 - 4t^2u^2$ and $p = 196s^4 + 16t^4 - 11u^4 - 112s^2t^2 + 28s^2u^2 - 8t^2u^2$ and $p = 100s^4 + 4t^4 - 11u^4 + 40s^2t^2 + 20s^2u^2 + 4t^2u^2$ in $y^2 = x^3 - 3px$([11]). Furthermore, the forms $p = 16h^4 + 3i^4 + j^4 + k^4 + l^4 + s^4 + t^4 + u^4 + v^4 + w^4 + \eta^4 + \iota^4 + \delta^4 + \lambda^4 - 8h^2i^2 + 8h^2j^2 + 8h^2k^2 + 8h^2l^2 + 8h^2s^2 + 8h^2t^2 + 8h^2u^2 + 8 \cdot h^2v^2 + 8h^2w^2 + 8h^2\eta^2 + 8h^2\iota^2 + 8h^2\delta^2 + 8h^2\lambda^2 - 2i^2j^2 - 2i^2k^2 - 2i^2l^2$

$-2i^2s^2-2i^2t^2-2i^2u^2-2i^2v^2-2i^2w^2-2i^2\eta^2-2i^2\iota^2-2i^2\delta^2-2i^2\lambda^2+2j^2k^2+2j^2l^2+2j^2s^2+2j^2t^2+2j^2u^2+2j^2v^2+2j^2w^2+2j^2\eta^2+2j^2\iota^2+2\cdot j^2\delta^2+2j^2\lambda^2+2k^2l^2+2k^2s^2+2k^2t^2+2k^2u^2+2k^2v^2+2k^2w^2+2k^2\eta^2+2k^2\iota^2+2k^2\delta^2+2k^2\lambda^2+2l^2s^2+2l^2t^2+2l^2u^2+2l^2v^2+2l^2w^2+2l^2\eta^2+2l^2\iota^2+2l^2\delta^2+2l^2\lambda^2+2s^2t^2+2s^2u^2+2s^2v^2+2s^2w^2+2s^2\eta^2+2s^2\iota^2+2s^2\delta^2+2s^2\lambda^2+2t^2u^2+2t^2v^2+2t^2w^2+2t^2\eta^2+2t^2\iota^2+2t^2\delta^2+2t^2\lambda^2+2u^2v^2+2u^2w^2+2u^2\eta^2+2u^2\iota^2+2u^2\delta^2+2u^2\lambda^2+2v^2w^2+2v^2\eta^2+2v^2\cdot\iota^2+2v^2\delta^2+2v^2\lambda^2+2w^2\eta^2+2w^2\iota^2+2w^2\delta^2+2w^2\lambda^2+2\eta^2\iota^2+2\eta^2\delta^2+2\eta^2\lambda^2+2\iota^2\delta^2+2\iota^2\lambda^2+2\delta^2\lambda^2$ and $p=6h^4+i^4+j^4+k^4+l^4+s^4+t^4+u^4+v^4+w^4+\eta^4+\iota^4+\delta^4+\lambda^4-4h^2i^2+4h^2j^2+4h^2k^2+4h^2l^2+4h^2s^2+4h^2t^2+4h^2u^2+4h^2v^2+4h^2w^2+4h^2\eta^2+4h^2\iota^2+4h^2\delta^2+4h^2\lambda^2-2i^2\cdot j^2-2i^2k^2-2i^2l^2-2i^2s^2-2i^2t^2-2i^2u^2-2i^2v^2-2i^2w^2-2i^2\eta^2-2i^2\cdot\iota^2-2i^2\delta^2-2i^2\lambda^2+2j^2k^2+2j^2l^2+2j^2s^2+2j^2t^2+2j^2u^2+2j^2v^2+2j^2\cdot w^2+2j^2\eta^2+2j^2\iota^2+2j^2\delta^2+2j^2\lambda^2+2k^2l^2+2k^2s^2+2k^2t^2+2k^2u^2+2\cdot k^2v^2+2k^2w^2+2k^2\eta^2+2k^2\iota^2+2k^2\delta^2+2k^2\lambda^2+2l^2s^2+2l^2t^2+2l^2u^2+2l^2v^2+2l^2w^2+2l^2\eta^2+2l^2\iota^2+2l^2\delta^2+2l^2\lambda^2+2s^2t^2+2s^2u^2+2s^2v^2+2\cdot s^2w^2+2s^2\eta^2+2s^2\iota^2+2s^2\delta^2+2s^2\lambda^2+2t^2u^2+2t^2v^2+2t^2w^2+2t^2\eta^2+2t^2\iota^2+2t^2\delta^2+2t^2\lambda^2+2u^2v^2+2u^2w^2+2u^2\eta^2+2u^2\iota^2+2u^2\delta^2+2u^2\lambda^2+2v^2w^2+2v^2\eta^2+2v^2\iota^2+2v^2\delta^2+2v^2\lambda^2+2w^2\eta^2+2w^2\iota^2+2w^2\delta^2+2\cdot w^2\lambda^2+2\eta^2\iota^2+2\eta^2\delta^2+2\eta^2\lambda^2+2\iota^2\delta^2+2\iota^2\lambda^2+2\delta^2\lambda^2$ in curve $y^2=x^3-2px$ ([12]) can be derived. Now we have notice that whether the rank is maintained or not in curve $y^2=x^3-2px$ when the prime is given as $p=Hu^4-Iu^2v^2+Kv^4$. That is, only the symbol of term for Iu^2v^2 is changed into negative and others are unchanged then, whether the rank is maintained or not. In this case, the residue of prime is not the subject to be considered. In this article, we approach to this point. For it, we needed to define several notations. First, we define that p_1 and p_2 are parity primes when it satisfies that $p_1=Hu^4+Iu^2v^2+Kv^4$ and $p_2=Hu^4-Iu^2v^2+Kv^4$. Here, we restrict to our treatment to curve $y^2=x^3-2px$.

