International Journal of Algebra, Vol. 20, 2026, no. 2, 71 - 80
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2026.92008

Elliptic Curve $y^2 = x^3 + pqx$ with Distinct Odd Primes

Shin-Wook Kim

Deokjin-gu, Songcheon, 54823
I-Park Apt
Jeonju, Jeonbuk, Korea

Abstract

Denote E_{pq} as an elliptic curve $y^2 = x^3 + pqx$ with distinct odd primes p and q then, we shall compute the rank of curve where p and q are composed of more than 10 variables and 15 terms.

Mathematics Subject Classification: 11A41, 11G05

Keywords: Distinct primes, elliptic curve

1 Introduction

In [3], the author showed that rank of E_{-2p}: $y^2 = x^3 - 2px$ with prime p as $p = 6h^4 + i^4 + j^4 + k^4 + l^4 + s^4 + t^4 + u^4 + v^4 + w^4 + \eta^4 + \iota^4 + \delta^4 + \lambda^4 - 4h^2i^2 + 4h^2j^2 + 4h^2k^2 + 4h^2l^2 + 4h^2s^2 + 4h^2t^2 + 4h^2u^2 + 4h^2v^2 + 4h^2w^2 + 4h^2\eta^2 + 4h^2\iota^2 + 4h^2\delta^2 + 4h^2\lambda^2 - 2i^2j^2 - 2i^2k^2 - 2i^2l^2 - 2i^2s^2 - 2i^2t^2 - 2i^2u^2 - 2i^2v^2 - 2i^2w^2 - 2i^2\eta^2 - 2i^2\iota^2 - 2i^2\delta^2 - 2i^2\lambda^2 + 2j^2k^2 + 2j^2l^2 + 2j^2s^2 + 2j^2t^2 + 2j^2u^2 + 2j^2v^2 + 2j^2w^2 + 2j^2\eta^2 + 2j^2\iota^2 + 2j^2\delta^2 + 2j^2\lambda^2 + 2 \cdot k^2l^2 + 2k^2s^2 + 2k^2t^2 + 2k^2u^2 + 2k^2v^2 + 2k^2w^2 + 2k^2\eta^2 + 2k^2\iota^2 + 2k^2\delta^2 + 2k^2\lambda^2 + 2l^2s^2 + 2l^2t^2 + 2l^2u^2 + 2l^2v^2 + 2l^2w^2 + 2l^2\eta^2 + 2l^2\iota^2 + 2l^2\delta^2 + 2l^2\lambda^2 + 2s^2t^2 + 2s^2u^2 + 2s^2v^2 + 2s^2w^2 + 2s^2\eta^2 + 2s^2\iota^2 + 2s^2\delta^2 + 2s^2\lambda^2 + 2t^2u^2 + 2t^2v^2 + 2t^2w^2 + 2t^2\eta^2 + 2t^2\iota^2 + 2t^2\delta^2 + 2t^2\lambda^2 + 2u^2v^2 + 2u^2w^2 + 2u^2\eta^2 + 2u^2\iota^2 + 2u^2\delta^2 + 2u^2\lambda^2 + 2v^2w^2 + 2v^2\eta^2 + 2v^2\iota^2 + 2v^2\delta^2 + 2v^2$

