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At least Rank 2 in Elliptic Curve

$y^2 = x^3 + pqx$

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Abstract

We appoint that E_{pq} is an elliptic curve $y^2 = x^3 + pqx$ with distinct odd primes p and q. If p is supposed as the form $p \equiv 1(mod\ 8)$ then, there can be given rank at least 2 according to defining prime q. For taking at least 2, generally primes are gotten as $p = Hu^4 + Iu^2v^2 + Kv^4$ and $q = H'u^4 + I'u^2v^2 + K'v^4 \cdots\cdots (BB)$. In this article, we will treat rank of curve E_{pq} where primes are composed of more terms than (BB).

Mathematics Subject Classification: 11A41, 11G05

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1 Introduction

In [4], the author showed that rank of $E_{-pq} : y^2 = x^3 - pqx$ is 2 where primes are given as $p = 9u^4 + 4v^4 + 16w^4 + 14u^2v^2 + 24u^2w^2 + 16v^2w^2$, $q = 9u^4 + 4v^4 + 16w^4 + 10u^2v^2 + 24u^2w^2 + 16v^2w^2\ w.i.u.v.w.1$ and $p \equiv 3(mod\ 16)$ and $q \equiv 15(mod\ 16)$. In [7], the author verified that rank of $y^2 = x^3 - pqx$ is 2 where primes are $p = c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + 3z^2 - 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2 \cdot c^2j + 2c^2k + 2c^2l + 2c^2s + 2c^2t + 2c^2u + 2c^2v + 2c^2w + 2c^2z - 2df - 2d \cdot g - 2dh - 2di - 2dj - 2dk - 2dl - 2ds - 2dt - 2du - 2dv - 2dw - 2dz +2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2f$

$\cdot z + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2j \cdot v + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2 \cdot lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + +2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz$ and $q = c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 - z^2 - 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2c^2l + 2c^2s + 2c^2t + 2c^2u + 2c^2v + 2c^2w + 2c^2z - 2df - 2dg - 2dh - 2di - 2dj - 2dk - 2dl - 2ds - 2dt - 2du - 2d \cdot v - 2dw - 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + +2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2 \cdot gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2j \cdot s + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2 \cdot kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2 \cdot tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz\ w.i.c.d.f.g..h.i.j.k.l.s.t.u.v.w.z.1$ and $p \equiv 11(mod\ 16)$ and $q \equiv 7(mod\ 16)$. In [8], the author verified that rank of $y^2 = x^3 + pqx$ is at least 2 where the primes are $p = 9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB - 8AC - 8A \cdot D - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4HJ + 4I \cdot J$ and $q = 7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 - 8AB - 8AC - 8AD - 8AF - 8AG - 8AH - 8AI - 8AJ + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4DF + 4DG + 4DH + 4DI + 4DJ + 4FG + 4FH + 4FI + 4FJ + 4GH + 4GI + 4GJ + 4HI + 4 \cdot HJ + 4IJ\ w.i.A.B.C.D.F.G.H.I.J.1 \cdots\cdots (ZZ)$ and $p \equiv 1(mod\ 8)$ and $q \equiv 7, 15(mod\ 16)$. In this paper, we shall calculate rank of elliptic curve $y^2 = x^3 + pqx$ where primes are more complex than(ZZ).

We take the notations as follows:

$w.i.A.B.C.D.F.G.H.I.J.K.L.1$: with integers A and B and C and D and F and G and H and I and J and K and L

$(A, B, C, D, F, G, H, I, J, K, L) = 1$.

$r^{\geq}4.4$: $2^r \geq \frac{4\cdot 4}{4}$([2]).

2 Investigation of Rank

We treat the rank where primes are $p \equiv 1(mod\ 8)$ and $q \equiv 7, 15(mod\ 16)$. Thus, from [6] it is enough that we only search the solutions of equations $2)N^2 = pM^4 + qe^4$ for Γ and $9)N^2 = 2pM^4 - 2qe^4$ for $\overline{\Gamma}$.

