

## At least Rank 2 in Elliptic Curve $y^2 = x^3 - 2px$ II

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### Abstract

Define  $E_{-2p}$  as an elliptic curve  $y^2 = x^3 - 2px$  then, we will research the rank of it and submit several examples.

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**Keywords:** Prime, elliptic curve

### 1 Introduction

In [6], the author verified that rank of  $y^2 = x^3 - 2px$  is at least 2 where prime is gotten as  $p = 16u^8 + 4v^8 + w^8 + 16u^4v^4 + 8u^4w^4 - 4v^4w^4$  w. i. u. v. w. 1 and  $p \equiv 1 \pmod{8}$ . If we take prime as  $p = 16u^8 + 16v^8 + w^8 + 32u^4v^4 + 8u^4w^4 - 8v^4w^4$  and searching the solutions of equations  $4)N^2 = -2M^4 + pe^4$  for  $\Gamma$  and  $4)N^2 = 8M^4 + pe^4$  for  $\bar{\Gamma}$  then, it doesn't induce a solution simply. If we suppose the terms  $16u^8$  and  $16v^8$  and  $w^8$  and  $32u^4v^4$  and  $8u^4w^4$  comprised of resultant then, calculations  $-8v^4w^4 - 2M^4$  and  $-8v^4w^4 + 8M^4$  must deduce the results which completes an inducement of square in resultant. But it is not simple treatment. And if this doesn't submit probability of inducement of square then, we should change the components of resultant. In searching the primes that satisfies the solutions of equations  $4)N^2 = -2M^4 + pe^4$  for  $\Gamma$  and  $4)N^2 = 8M^4 + pe^4$  for  $\bar{\Gamma}$  we need not to maintain the coefficient of  $u^8$  as 16. Our aim is attaining square in resultant and so its bigness is meaningless. In this article, we approach to more consequence of rank at least 2 in curve  $y^2 = x^3 - 2px$ . From [4], it is sufficient that we find the solutions of two equations  $4)N^2 = -2M^4 + pe^4$  for  $\Gamma$  and  $4)N^2 = 8M^4 + pe^4$  for  $\bar{\Gamma}$ .

## 2 Computation of Rank

The notation *w. i. u. v. w. 1* is in [5] and *w. i. u. v. 1* and  $r \geq 4.4$  and  $r2.4$  are in [3].

**Theorem 2.1.** (1). Denote prime  $p$  as  $p = 64u^8 + 4v^8 + w^8 + 32u^4v^4 + 16u^4w^4 - 4v^4w^4$  *w. i. u. v. w. 1* and  $p \equiv 1 \pmod{8}$  in elliptic curve  $E_{-2p} : y^2 = x^3 - 2px$  then, we confront to the consequence

$$\begin{aligned} & \text{rank}(E_{-2(64u^8+4v^8+w^8+32u^4v^4+16u^4w^4-4v^4w^4)}(Q)) \\ & \geq \text{rank}(E_{-(11u^4+v^4+w^4+6u^2v^2+6u^2w^2+2v^2w^2)}(7u^4+v^4+w^4+6u^2v^2+6u^2w^2+2v^2w^2)(Q)) \\ & > \text{rank}(E_{-5(451195u^4+97200u^2v^2+162v^4)}(Q)). \end{aligned}$$

where  $E_{-pq} : y^2 = x^3 - pqx$  is the curve as  $p = 11u^4 + v^4 + w^4 + 6u^2v^2 + 6u^2w^2 + 2v^2w^2$  and  $q = 7u^4 + v^4 + w^4 + 6u^2v^2 + 6u^2w^2 + 2v^2w^2$  *w. i. u. v. w. 1* and  $p \equiv 11 \pmod{16}$  and  $q \equiv 7 \pmod{16}$  and  $E_{-5p} : y^2 = x^3 - px$  is given as  $p = 451195u^4 + 97200u^2v^2 + 162v^4$  *w. i. u. v. 1* and  $p \equiv 11 \pmod{16}$ .

*Proof.* Equation 4) for  $\Gamma$  is

$$N^2 = -2M^4 + (64u^8 + 4v^8 + w^8 + 32u^4v^4 + 16u^4w^4 - 4v^4w^4)e^4.$$

Suppose that  $e = 1$ .

The square terms with one variable in coefficient of  $e^4$  are  $64u^8$  and  $4v^8$  and  $w^8$ .

Therefore, in selecting the components of  $N$  we decide as  $8u^4$  and  $2v^4$ .

And we consider final one as  $\pm w^4$ .

The existence of term  $-4v^4w^4$  leaves the possibility that symbol of term for  $w^4$  is negative.

The important thing is calculation with  $-2M^4$  should derive last component as square.

From the determination of  $8u^4$  and  $2v^4$  we can obtain the terms  $64u^8$  and  $4v^8$  and  $32u^4v^4$ .