$\cdot\lambda^2 + 2w^2\eta^2 + 2w^2\iota^2 + 2w^2\delta^2 + 2w^2\lambda^2 + 2\eta^2\iota^2 + 2\eta^2\delta^2 + 2\eta^2\lambda^2 + 2\iota^2\delta^2 + 2\iota^2\lambda^2 + 2\delta^2\lambda^2 \cdots\cdots (AA)$ with integers $h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda$ and $(h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda) = 1$ and $p \equiv 11(mod\ 16)$ is 1. In curve $E_{-2p}: y^2 = x^3 - 2px$ we can obtain many results of rank 1 where the prime is the form $p = Hu^4 + Iu^2v^2 + Kv^4 \cdots\cdots (AB)$. If we force to choose one of forms (AA) and (AB) then, taking latter one is effective. But we also pursue the form (AA). This is involved to examples. The examples of (AA) are given as 443, 43051([3]). Between these primes there are 491, 571, $\cdots\cdots\cdots\cdots\cdots\cdots$, 39371, $\cdots\cdots\cdots$ with $p \equiv 11(mod\ 16)$. Without result of (AA) we cannot group into a single category the numbers 443, 43051. Namely, these two primes cannot be correlated to same category except the form $p \equiv 11(mod\ 16)$. It is similar in curve $E_{-pq}: y^2 = x^3 - pqx$. In [5], the author verified that rank of E_{-pq} is 2 where primes are $p = c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + 3z^2 - 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2 \cdot c^2l + 2c^2s + 2c^2t + 2c^2u + 2c^2v + 2c^2w + 2c^2z - 2df - 2dg - 2dh - 2di -2dj - 2dk - 2dl - 2ds - 2dt - 2du - 2dv - 2dw - 2dz + 2fg + 2fh + 2 \cdot fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi +2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2h \cdot k + 2hl + 2hs + 2ht + 2hu + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2i \cdot t + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz +2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw +2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + +2tz + 2uv + 2u \cdot w + 2uz + 2vw + 2vz + 2wz$ and $q = c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 +l^2 + s^2 + t^2 + u^2 + v^2 + w^2 - z^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2 \cdot c^2j + 2c^2k + 2c^2l + 2c^2s + 2c^2t + 2c^2u + 2c^2v + 2c^2w + 2c^2z - 2df - 2d \cdot g - 2dh - 2di - 2dj - 2dk - 2dl - 2ds - 2dt - 2du - 2dv - 2dw - 2dz +2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + +2fv + 2fw + 2 \cdot fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz +2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik +2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2j \cdot v + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2 \cdot lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz\ w.i.c.d.f.g.h.i.j.k.l.s.t.u.v. w.z.1 \cdots\cdots (BB)$ and $p \equiv 11(mod\ 16)$ and $q \equiv 7(mod\ 16)$ and examples are 443, 439 and 56171, 56167. Between these primes there are prime numbers with $p \equiv 11(mod\ 16)$ and $q \equiv 7(mod\ 16)$ as 827 , $\cdots\cdots\cdots\cdots\cdots$, 50891 , $\cdots\cdots\cdots\cdots\cdots$ and 1319, $\cdots\cdots\cdots\cdots\cdots$,$48407$, $\cdots\cdots\cdots\cdots\cdots$. Henceforth, without (BB) it is difficult to relate the primes 443, 56171 and 439, 56167. In [4], the author proved that rank of $E_{pq}: y^2 = x^3 + pqx$ is at least 2 where primes are $p = 9s^2 + 2t^2 + 2u^2 + 2v^2 + 2w^2 + 8st + 8su + 8sv + 8ww + 4tu + 4tv + 4tw +4uv + 4uw + 4vw$ and $q = 7s^2 + 2t^2 + 2u^2 + 2v^2 + 2w^2 + 8st + 8su + 8sv + 8sw + 4tu + 4tv + 4tw + 4uv + 4uw + 4vw\ w.i.s.t.u.v.w.1$ and $p \equiv 1(mod\ 8)$ and $q \equiv 7(mod\ 16)$. But compared with the case of E_{-pq} the number of terms in primes are less. There is no certain number that which is many terms in prime. In this article, we shall treat rank of curve E_{pq} where distinct odd primes are more than 5 variables and 15 terms. Put E as an elliptic curve $y^2 = x^3 + ax^2 + bx$ and Γ as the set of rational points on E and take $\bar{E}$ is the curve $y^2 = x(x^2 - 2ax + a^2 - 4b)$ and $\bar{\Gamma}$ as the set of rational points on $\bar{E}$. For the set Γ and $\bar{\Gamma}$ we take $N^2 = b_1M^4 + aM^2e^2 + b_2e^4$ and $N^2 = b_1M^4 - 2aM^2e^2 + b_2e^4$ as relating equation in section 6 of Chapter III in [6] and in [2]. Lastly, we acquire that $2^r = \frac{\#\alpha(\Gamma)\#\bar{\alpha}(\bar{\Gamma})}{4}$ where r denotes the rank of E.