2 Form $p=Hu^4-Iu^2v^2+Kv^4$

In section 2, we access to rank of curve. For E_{-2p} it is enough that we only find the solution of equation $4)N^2=-2M^4+pe^4$ for Γ from [3] and for E_{-pq} it is sufficient that we find the solutions of $3)N^2=pM^4-qe^4$ for Γ and $5)N^2=2pM^4+2qe^4$ for $\bar{\Gamma}$ from [5]. And the notations $r4.2$ and $r4.4$ and $w.i.u.v.1$ are in [9] and $w.i.u.v.w.1$ is in [14].

Theorem 2.1. Assume that E_{-2p} is an elliptic curve $y^2=x^3-2px$ where p is a prime and E_{-pq} is an elliptic curve $y^2=x^3-pqx$ where p and q are distinct odd primes such that $p=27u^4+v^4+w^4+10u^2v^2+10u^2w^2+2v^2w^2$ and $q=23u^4+v^4+w^4+10u^2v^2+10u^2w^2+2v^2w^2$ $w.i.u.v.w.1$ and $p\equiv 3(mod\ 16)$, $q\equiv 15(mod\ 16)$ then, following result is derived:

(1). We appoint that two distinct primes p_1 and p_2 are parity primes as $p_1 = 243u^4 + 36u^2v^2 + 4v^4$ with integers u and v and $(u\ ,\ v) = 1$ and $p_1 \equiv 11(mod\ 16)$ and $p_2 = 243u^4 - 36u^2v^2 + 4v^4\ w.i.u.v.1$ and $p_2 \equiv 3(mod\ 16)$ in curve E_{-2p} then, there educed that

$$rank(E_{-2(243u^4+36u^2v^2+4v^4)}(Q)) = rank(E_{-2(243u^4-36u^2v^2+4v^4)}(Q))$$

$$< rank(E_{-(27u^4+v^4+w^4+\cdots+10u^2w^2+2v^2w^2)(23u^4+v^4+w^4+\cdots+10u^2w^2+2v^2w^2)}(Q)).$$

(2). Set two distinct primes p_1 and p_2 are parity primes as $p_1 = 49u^4 + 28u^2v^2 + 6v^4\ w.i.u.v.1$ and $p_1 \equiv 3(mod\ 16)$ and $p_2 = 49u^4 - 28u^2v^2 + 6v^4\ w.i.u.v.1$ and $p_2 \equiv 11(mod\ 16)$ in E_{-2p} then, we are faced with

$$rank(E_{-2(49u^4+28u^2v^2+6v^4)}(Q)) = rank(E_{-2(49u^4-28u^2v^2+6v^4)}(Q))$$

$$< rank(E_{-(27u^4+v^4+w^4+\cdots+10u^2w^2+2v^2w^2)(23u^4+v^4+w^4+\cdots+10u^2w^2+2v^2w^2)}(Q)).$$

