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Several Results Correlated to Strongly Irreducible Ideal in Commutative Ring

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Abstract

Suppose that P_i is strongly irreducible ideal which contains ideal I of commutative ring R with identity. Then, we investigate whether the union of $\sqrt{P_i}/I$ in R/I is strongly irreducible or not. In addition, we will treat that whether intersection of strongly irreducible is strongly irreducible or not.

Mathematics Subject Classification: 13A15, 13C05

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1 Introduction

In [3], the authors defined irreducible ideal and strongly irreducible ideal of a commutative ring R and treated some properties related to prime ideal and principal ideal and primary ideal of R. They, showed that under the assumption that (R, M) is a quasi local ring and I is a strongly irreducible M-primary ideal of R and $I \subset I{:_R} M$, then, $I{:_R} M$ is a principal ideal and $I = (I{:_R} M)M$. In [1], Atani applied the properties of strongly irreducible ideal to R-module. He defined that R-submodule N is strongly irreducible if for submodules N_1, N_2 of M, $N_1 \cap N_2$ applies that $N_1 \subseteq N$ or $N_2 \subseteq N$ and proved that under the assumption that R is a ring and M is multiplication R-module and N is a prime submodule of M, then N is strongly irreducible([1]). In [2], Darani considered condition of iff to irreducibleness and principal quotient-chain condition. In [Lemma 3 in 1·8 of

Chap 1, 5], Northcott showed that if R is Noetherian then, every irreducible ideal is primary. In this paper, we consider whether the union of $\sqrt{P_i}/I$ in R/I is strongly irreducible or not and we shall investigate whether intersection of strongly irreducible is strongly irreducible or not.

Assume that R is a commutative ring with identity. For ideal I of R define radical of I as $\sqrt{I} = \{a \in R: a^n \in I$ for some positive integer $n\}$ then, it also became ideal of R. And for ideals I, A, B we say that I is strongly irreducible if $A \cap B \subseteq I$, then $A \subseteq I$ or $B \subseteq I$. And suppose that I is irreducible if $A \cap B = I$, then $A = I$ or $B = I$.

2 Radical of Prime ideal

Now we treat following result:

Theorem 2.1. (1). We appoint that $P_i(i = 1, 2, \cdots\cdots, n)$ are prime ideals in R and ideal I of R is contained to P_i then, $\bigcup_{i=1}^{n}(\sqrt{P_i}/I)$ is a strongly irreducible in R/I.

(2). Assume that $I_1, I_2, \cdots\cdots, I_n$ are strongly irreducible ideals of R then, $\bigcup_{i=1}^{n} I_i$ is irreducible.

(3). Take I as a prime ideal of R then, $\sqrt{I}$ is strongly irreducible.

(4). If I is irreducible ideal in R then, we cannot say that I is strongly irreducible.

(5). Set I_1 and I_2 are strongly irreducible ideals of R then, $I_1 \cap I_2$ is not strongly irreducible.

(6). Suppose that P_1 and P_2 are prime ideals in R then, we cannot say that $P_1 \cap P_2$ is prime ideal.

Proo$f$. (1). For prime ideals P_1, P_2 we must show that $P_1 \cup P_2$ is a prime ideal.
Assume that $ab \in P_1 \cup P_2$ then, $ab \in P_1$ or $ab \in P_2$.
Since P_1 and P_2 are prime ideals there exist three cases as follows:

(a). $a \in P_1, b \notin P_1$, (b). $a \notin P_1, b \in P_1$, (c). $a \in P_1, b \in P_1$ or

1) $a \in P_2, b \notin P_2$, 2) $a \notin P_2, b \in P_2$, 3) $a \in P_2, b \in P_2$.

Now from above, we confront to following results:

(a), 1): $a \in P_1 \cup P_2, b \notin P_1 \cup P_2$.

(a), 2): $a \in P_1 \subseteq P_1 \cup P_2, b \in P_2 \subseteq P_1 \cup P_2$.

(a), 3): $a \in P_1 \cup P_2$, $b \in P_2 \subseteq P_1 \cup P_2$.

(b), 1): $a \in P_2 \subseteq P_1 \cup P_2$, $b \in P_1 \subseteq P_1 \cup P_2$.

(b), 2): $a \notin P_1 \cup P_2$, $b \in P_1 \cup P_2$.

(b), 3): $a \in P_2 \subseteq P_1 \cup P_2$, $b \in P_1 \cup P_2$.

(c), 1): $a \in P_1 \cup P_2$, $b \in P_1 \subseteq P_1 \cup P_2$.

(c), 2): $a \in P_1 \subseteq P_1 \cup P_2$, $b \in P_1 \cup P_2$.

(c), 3): $a \in P_1 \cup P_2$, $b \in P_1 \cup P_2$.