Without deciding the symbol of  $\pm w^4$  we can get the square  $w^8$ .

In numerical value  $-2M^4 - 4v^4w^4$  the symbol is decided as negative and if we choose the value  $M$  as  $2vw$  then, we confront to  $-32v^4w^4 - 4v^4w^4 = -36v^4w^4$ .

This cannot be component of square since there is factor 3.

Whereas in arithmetical value  $-2M^4 + 16u^4w^4$  if we take  $M$  as  $2uw$  then, we are faced with  $-32u^4w^4 + 16u^4w^4 = -16u^4w^4$ .

And this is matched if we decide the component as  $-w^4$ .

Next, from  $2v^4$  and  $-w^4$  there comes that  $-4v^4w^4$ .

Henceforth, we gain the integer  $N$  as  $8u^4 + 2v^4 - w^4$ .

Consequently, the triple  $(2uw, 1, 8u^4 + 2v^4 - w^4)$  satisfies the solution of relating equation 4).

Whence, it implies that  $\#\alpha(\Gamma) \geq 4$ .

Next, in second equation 4) for  $\bar{\Gamma}$  is given

$$4)N^2 = 8M^4 + (64u^8 + 4v^8 + w^8 + 32u^4v^4 + 16u^4w^4 - 4v^4w^4)e^4.$$

We take that  $e = 1$ .

The term is given as  $8M^4$ .

As we did in the above we maintain the two components as  $8u^4$  and  $2v^4$ .

If the last component is gotten as  $w^4$  and take  $M$  as  $vw$  then, from the calculation  $8v^4w^4 - 4v^4w^4 = 4v^4w^4$  we attain the square

$$64u^8 + 4v^8 + w^8 + 32u^4v^4 + 16u^4w^4 + 4v^4w^4.$$

Wherefore, the solution is deduced as  $(vw, 1, 8u^4 + 2v^4 + w^4)$ .

Thus, it gives that  $\#\bar{\alpha}(\bar{\Gamma}) = 4$ .

To conclude, we gain  $r \geq 4.4$ .

On this account, we reach the consequence

$$\text{rank}(E_{-2(64u^8+4v^8+w^8+32u^4v^4+16u^4w^4-4v^4w^4)}(Q)) \geq 2.$$

In the next step, we treat the rank of  $y^2 = x^3 - pqx$ .

From [3] it is enough that we only find the solutions of equations

$$3)N^2 = (11u^4 + v^4 + w^4 + 6u^2v^2 + 6u^2w^2 + 2v^2w^2)M^4 - (7u^4 + v^4 + w^4 + 6u^2v^2 + 6u^2w^2 + 2v^2w^2)e^4 \text{ for } \Gamma \text{ and}$$

$$5)N^2 = 2(11u^4 + v^4 + w^4 + 6u^2v^2 + 6u^2w^2 + 2v^2w^2)M^4 + 2(7u^4 + v^4 + w^4 + 6u^2v^2 + 6u^2w^2 + 2v^2w^2)e^4 \text{ for } \bar{\Gamma}.$$

We appoint that  $M = e = 1$  in equation 3) then, we acquire that

$$(11u^4 + v^4 + w^4 + 6u^2v^2 + 6u^2w^2 + 2v^2w^2) - (7u^4 + v^4 + w^4 + 6u^2v^2 + 6u^2w^2 + 2v^2w^2) = 4u^4.$$

Hence, we have the solution as  $(1, 1, 2u^2)$ .

Next, let  $M = e = 1$  in equation 5) then, it comes that

$$\begin{aligned}
& (22u^4 + 2v^4 + 2w^4 + 12u^2v^2 + 12u^2w^2 + 4v^2w^2) \\
& + (14u^4 + 2v^4 + 2w^4 + 12u^2v^2 + 12u^2w^2 + 4v^2w^2) \\
& = 36u^4 + 4v^4 + 4w^4 + 24u^2v^2 + 24u^2w^2 + 8v^2w^2.
\end{aligned}$$

It educes the solution as  $(1, 1, 6u^4 + 2v^4 + 2w^4)$ .

Whence, we have that  $\#\alpha(\Gamma) = \#\bar{\alpha}(\bar{\Gamma}) = 4$ .

Accordingly, there induced the consequence

$$\begin{aligned}
& \text{rank}(E_{-(11u^4+v^4+w^4+6u^2v^2+6u^2w^2+2v^2w^2)}(7u^4+v^4+w^4+6u^2v^2+6u^2w^2+2v^2w^2)(Q)) \\
& = 2.
\end{aligned}$$

Third, we compute the rank of  $y^2 = x^3 - 5px$ .

Due to [2] we ought to find the solution of

$$5)N^2 = 10M^4 + 2(451195u^4 + 97200u^2v^2 + 162v^4)e^4 \text{ for } \bar{\Gamma}.$$

It is rewritten as

$$5)N^2 = 10M^4 + (902390u^4 + 194400u^2v^2 + 324v^4)e^4.$$

The square  $324v^4$  is in coefficient of  $e^4$ .