Theorem 2.1. (1). Define distinct odd primes p and q as $p = 9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 8AB - 8AC - 8A \cdot D - 8AF - 8AG - 8AH - 8AI - 8AJ - 8AK - 8AL + 4BC + 4BD + 4BF + 4 \cdot BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL$ and $q = 7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 8AB - 8AC - 8AD - 8AF - 8AG - 8AH - 8AI - 8AJ - 8AK - 8AL + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL\, w.i.A.B.C.D.F.G.H.I.J.K.L.1$ and $p \equiv 1 (mod\ 8)$ and $q \equiv 7, 15 (mod\ 16)$ in $E_{pq}: y^2 = x^3 + pqx$ then, there derived the consequence

$$rank(E_{(9A^2+\cdots\cdots-8AB-\cdots\cdots+4BC+\cdots\cdots+4KL)(7A^2+\cdots\cdots-8AB-\cdots\cdots+4BC+\cdots\cdots+4KL)}(Q)) \geq 2.$$

(2). Let distinct odd primes p and q be $p = 9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 + 8AB + 8AC + 8AD + 8AF + 8AG + 8 \cdot AH + 8AI + 8AJ + 8AK + 8AL + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL$ and $q = 7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 + 8AB + 8AC + 8AD + 8AF + 8A \cdot G + 8AH + 8AI + 8AJ + 8AK + 8AL + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4 \cdot CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4HK + 4 \cdot HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL\, w.i.A.B.C.D.F.G.H.I.J.K.L.1$ and $p \equiv 1 (mod\ 8)$ and $q \equiv 15 (mod\ 16)$ in $E_{pq}: y^2 = x^3 + pqx$ then, we conclude that

$$rank(E_{(9A^2+\cdots\cdots+8AB+\cdots\cdots+4BC+\cdots\cdots+4KL)(7A^2+\cdots\cdots+8AB+\cdots\cdots+4BC+\cdots\cdots+4KL)}(Q)) \geq 2.$$

(3). Denote odd primes p and q as $p = 33A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 16AB - 16AC - 16AD - 16AF - 16A \cdot G - 16AH - 16AI - 16AJ - 16AK - 16AL + 4BC + 4BD + 4BF + 4BG + 4 \cdot BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4$

$\cdot CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4 \cdot HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL$ and $q = 31A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 16AB - 16AC - 16AD - 16AF - 16AG - 16AH - 16AI - 16AJ - 16AK - 16AL + 4BC + 4B \cdot D + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4 \cdot CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4 \cdot GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL\, w.i.A .B.C.D.F.G.H.I.J.K.L.1$ and $p \equiv 1 (mod\, 8)$ and $q \equiv 15 (mod\, 16)$ in $E_{pq}: y^2 = x^3 + pqx$ then, there derived that

$$rank(E_{(33A^2+\cdots\cdots-16AB-\cdots\cdots+\cdots\cdots+4KL)(31A^2+\cdots\cdots-16AB-\cdots\cdots+\cdots\cdots+4KL)}(Q)) \geq 2.$$

(4). We appoint that distinct odd primes p and q as $p = 33A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 + 16AB + 16AC + 16AD + 16AF + 16AG + 16AH + 16AI + 16AJ + 16AK + 16AL + 4BC + 4BD + 4BF +4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI +4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG +4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL$ and $q = 31A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 + 16AB + 16 \cdot AC + 16AD + 16AF + 16AG + 16AH + 16AI + 16AJ + 16AK + 16AL + 4B \cdot C + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4 \cdot CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4 \cdot GK + 4GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4K \cdot L\, w.i.A.B.C.D.F.G.H.I.J.K.L.1$ and $p \equiv 1 (mod\, 8)$ and $q \equiv 7 (mod\, 16)$ in curve $E_{pq}: y^2 = x^3 + pqx$ then, it deduces that

$$rank(E_{(33A^2+\cdots\cdots+16AB+\cdots\cdots+\cdots\cdots+4KL)(31A^2+\cdots\cdots+16AB+\cdots\cdots+\cdots\cdots+4KL)}(Q)) \geq 2.$$

Proo$f$. (1). First, we notice equation for Γ. It is given as $2)N^2 = (9A^2 + 2B^2 + 2 \cdot C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 8AB - 8AC - 8A \cdot D - 8AF - 8AG - 8AH - 8AI - 8AJ - 8AK - 8AL + 4BC + 4BD + 4BF + 4 \cdot BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4 \cdot HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL)M^4 + (7A^2 + 2B^2 +2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 8AB - 8AC - 8 \cdot AD - 8AF - 8AG - 8AH - 8AI - 8AJ - 8AK - 8AL + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4$