Henceforth, we have the inclusion

$$a \in P_1 \cup P_2 \text{ or } b \in P_1 \cup P_2.$$

Thus, $P_1 \cup P_2$ is a prime ideal$\cdots\cdots (U)$.
Next, we appoint that $P_1 \cup P_2 \cup \cdots\cdots \cup P_{n-1}$ is a prime ideal.
Then, the union

$$(P_1 \cup P_2 \cup \cdots\cdots\cdots\cdots \cup P_{n-1}) \cup P_n = (P_1 \cup P_2 \cup \cdots\cdots\cdots\cdots \cup P_n) \cdots\cdots (V)$$

is also prime ideal from (U).
Now it is necessary to shows that if P_i is a prime ideal of R then, there comes $P_i = \sqrt{P_i}$.
Set $a \in P_i (i = 1, 2, \cdots\cdots, n)$ then, we have that $a = a^1 \in P_i$ and thus $P_i \subseteq \sqrt{P_i}$.
Conversely, let $a \in \sqrt{P_i}$ then, we attain that $a^n \in P_i$ for some positive integer n.
Therefore, we obtain that $a^n = aa^{n-1} \in P_i$, hence it follows that

$$a \in P_i \text{ or } a^{n-1} \in P_i.$$

Thus, from $a^{n-1} \in P_i$ we know that $a^{n-1} = aa^{n-2} \in P_i$.
And hence it implies that $a \in P_i$ or $a^{n-2} \in P_i$.
Proceeding this gives that $a \in P_i$.
Accordingly, we acquire that $\sqrt{P_i} \subseteq P_i$.
Whence, $\bigcup_{i=1}^{n} \sqrt{P_i}$ is a prime ideal of R by (V).
Thereby, from [lemma 2.2(2), 3], there comes that

$$\bigcup_{i=1}^{n} \sqrt{P_i} : \text{strongly irreducible.}$$

Next, from [lemma 2.2(8), 3], $(\bigcup_{i=1}^{n} \sqrt{P_i}) / I \cdots\cdots (UU)$ is strongly irreducible in

R/I where each $P_i (i = 1, 2, \cdots\cdots, n)$ contains ideal I.
In the next step, we show that $(\bigcup_{i=1}^{n} \sqrt{P_i})\, /I = \bigcup_{i=1}^{n}(\sqrt{P_i}/I)$.
Take $x + I \in (\bigcup_{i=1}^{n} \sqrt{P_i})\, /I$ then, it derives that $x \in \bigcup_{i=1}^{n} \sqrt{P_i}$.
And so it induces that

$$x \in \sqrt{P_1} \text{ or } x \in \sqrt{P_2} \text{ or } \cdots\cdots\cdots\cdots \text{ or } x \in \sqrt{P_n}.$$

Whence, we attain that

$$x + I \in \sqrt{P_1}/\mathrm{I} \text{ or } x + I \in \sqrt{P_2}/I \text{ or } \cdots\cdots\cdots\cdots \text{ or } x + I \in \sqrt{P_n}/I.$$

Hence, it gives that $x + I \in \bigcup_{i=1}^{n}(\sqrt{P_i}/I)$.
So we face the inclusion $(\bigcup_{i=1}^{n} \sqrt{P_i})\, /I \subseteq \bigcup_{i=1}^{n}(\sqrt{P_i}/I)$.
Conversely, we suppose that $x + I \in \bigcup_{i=1}^{n}(\sqrt{P_i}/I)$.
Then, it follows that

$$x + I \in \sqrt{P_1}/I \text{ or } x + I \in \sqrt{P_2}/I \text{ or } \cdots\cdots\cdots\cdots \text{ or } x + I \in \sqrt{P_n}/I.$$

On this account, we get that

$$x \in \sqrt{P_1} \text{ or } x \in \sqrt{P_2} \text{ or } \cdots\cdots\cdots\cdots \text{ or } x \in \sqrt{P_n}.$$

For this reason, it induces that $x \in \bigcup_{i=1}^{n} \sqrt{P_i}$.
Wherefore, it shows that $x + I \in (\bigcup_{i=1}^{n} \sqrt{P_i})/I$.
Consequentially, it induces that

$$\bigcup_{i=1}^{n}(\sqrt{P_i}/I) \subseteq (\bigcup_{i=1}^{n} \sqrt{P_i})\, /I.$$

Finally, we conclude that $\bigcup_{i=1}^{n}(\sqrt{P_i}/I)$ is strongly irreducible from (UU).
(2). First we show that $I_1 \cup I_2$ is strongly irreducible.
Let $A \cap B \subset I_1 \cup I_2$ then, it comes that

$$A \cap B \subset I_1 \text{ or } A \cap B \subset I_2.$$

Since both I_1 and I_2 are strongly irreducible we gain

$$A \subset I_1 \text{ or } B \subset I_1 \text{ or } A \subset I_2 \text{ or } B \subset I_2.$$