Proo$f$. (1). From [3] we get that

$$rank(E_{-2(243u^4+36u^2v^2+4v^4)}(Q)) = 1.$$

Next, for prime $p_2 = 243u^4 - 36u^2v^2 + 4v^4$ relating equation for Γ is given as

$$4)N^2 = -2M^4 + (243u^4 - 36u^2v^2 + 4v^4)e^4 \text{ from [3].}$$

Suppose that two terms $-36u^2v^2$ and $4v^4$ are consisted of resultant.
We needed to regard arithmetical value

$$-2M^4 + 243u^4e^4.$$

Take e as 1 then, we see that $-2M^4 + 243u^4$ and it should be $81u^4$.
Hence, we confront to

$$2M^4 = 162u^4.$$

Whence, it yields that $M = 3u$.
In addition, from

$$-2(3u)^4 + 243u^4 - 36u^2v^2 + 4v^4$$

$$= 81u^4 - 36u^2v^2 + 4v^4$$

there induced that $N = 9u^2 - 2v^2$.

Thus, the triple $(3u, 1, 9u^2 - 2v^2)$ is deduced as the solution.
Accordingly, we attain that $r4.2$.
Therefore, we reach that

$$rank(E_{-2(243u^4-36u^2v^2+4v^4)}(Q)).$$

Now we treat rank of curve E_{-pq}.
There is given relating equation as

$3) N^2 = (27u^4 + v^4 + w^4 + 10u^2v^2 + 10u^2w^2 + 2v^2w^2)M^4 - (23u^4 + v^4 + w^4 + 10u^2v^2 + 10u^2w^2 + 2v^2w^2)e^4$ for Γ and

$5) N^2 = 2(27u^4 + v^4 + w^4 + 10u^2v^2 + 10u^2w^2 + 2v^2w^2)M^4 + 2(23u^4 + v^4 + w^4 + 10u^2v^2 + 10u^2w^2 + 2v^2w^2)e^4$ for $\bar{\Gamma}$.

Triples $(1, 1, 2u^2)$ and $(1, 1, 10u^2 + 2v^2 + 2w^2)$ is educed as the solutions of above equations.
Thus, we acquire that $r4.4$.
Whence, there deduced that

$$rank(E_{-(27u^4+v^4+w^4+\cdots+2v^2w^2)(23u^4+v^4+w^4+\cdots+2v^2w^2)}(Q)) \cdots\cdots (\text{아}).$$

Hence, the proof is completed.
(2). Next, we treat for primes $p_1 = 49u^4 + 28u^2v^2 + 6v^4$ and $p_2 = 49u^4 - 28u^2v^2 + 6v^4$.
Equations are gotten as

$4) N^2 = -2M^4 + (49u^4 + 28u^2v^2 + 6v^4)e^4$ for Γ and

$4) N^2 = -2M^4 + (49u^4 - 28u^2v^2 + 6v^4)e^4$ for Γ due to [3].

We got solutions as

$(1, 1, 7u^2 + 2v^2)$ and $(1, 1, 7u^2 - 2v^2)$ respectively.

Thereby, it is induced that $r4.2$ in both p_1 and p_2.
On this account, we confront to

$$rank(E_{-2(49u^4+28u^2v^2+6v^4)}(Q))$$

$$= rank(E_{-2(49u^4-28u^2v^2+6v^4)}(Q)) = 1.$$

For this reason, we achieved the proof from (아). □

In above, we took parity primes $p_1 = 243u^4 + 36u^2v^2 + 4v^4$ and $p_2 = 243u^4 - 36u^2v^2 + 4v^4$, $p_1 = 49u^4 + 28u^2v^2 + 6v^4$ and $p_2 = 49u^4 - 28u^2v^2 + 6v^4$. Meanwhile, the forms of residues are $p_1 \equiv 11(mod\,16)$ and $p_2 \equiv 3(mod\,16)$, $p_1 \equiv 3(mod\,16)$ and $p_2 \equiv 11(mod\,16)$ respectively. Even if residues are different relating equations are the same, thus parity primes could be induced.