We assume several notations as follows:

$w.i.A.B.C.D.F.G.H.I.J.1$: with integers A and B and C and D and F and G and H and I and J and $(A, B, C, D, F, G, H, I, J) = 1$.

$r^{\geq}4.4$: $2^r \geq \frac{4\cdot 4}{4}$([2]).

2 Composition of Primes

Now we consider the rank of curve.

Theorem 2.1. (1). Set the primes p and q as $p = 9A^2 + 2B^2 + 2C^2 + 2D^2 + 2 \cdot F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB - 8AC - 8AD - 8AF - 8AG - 8AH - 8 \cdot AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ$ and $q = 7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB - 8AC - 8AD - 8AF - 8 \cdot AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ$ $w.i.A.B.C.D.F.G.H.I.J.1$ and $p \equiv 1 (mod\ 8)$ and $q \equiv 7, 15 (mod\ 16)$ in $E_{pq}: y^2 = x^3 + pqx$ then, we gain

$$rank(E_{(9A^2+2B^2+\cdots\cdots-8AB-\cdots\cdots\cdots\cdots+4IJ)(7A^2+2B^2+\cdots\cdots-8AB-\cdots\cdots\cdots\cdots+4IJ)}(Q)) \geq 2.$$

(2). Let the primes p and q be $p = 9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 8AB + 8AC + 8AD + 8AF + 8AG + 8AH + 8AI + 8AJ + 4 \cdot BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ$ and $q = 7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 8AB + 8AC + 8AD + 8AF + 8AG + 8AH +$

$8AI + 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ\ w.i.A.B.C.D.F.G.H.I.J.$ 1 and $p \equiv 1(mod\ 8)$ and $q \equiv 15(mod\ 16)$ in $E_{pq}: y^2 = x^3 + pqx$ then, there derived that

$$rank(E_{(9A^2+2B^2+\cdots\cdots+8AB+\cdots\cdots\cdots+4IJ)(7A^2+2B^2+\cdots\cdots+8AB+\cdots\cdots\cdots\cdots+4IJ)}(Q)) \geq 2.$$

(3). We assign that primes p and q as $p = 3A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 4AB + 4AC + 4AD + 4AF + 4AG + 4AH + 4AI + 4 \cdot AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ$ and $q = A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 4AB + 4AC + 4AD + 4AF + 4AG + 4AH + 4 \cdot AI + 4AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ\ w.i.A.B.C.D.F.G.H.I.J.1$ and $p \equiv 3,11(mod\ 16)$ and $q \equiv 1(mod\ 8)$ in $E_{pq}: y^2 = x^3 + pqx$ then, we get the consequence

$$rank(E_{(3A^2+2B^2+\cdots\cdots+4AB+\cdots\cdots\cdots\cdots+4IJ)(A^2+2B^2+\cdots\cdots+4AB+\cdots\cdots\cdots\cdots+4IJ)}(Q)) \geq 2.$$

Proof. (1). From [4] we only should find the solutions of

$2)N^2 = (9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB - 8 \cdot AC - 8AD - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4 \cdot HJ + 4IJ)M^4 + (7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB - 8AC - 8AD - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + +4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ)e^4$ for Γ and

$9)N^2 = 2(9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB$

$-8AC - 8AD - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG$

$+4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG$

$+4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI +$

$4HJ + 4IJ)M^4 - 2(7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2$

$-8AC - 8AD - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG$

$+4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG$

$+4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI +$

$4HJ + 4IJ)e^4$

for $\bar{\Gamma}$.