And the term  $194400u^2v^2$  is factored as

$$2 \cdot 5400u^2 \cdot 18v^2.$$

Wherefore, in numerical value  $10M^4 + 902390u^4e^4$  we assume that  $e = 1$ .

Thus, it should be valid that

$$10M^4 + 902390u^4 = 29160000u^4.$$

So we face the relation as

$$10M^4 = 28257610u^4.$$

Hence, it comes that  $M^4 = 2825761u^4$ .

The number 2825761 is larger than 2560000.

2560000 is quartic of 40, thus we choose 41.

Then, quartic of  $41u$  is given as  $2825761u^4$ .

Consequentially, the value  $M$  is deduced as  $41u$ .

Furthermore, from the computation

$$\begin{aligned} & 10(41u)^4 + 902390u^4 + 194400u^2v^2 + 324v^4 \\ & = 29160000u^4 + 194400u^2v^2 + 324v^4 \end{aligned}$$

we obtain that  $N = 5400u^2 + 18v^2$ .

Therefore, we attain the solution as

$$(41u, 1, 5400u^2 + 18v^2).$$

And so there derived that  $\#\bar{\alpha}(\bar{\Gamma}) = 4$ .

And it shows that  $r_{2.4}$ .

Thereby, we achieved the proof.  $\square$

The prime is gotten as  $p = 64u^8 + 4v^8 + w^8 + 32u^4v^4 + 16u^4w^4 - 4v^4w^4$ . Compared with the form  $p = 16u^8 + 4v^8 + w^8 + 16u^4v^4 + 8u^4w^4 - 4v^4w^4$  ([6]) two squares  $4v^8, w^8$  and the term  $-4v^4w^4$  are maintained. The first square is changed into  $64u^8$ , thus other two terms  $16u^4v^4$  and  $8u^4w^4$  are also changed into  $32u^4v^4$  and  $16u^4w^4$ .

**Remark 2.2.** In  $E_{3p}$  if we take the prime as  $p = u^8 + 64v^8 + 1024w^8 + 16u^4v^4 + 128u^4w^4 - 512v^4w^4$  then, for equation 12)  $N^2 = -12M^4 + pe^4$  for  $\bar{\Gamma}$  ([4]) if we take  $M$  and  $e$  as  $2uw$  and 1 then, we are confronted with  $-12(2uw)^4 + u^8 + 64v^8 + 1024w^8 + 16u^4v^4 + 128u^4w^4 - 512v^4w^4 = u^8 + 64v^8 + 1024w^8 + 16u^4v^4 - 192u^4w^4 + 128u^4w^4 - 512v^4w^4 = u^8 + 64v^8 + 1024w^8 + 16u^4v^4 - 64u^4w^4 - 512v^4w^4$  and hence we have the solution as  $(2uv, 1, u^4 + 8v^4 - 32w^4)$  meanwhile for equation 2)  $N^2 = 3M^4 + pe^4$  for  $\Gamma$  searching the solution is not simple treatment. If we take  $M$  and  $e$  as  $u^2$  and 1 then, we gain  $3(u^2)^4 + u^8 + 64v^8 + 1024w^8 + 16u^4v^4 + 128u^4w^4 - 512v^4w^4 = 4u^8 + 64v^8 + 1024w^8 + 16u^4v^4 + 128u^4w^4 - 512v^4w^4$  but it is not a square and if we take  $M$  and  $e$  as  $2uv$  and 1 then, we face that  $3(2uv)^4 + u^8 + 64v^8 + 1024w^8 + 16u^4v^4 + 128u^4w^4 - 512v^4w^4 = u^8 + 64v^8 + 1024w^8 + 64u^4v^4 + 128u^4w^4 - 512v^4w^4$  and it is neither a square.

**Remark 2.3.** In above the bigness of coefficient of  $u^4$  in prime  $p$  in curve  $E_{-5p}$  is larger than the results in [2]. Hence, it also derives the coefficient 97200 of  $u^2v^2$  which is large in relative.

**Remark 2.4.** In  $E_{-5p}$  we also can attain rank correlated to 2 but not so much as in  $E_{-2p}$  and it doesn't yet found the form as in  $p = Hu^8 + Iv^8 + Kw^8 + Lu^4v^4 + Su^4w^4 + Uv^4w^4$ .

### 3 Examples

In section 3, we suggest examples. Primality was taken by [1].

We have examples as follows:

$(p, u, v, w)$ : (113, 1, 1, 1) and (4206593, 4, 1, 1).

$(p, q, u, v, w)$ : (443, 439, 1, 3, 3) and (2971, 2647, 3, 5, 1).

$(p, u, v)$ : (842587, 1, 2) and (4160347, 1, 6) and (48950107, 1, 18).

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