Remark 2.2. In E_{-2p} if we take $p_1 = 13u^4 + 100u^2v^2 + 200v^4$ and $p_1 \equiv 13(mod\,16)$ then, rank is given as 1([6]). Take $p_2 = 13u^4 - 100u^2v^2 + 200v^4$ and $p_2 \equiv 5(mod\,16)$ then, solution of equation $2)N^2 = -M^4 + 2(13u^4 - 100u^2v^2 + 200v^4)e^4$ for Γ and $(u, 1, 5u^2 - 20v^2)$ is gotten as the solution and example is (353, 3, 1).

3 Examples

In this section, we regard examples of theorem 2.1.
Primality was taken by [1].
Examples are gotten as follows:

(p_2, u, v): (211, 1, 1) and (150979, 5, 1) and (581683, 7, 1) and

(3553411, 11, 1).

(p, q, u, v, w): (227, 223, 1, 3, 1) and (69827, 60223, 7, 3, 1).

(p_1, u, v): (787, 1, 3) and (91283, 1, 11).

(p_2, u, v): (687403, 11, 3) and (482347, 11, 9).

References

[1] C. Caldwell, http://primes.utm.edu/curios/includes/primetest.php.

[2] S. W. Kim, Ranks of elliptic curves $y^2 = x^3 \pm 4px$, *Int. J. of Algebra,* **9** (2015), 283-290. https://doi.org/10.12988/ija.2015.5421

[3] S. W. Kim, Crucial function of prime's form, *Int. J. of Algebra,* **10** (2016), 283-290. https://doi.org/10.12988/ija.2016.6428

[4] S. W. Kim, Searching the ranks of elliptic curves $y^2 = x^3 - px$, *Int. J. of Algebra,* **12** (2018), 311-318. https://doi.org/10.12988/ija.2018.8934

[5] S. W. Kim, Different odd primes in curve $y^2 = x^3 - pqx$, *Far East J. Math. Sci. (FJMS),* **107** (2018), 155-165. https://doi.org/10.17654/ms107010155

[6] S. W. Kim, Various forms in components of primes, *Int. J. of Algebra,* **13** (2019), 59-72. https://doi.org/10.12988/ija.2019.913

[7] S. W. Kim, Algebraic structure in elliptic curves $y^2 = x^3 - 5px$, *Int. J. of Algebra,* **13** (2019), 143-152. https://doi.org/10.12988/ija.2019.9212

[8] S. W. Kim, Ranks of several elliptic curves $y^2 = x^3 \mp 3px$, *Int. J. of Algebra,* **15** (2021), 191-203. https://doi.org/10.12988/ija.2021.91565

[9] S. W. Kim, Enumeration in ranks of various elliptic curves $y^2 = x^3 \pm Ax$, *Int. J. of Algebra,* **14** (2020), 139-162. https://doi.org/10.12988/ija.2020.91250

[10] S. W. Kim, Ranks in some elliptic curves $y^2 = x^3 \pm Apx$, *JP J. of Algebra, Number Theory and Applications,* **51** (2021), 223-248. https://doi.org/10.17654/nt051020223

[11] S. W. Kim, Ranks in elliptic curves of the forms $y^2 = x^3 + Ax^2 + Bx$, *Int. J. of Algebra,* **16** (2022), 109-218. https://doi.org/10.12988/ija.2022.91726

[12] S. W. Kim, Composition of primes in elliptic curves $y^2 = x^3 - 2px$, *Int. J. of Algebra,* **17** (2023), 105-112. https://doi.org/10.12988/ija.2023.91743

[13] S. W. Kim, Ranks in curves $y^2 = x^3 - 2px$ with primes, *Int. J. of Algebra,* **17** (2023), 113-120. https://doi.org/10.12988/ija.2023.91744

[14] S. W. Kim, Distinct primes in elliptic curve $y^2 = x^3 - pqx$, *Int. J. of Algebra,* **18** (2024), 41-48. https://doi.org/10.12988/ija.2024.91854

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