$\cdot HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL)e^4$. In coefficients of M^4 and e^4 there exist $9A^2$ and $7A^2$. Therefore, if we take $M = e = 1$ then, we can obtain square $16A^2$. Next, from the terms $-8AB$ to $-8AL \cdots\cdots (XX)$ are negative and other are positive. Hence, in choosing the components of N the negative one must exist. We take first component as $4A$. The terms from $2B^2$ to $2L^2$ are in both coefficients. Now there are negative terms and positive terms. We select other components as $-2B$, $-2C$, $-2D$, $-2F$, $-2G$, $-2H$, $-2I$, $-2J$, $-2K$, $-2L \cdots\cdots (VV)$. Now taking a pair $4A$ with (VV) then, there comes that $-16AB$ and $-16AC$ and $-16AD$ and $-16AF$ and $-16AG$ and $-16AH$ and $-16AI$ and $-16AJ$ and $-16AK$ and $-16AL$. These can be gotten from addition of itself in (XX) which were in coefficients of M^4 and e^4. Next, doing a pair in each case in (VV) there deduced that $8BC$ and $8BD$ and $8BF$ and $8BG$ and $8BH$ and $8BI$ and $8BJ$ and $8BK$ and $8BL$. It is also can be induced from addition of itself from $4BC$ to $4BL$ in both coefficients of M^4 and e^4. Remanent terms also can be given similarly. Wherefore, we get the solution as $(1, 1, 4A - 2B - 2C - 2D - 2F - 2G - 2H - 2I - 2J - 2K - 2L)$. Thus, we have that $\#\alpha(\Gamma) = 4$.

In the next step, we consider equation for $\bar{\Gamma}$. It is given as 9) $N^2 = 2(9A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 8AB - 8AC - 8 \cdot AD - 8AF - 8AG - 8AH - 8AI - 8AJ - 8AK - 8AL + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4 \cdot HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL)M^4 - 2(7A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 8AB - 8AC - 8 \cdot AD - 8AF - 8AG - 8AH - 8AI - 8AJ - 8AK - 8AL + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4 \cdot HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL)e^4$. Substitute 1 into both M and e derives that $(18A^2 + 4B^2 + 4C^2 + 4D^2 + 4F^2 + 4G^2 + 4H^2 + 4I^2 + 4J^2 + 4K^2 + 4L^2 - 16AB - 16AC - 16AD - 16AF - 16AG - 16AH - 16AI - 16AJ - 16AK - 16AL + 8BC + 8BD + 8BF + 8BG + 8BH + 8BI + 8BJ + 8BK + 8BL + 8CD + 8CF + 8CG + 8CH + 8CI + 8CJ + 8CK + 8CL + 8DF + 8DG + 8DH + 8DI + 8DJ + 8DK + 8DL + 8FG + 8FH + 8FI + 8FJ + 8FK + 8FL + 8GH + 8GI + 8GJ + 8GK + 8GL + 8HI + 8HJ + 8HK + 8HL + 8IJ + 8IK + 8IL + 8JK + 8JL + 8KL) - (14A^2 + 4B^2 + 4C^2 + 4D^2 + 4F^2 + 4 \cdot G^2 + 4H^2 + 4I^2 + 4J^2 + 4K^2 + 4L^2 - 16AB - 16AC - 16AD - 16AF - 16A \cdot G - 16AH - 16AI - 16AJ - 16AK - 16AL + 8BC + 8BD + 8BF + 8BG + 8 \cdot BH + 8BI + 8BJ + 8BK + 8BL + 8CD + 8CF + 8CG + 8CH + 8CI + 8CJ + 8 \cdot CK + 8CL + 8DF + 8DG + 8DH + 8DI + 8DJ + 8DK + 8DL + 8FG + 8FH + 8FI + 8FJ + 8FK + 8FL + 8GH + 8GI + 8GJ + 8GK + 8GL + 8HI + 8HJ + 8 \cdot HK + 8HL + 8IJ + 8IK + 8IL + 8JK + 8JL + 8KL) = 4A^2$. Hence, we got the solution as $(1, 1, 2A)$. And it implies that $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$. In conclusion, we accomplished the proof from $r^{\geq}4.4$.

(2). Compared with (1) the difference in prims are symbols of terms from $8AB$ to $8AL$ in primes p and q. Thus, the solution of equation 2) is induced as (1, 1, $4A + 2B + 2C + 2D + 2F + 2G + 2H + 2I + 2J + 2K + 2L$). And it shows that $\#\alpha(\Gamma) = 4$. Next, it is clear that solution of 9) is educed as $(1, 1, 2A)$. Hence, we also take that $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$. Consequentially, we achieved the proof from $r^{\geq}4.4$.