In equation 2) the terms $9A^2$ and $7A^2$ are in coefficients of M^4 and e^4.
Thus, we can imagine the square $16A^2$ after selecting the values M and e as 1.
Now the remaining terms in coefficients of M^4 and e^4 is common numerical value $2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB - 8AC - 8AD - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH +4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH +4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ$.
Henceforth, in determining the components of N we take first one as $4A$.
And from the negative terms we choose others as $-2B$ and $-2C$ and $-2D$ and $-2F$ and $-2G$ and $-2H$ and $-2I$ and $-2J$.
Next, from taking a pair from $-2B$ to $-2J$ we can obtain other terms from $8BC$ to $8IJ$.
Consequently, we get the solution of 2) as $(1, 1, 4A - 2B - 2C - 2D - 2F - 2G - 2H - 2I - 2J)$.
Thereby, we take that $\#\alpha(\Gamma) = 4$.
In the next step, in equation 5) we assume that $M = e = 1$ then, there comes that

$2(9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB - 8AC$

$-8AD - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH$

$+4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH$

$+4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ +$

$4IJ) - 2(7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AC - 8$

$\cdot AD - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH +$

$4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH +$

$4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4I$

$\cdot J) = 18A^2 - 14A^2 = 4A^2.$

Wherefore, $(1, 1, 2A)$ is given as the solution of 5).
And so we reach that $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$.
Therefore, we accomplished the proof from $r^{\geq}4.4$
(2). Due to [4], there are equations as

$2)N^2 = (9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 8AB + 8$

$\cdot AC + 8AD + 8AF + 8AG + 8AH + 8AI + 8AJ + 4BC + 4BD + 4BF + 4BG$

$+4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG$

$+4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI$

$+4HJ + 4IJ)M^4 + (7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2$

$+8AB + 8AC + 8AD + 8AF + 8AG + 8AH + 8AI + 8AJ + 4BC + 4BD + 4BF$

$+4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF$

$+4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ$

$+4HI + 4HJ + 4IJ)e^4$ for Γ and

$9)N^2 = 2(9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 8AB +$

$8AC + 8AD + 8AF + 8AG + 8AH + 8AI + 8AJ + 4BC + 4BD + 4BF + 4BG +$

$4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG +$

$4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4$

$\cdot HJ + 4IJ)M^4 - 2(7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2$

$+8AB + 8AC + 8AD + 8AF + 8AG + 8AH + 8AI + 8AJ + 4BC + 4BD + 4BF$

$+4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF$

$+4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ +$

$4HI + 4HJ + 4IJ)e^4$ for $\bar{\Gamma}$.

The difference in primes between (1) and (2) is negative terms became positive terms.

Henceforth, the we got the solutions $(1, 1, 4A + 2B + 2C + 2D + 2F + 2G + 2H + 2I + 2J)$ and $(1, 1, 2A)$ respectively.

Whence, we take that $\#\alpha(\Gamma) = 4$ and $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$.

On that account, we are confronted with $r^{\geq}4.4$.

So it completes the proof.

(3). First, we treat the primes $p \equiv 3(mod\ 16)$ and $q \equiv 1(mod\ 8)$.

In [2], the equations which take a solution are given as $2)N^2 = pM^4 + qe^4$ for Γ and $10)N^2 = -2pM^4 + 2qe^4$ for $\bar{\Gamma}$.

In it p and q are the forms $p \equiv 1(mod\ 8)$ and $q \equiv 3(mod\ 16)$.

But here p and q are gotten as $p \equiv 3(mod\ 16)$ and $q \equiv 1(mod\ 8)$.

Whence, we maintain $2)N^2 = pM^4 + qe^4$ but for equation 10) it is changed into $9)N^2 = 2pM^4 - 2qe^4$.

Thus, we should find the solution of relating equations $2)N^2 = (3A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 4AB + 4AC + 4AD + 4AF + 4 \cdot AG + 4AH + 4AI + 4AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ)M^4 + (A^2 +2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 4AB + 4AC + 4AD + 4 \cdot AF + 4AG + 4AH + 4AI + 4AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ)e^4$ for Γ and $9)N^2 = 2(3A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 4AB + 4AC + 4AD + 4AF + 4AG + 4AH + 4AI + 4AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ)M^4 - 2(A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 +2I^2 + 2J^2 + 4AC + +4AD + 4AF + 4AG + 4AH + 4AI + 4AJ + 4BC + 4BD +4BF + 4BG + 4BH + 4BI + 4BJ + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF +4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4IJ)e^4$ for $\bar{\Gamma}$.