(3). Equation for Γ is gotten as

$$2)N^2 = (33A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 16AB - 16AC - 16AD - 16AF - 16AG - 16AH - 16AI - 16AJ - 16A \cdot K - 16AL + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4J \cdot K + 4JL + 4KL)M^4 + (31A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 16AB - 16AC - 16AD - 16AF - 16AG - 16AH - 16AI - 16AJ - 16AK - 16AL + 4BC + 4BD + 4BF + 4BG + 4BH + 4BI + 4BJ + 4 \cdot BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4 \cdot IK + 4IL + 4JK + 4JL + 4KL)e^4.$$

In coefficients of M^4 and e^4 there are $33A^2$ and $31A^2$. If we suppose as $M = e = 1$ then, we can attain a square $64A^2$. The terms from $-16AB$ to $-16AL \cdots\cdots (XX')$ are negative and remaining terms are positive. First, we choose component of N as $8A$. There exist terms from $2B^2$ to $2L^2$ in both coefficients of M^4 and e^4. As we did (1) in the above, we select other components as $-2B, -2C, -2D, -2F, -2G, -2H, -2I, -2J, -2K, -2L \cdots\cdots (VV')$. Now doing a pair $8A$ with (VV') then, we are faced with $-32AB$ and $-32AC$ and $-32AD$ and $-32AF$ and $-32AG$ and $-32AH$ and $-32AI$ and $-32AJ$ and $-32AK$ and $-32AL$. These are derived from addition of itself in (XX') which were in coefficients of M^4 and e^4. Next, remaining terms can be deduced similarly as we did in (1) in the above. Accordingly, the solution is gotten as (1, 1, $8A - 2B - 2C - 2D - 2F - 2G - 2H - 2I - 2J - 2K - 2L$). Whence, we have $\#\alpha(\Gamma) = 4$.

Next, equation for $\bar{\Gamma}$ is gotten as $9)N^2 = 2(33A^2 + 2B^2 + 2C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 16AB - 16AC - 16AD - 16AF - 16A \cdot G - 16AH - 16AI - 16AJ - 16AK - 16AL + 4BC + 4BD + 4BF + 4BG + 4 \cdot BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4CH + 4CI + 4CJ + 4 \cdot CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4GL + 4HI + 4HJ + 4 \cdot HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL)M^4 - 2(31A^2 + 2B^2 + 2 \cdot C^2 + 2D^2 + 2F^2 + 2G^2 + 2H^2 + 2I^2 + 2J^2 + 2K^2 + 2L^2 - 16AB - 16AC - 16AD - 16AF - 16AG - 16AH - 16AI - 16AJ - 16AK - 16AL + 4BC + 4B \cdot D + 4BF + 4BG + 4BH + 4BI + 4BJ + 4BK + 4BL + 4CD + 4CF + 4CG + 4$

$\cdot CH + 4CI + 4CJ + 4CK + 4CL + 4DF + 4DG + 4DH + 4DI + 4DJ + 4DK + 4DL + 4FG + 4FH + 4FI + 4FJ + 4FK + 4FL + 4GH + 4GI + 4GJ + 4GK + 4 \cdot GL + 4HI + 4HJ + 4HK + 4HL + 4IJ + 4IK + 4IL + 4JK + 4JL + 4KL)e^4$.

Replace 1 into both M and e derives that $(66A^2 + 4B^2 + 4C^2 + 4D^2 + 4F^2 + 4 \cdot G^2 + 4H^2 + 4I^2 + 4J^2 + 4K^2 + 4L^2 - 32 \cdot AB - 32AC32AD - 32AF - 32AG -32AH - 32AI - 32AJ - 32AK - 32AL + 8BC + 8BD + 8BF + 8BG + 8BH +8BI + 8BJ + 8BK + 8BL + 8CD + 8CF + 8CG + 8CH + 8CI + 8CJ + 8CK + 8CL + 8DF + 8DG + 8DH + 8DI + 8DJ + 8DK + 8DL + 8FG + 8FH + 8FI + 8FJ + 8FK + 8FL + 8GH + 8GI + 8GJ + 8GK + 8GL + 8HI + 8HJ + 8HK + 8 \cdot HL + 8IJ + 8IK + 8IL + 8JK + 8JL + 8KL) - (62A^2 + 4B^2 + 4C^2 + 4D^2 + 4 \cdot F^2 + 4G^2 + 4H^2 + 4I^2 + 4J^2 + 4K^2 + 4L^2 - 32AB - 32AC - 32AD - 32AF -32AG - 32AH - 32AI - 32AJ - 32AK - 32AL + 8BC + 8BD + 8BF + 8BG +8BH + 8BI + 8BJ + 8BK + 8BL + 8CD + 8CF + 8CG + 8CH + 8CI + 8CJ + 8CK + 8CL + 8DF + 8DG + 8DH + 8DI + 8DJ + 8DK + 8DL + 8FG + 8FH +8FI + 8FJ + 8FK + 8FL + 8GH + 8GI + 8GJ + 8GK + 8GL + 8HI + 8HJ + 8HK + 8HL + 8IJ + 8IK + 8IL + 8JK + 8JL + 8KL) = 66A^2 - 62A^2 = 4A^2$. Thereby, we attain the solution as $(1, 1, 2A)$ and hence $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$ is given.

On that account, we acquire the conclusion as $r^{\geq}4.4$. Hence, the consequence $rank(E_{(33A^2+\cdots\cdots-16AB-\cdots\cdots+\cdots\cdots+4KL)(31A^2+\cdots\cdots-16AB-\cdots\cdots+\cdots\cdots+4KL)}(Q)) \geq 2$ is given.

(4). The difference between (3) and (4) in primes are symbols from $16AB$ to $16AL$. Therefore, the solution of 2) is gotten as $(1, 1, 8A + 2B + 2C + 2D + 2F + 2G + 2H + 2I + 2J + 2K + 2L)$. Henceforth, it deduces that $\#\alpha(\Gamma) = 4$. Next, the solution of 9) is educed as $(1, 1, 2A)$, thus we reach the conclusion $\#\bar{\alpha}(\bar{\Gamma}) \geq 4$. Now from the fact $r^{\geq}4.4$ we obtain the consequence as $rank(E_{(33A^2+\cdots\cdots\cdots+4KL)(31A^2+\cdots\cdots+4KL)}(Q)) \geq 2$. □

In (1), (2) and (3), (4) the coefficients of M^4 and $e^4$are same in absolute values but different symbols in terms. Namely, in (1) from $-8AB$ to $-8AL$ are negative meanwhile in (2) the symbols are positive. Other terms are all same. In (3) from the term $-16AB$ to $-16AL$ it is negative whereas in (4), the values are positive. And remaining terms are same.

Remark 2.2. In [3], the author treated rank of curve $E_{14p}: y^2 = x^3 + 14px$. If it satisfies that p is the form $p = u^8 + 60u^4v^4 + 4v^8 \quad w.i.u.v.1$ and $p \equiv 1 (mod\ 8)$ then, rank is at least 2([3]). The form E_{14p} is not a orthodox curve. It is neither secondary curve. It is in between the curves. It is necessary to note that rank is at least 2 because in generalization of rank in elliptic curve from the value 2 it is meaningful. Rank 1 is starting point that $E(Q)$ is infinite but in practical computation we can confront to rank 1 often whereas from 2 it is difficult to get in relative.

Remark 2.3. In [5], the author treated rank of curve $y^2 = x^3 - pqstx$ and $y^2 =$

$x^3 - pqsx$. Even if the results are at least rank 1 it needs to consider because it is the first result in forms whose numbers of coefficients of x are more than 2(except the curves $y^2 = x^3 \pm 2pqx$).

3 Examples

In this section we note to examples of previous theorem.
We can check primality by [1].
Now the examples from theorem 2.1(1) to (4) are the followings:

$(p, q, A, B, C, D, F, G, H, I, J, K, L)(q \equiv 7(mod\ 16))$:

(409, 167, 11, 1, ⋯⋯⋯, 1) and (5449, 3767, 29, 1, ⋯⋯⋯ , 1) and

(80489, 60887, 99, 1, ⋯⋯⋯, 1).

$(q \equiv 15(mod\ 16))$:

(3121, 2063, 23, 1, ⋯⋯⋯ , 1) and (30881, 22943, 63, 1, ⋯⋯⋯, 1).

$(p, q, A, B, C, D, F, G, H, I, J, K, L)$:

(12641, 10463, 33, 1, ⋯⋯⋯, 1) and (40961, 33023, 63, 1, ⋯⋯⋯, 1).

$(p, q, A, B, C, D, F, G, H, I, J, K, L)$:

(146273, 136751, 69, 1, ⋯⋯⋯, 1) and (270737, 253439, 93, 1, ⋯⋯⋯, 1).

$(p, q, A, B, C, D, F, G, H, I, J, K, L)$:

(56633, 53591, 39, 1, ⋯⋯⋯, 1) and (141257, 133319, 63, 1, ⋯⋯⋯, 1).

In above '⋯⋯⋯' means that $C = D = F = G = H = I = J = K = 1$.

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