Now the triples $(1, 1, 2A + 2B + 2C + 2D + 2F + 2G + 2H + 2I + 2J)$ and $(1, 1, 2A)$ satisfies the solution of equations respectively.

Therefore, we acquire that $\#\alpha(\Gamma) = 4$ and $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$.

For this reason, we gain $r^{\geq}4.4$.

Second, we consider the case $p \equiv 11(mod\ 16)$ and $q \equiv 1(mod\ 8)$.

Relating equations for Γ are given as $1)N^2 = M^4 + (16k + 11)(8k' + 1)e^4$ and

2)$N^2 = (16k + 11)M^4 + (8k' + 1)e^4$.

As in first case in the above, 2) takes a solution $(1, 1, 2A + 2B + 2C + 2D + 2F + 2G + 2H + 2I + 2J)$.

Hence, we have that $\#\alpha(\Gamma) = 4$.

Next, relating equations for $\bar{\Gamma}$ are 1)$N^2 = M^4 - 4(16k + 11)(8k' + 1)e^4$ and 2)$N^2 = -M^4 + 4(16k + 11)(8k' + 1)e^4$ and 3)$N^2 = 2M^4 - 2(16k + 11)(8k' + 1)e^4$ and 4)$N^2 = -2M^4 + 2(16k + 11)(8k' + 1)e^4$ and 5)$N^2 = 4M^4 - (16k + 11)(8k' + 1)e^4$ and 6)$N^2 = -4M^4 + (16k + 11)(8k' + 1)e^4$ and 7)$N^2 = (16k + 11)M^4 - 4(8k' + 1)e^4$ and 8)$N^2 = -(16k + 11)M^4 + 4(8k' + 1)e^4$ and 9)$N^2 = 2(16k + 11)M^4 - 2(8k' + 1)e^4$ and 10)$N^2 = -2(16k + 11)M^4 + 2(8k' + 1)e^4$ and 11)$N^2 = 4(16k + 11)M^4 - (8k' + 1)e^4$ and 12)$N^2 = -4(16k + 11)M^4 + (8k' + 1)e^4$.

In modulo 8 in equation 2) yields that $1 \equiv N^2 \equiv 7M^4 + 4e^4 \equiv 7, 3(mod\, 8)$ and the sides do not match.

In modulo p in 3) educes that $N^2 \equiv 2M^4(mod\, p)$ but we also have $\left(\frac{2M^4}{p}\right) = -1$, thus a contradiction is emerged.

Cutting down on 6) by 4 implies that $1 \equiv N^2 \equiv 11e^4 \equiv 3(mod\, 4)$ and both sides are unmatched.

In modulo 4 in 7) deduces that $1 \equiv N^2 \equiv 11M^4 \equiv 3(mod\, 4)$ and two sides are unmatched.

As in first case in the above equation 9) has a solution $(1, 1, 2A)$.

Equation 11) has no solution since cutting down on it by 4 gives that $1 \equiv N^2 \equiv -e^4 \equiv 3(mod\, 4)$ and two sides LHS and RHS are unmatched.

Therefore, the conclusion $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$.

Accordingly, the proof is done. □

Above consequences shows that variables D, F, G, H, I, J are added to existing results of at least rank 2 in [4] where in it the variables are s, t, u. Resultantly, there educed 45 terms. Even if above variables in primes are not 10 variables we attained the forms of primes which are not least number of terms. There is left probability that we can obtain 10 or more number of variables in primes in curve E_{pq} whose rank is at least 2.

Remark 2.2. In [2], the primes were $p = Hu^4 + Iu^2v^2 + Kv^4$ and $q = H'u^4 + I'u^2v^2 + K'v^4$ and $p \equiv 1(mod\, 8)$ and $q \equiv 3(mod\, 16)$ and in [4] the primes were $p = 9s^2 + 2t^2 + 2u^2 + 2v^2 + 2w^2 - 8st - 8su - 8sv - 8sw + 4tu + 4tv + 4tw + 4uv + 4uw + 4vw$ and $q = 7s^2 + 2t^2 + 2u^2 + 2v^2 + 2w^2 - 8st - 8su - 8sv - 8sw + 4tu + 4tv + 4tw + 4uv + 4uw + 4vw$ $w.i.s.t.u.v.w.1$ and $p \equiv 1(mod\, 8)$ and $q \equiv 7,\ 15(mod\, 16)$ and $p = 9s^2 + 2t^2 + 2u^2 - 8st - 8su + 4tu$ and $q = 7s^2 + 2t^2 + 2u^2 - 8st - 8su + 4tu$ $w.i.s.t.u.1$ and $p \equiv 1(mod\, 8)$ and $q \equiv 15(mod\, 16)$. Compared with [2] the numbers of primes were increased and simultaneously q is changed from $q \equiv 3(mod\, 16)$ into $q \equiv 7,\ 15(mod\, 16)$ in (1) and (2).

3 Examples

In this section, we suggest examples. Primality was taken by [1].

There comes examples of theorem 2.1(1):

$$(p, q, A, B, C, D, F, G, H, I, J)(q \equiv 7(mod\ 16)):$$

(1193, 743, 15, 1, 1, 1, 1, 1, 1, 1, 1), (7817, 5639, 33, 1, 1, 1, 1, 1, 1, 1, 1),

(31817, 23879, 63 ,1, 1, 1, 1, 1, 1, 1, 1);

$$(p, q, A, B, C, D, F, G, H, I, J)(q \equiv 15(mod\ 16)):$$

(2753, 1871, 21, 1, 1, 1, 1, 1, 1, 1, 1), (15473, 11423, 45, 1, 1, 1, 1, 1, 1, 1, 1).

We got examples of theorem 2.1(2):

$$(p, q, A, B, C, D, F, G, H, I, J):$$

(401, 383, 3, 1, 1, 1, 1, 1, 1, 1, 1), (26801, 21599, 51, 1, 1, 1, 1, 1, 1, 1, 1),

There derived examples of theorem 2.1(3):

$$(p, q, A, B, C, D, F, G, H, I, J)(p \equiv 3(mod\ 16)):$$

(499, 401, 7, 1, 1, 1, 1, 1, 1, 1, 1), (2803, 1553, 25, 1, 1, 1, 1, 1, 1, 1, 1),

(4003, 2081, 31 ,1, 1, 1, 1, 1, 1, 1, 1);

$$(p, q, A, B, C, D, F, G, H, I, J)(p \equiv 11(mod\ 16)):$$

(251, 233, 3, 1, 1, 1, 1, 1, 1, 1, 1), (15739, 6761, 67, 1, 1, 1, 1, 1, 1, 1, 1).

References

[1] C. Caldwell, http://primes.utm.edu/curios/includes/primetest.php.

[2] S. W. Kim, Enumeration in ranks of various elliptic curves $y^2 = x^3 \pm Ax$, *Int. J. of Algebra,* **14** (2020), 139-162. https://doi.org/10.12988/ija.2020.91250

[3] S. W. Kim, Composition of primes in elliptic curves $y^2 = x^3 - 2px$, *Int. J. of Algebra,* **17** (2023), 105-112. https://doi.org/10.12988/ija.2023.91743

[4] S. W. Kim, Calculation of rank in elliptic curve $y^2 = x^3 + pqx$, *Int. J. of Algebra,* **19** (2025), 69-77. https://doi.org/10.12988/ija.2025.91957

[5] S. W. Kim, Compositions of distinct primes in elliptic curve $y^2 = x^3 - pqx$, *Int. J. of Algebra,* **18** (2024), 105-116. https://doi.org/10.12988/ija.2024.91862

[6] J. H. Silverman and J. Tate, *Rational points on elliptic curves,* Springer, New York, (1992), https://doi.org/10.1007/978-1-4757-4252-7

Received: January 1, 2026; Published: February 3, 2026