Whence, it were given that

$$A \subset I_1 \cup I_2 \text{ or } B \subset I_1 \cup I_2.$$

So it implies that $I_1 \cup I_2$ is strongly irreducible$\cdots\cdots\cdots(*)$.
Next, set $\bigcup_{i=1}^{n-1} I_i$ is strongly irreducible then, we have that $(\bigcup_{i=1}^{n-1} I_i) \cup I_n$ is strongly irreducible due to $(*)$.
Wherefore, $\bigcup_{i=1}^{n} I_i$ is strongly irreducible.
Now from [lemma 2.2(1), 3] $\bigcup_{i=1}^{n} I_i$ is irreducible.
(3). Since I is prime ideal we get that $I = \sqrt{I}$.
Let $A \cap B \subseteq \sqrt{I}(= I)$ then, we gain $A \subseteq I$ or $B \subseteq I$ it is because by [lemma 2.2(2), 3] I is strongly irreducible.
Thereby, it is gotten that $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$.
Wherefore, $\sqrt{I}$ is strongly irreducible.
(4). Ideal I is irreducible iff if $A \cap B = I$ then, $A = I$ or $B = I$.
And $A = I$ or $B = I$ implies that $A \subseteq I$ or $B \subseteq I$.
Let $A \cap B \subseteq I$ but this doesn't assure that $A \cap B = I$.
Henceforth, we cannot conclude that I is strongly irreducible.
(5). Assume that $A \cap B \subseteq I_1 \cap I_2$ then, we gain

$$A \cap B \subseteq I_1 \text{ and } A \cap B \subseteq I_2.$$

Since I_1 and I_2 are strongly irreducible we take that

$$A \subseteq I_1 \text{ or } B \subseteq I_1 \text{ and } A \subseteq I_2 \text{ or } B \subseteq I_2.$$

Now we face following cases:

1). $A \subseteq I_1, B \nsubseteq I_1$, 2). $A \nsubseteq I_1, B \subseteq I_1$, 3). $A \subseteq I_1, B \subseteq I_1$.

(a). $A \subseteq I_2, B \nsubseteq I_2$, (b). $A \nsubseteq I_2, B \subseteq I_2$, (c). $A \subseteq I_2, B \subseteq I_2$.

Cooperating each case in the above shows that

1), (a): $A \subseteq I_1 \cap I_2, B \nsubseteq I_1 \cap I_2$.

1), (b): $A \subseteq I_1, B \subseteq I_2$.

1), (c): $A \subseteq I_1 \cap I_2, B \subseteq I_2$.

2), (a): $A \subseteq I_2, B \subseteq I_1$.

2), (b): $A \nsubseteq I_1 \cap I_2, B \subseteq I_1 \cap I_2$.

2), (c): $A \subseteq I_2, B \subseteq I_1 \cap I_2$.

3), (a): $A \subseteq I_1 \cap I_2, B \subseteq I_1$.

3), (b): $A \subseteq I_1, B \subseteq I_1 \cap I_2$.

3), (c): $A \subseteq I_1 \cap I_2, B \subseteq I_1 \cap I_2$.

From above result, due to the case 1), (b) and 2), (a) we cannot say that $I_1 \cap I_2$ is strongly irreducible.

(6). Take P_1 and P_2 are prime ideals of R and set $ab \in P_1 \cap P_2$ then, we have $ab \in P_1$ and $ab \in P_2$.

It gives that $a \in P_1$ or $b \in P_1$ and $a \in P_2$ or $b \in P_2$ since P_1 and P_2 are prime ideals.

Then, there comes following cases:

1). $a \in P_1, b \notin P_1$, 2). $a \notin P_1, b \in P_1$, 3). $a \in P_1, b \in P_1$ and

a). $a \in P_2, b \notin P_2$, b). $a \notin P_2, b \in P_2$, c). $a \in P_2, b \in P_2$.

Now cooperating each case in above deduces that

1), (a): $a \in P_1 \cap P_2, b \notin P_1 \cap P_2$.

1), (b): $a \in P_1, b \in P_2$.

1), (c): $a \in P_1 \cap P_2, b \in P_2$.

2), (a): $a \in P_2, b \in P_1$.

2), (b): $a \notin P_1 \cap P_2, b \in P_1 \cap P_2$.

2), (c): $a \in P_2, b \in P_1 \cap P_2$.

3), (a): $a \in P_1 \cap P_2, b \in P_1$.

3), (b): $a \in P_1, b \in P_1 \cap P_2$.

3), (c): $a \in P_1 \cap P_2, b \in P_1 \cap P_2$.

In the above due to cases of 1), (b) and 2), (a) we cannot conclude that $a \in P_1 \cap P_2$ or $b \in P_1 \cap P_2$.

Consequentially, even though P_1 and P_2 are prime ideals of R we cannot assert that $P_1 \cap P_2$ is prime ideal. □

In above (1) and (2) we showed that union of strongly irreducible ideal is also strongly irreducible. In (2) we showed it by original method and in (1) we used prime ideal.

Remark 2.2. Generally, if I strongly irreducible then, we cannot conclude that $\sqrt{I}$ is strongly irreducible but in (3) I was assumed as a prime ideal, thus we can obtain that $\sqrt{I}$ is strongly irreducible.

Remark 2.3. In [4], the authors defined that ideal I of R is 2-ireducible if whenever $I = J \cap K \cap L$ for ideals I, J and K of R then, either $I = J \cap K$ or $I = J \cap L$ or $I = K \cap L$ and also it is 2-strongly irreducible if for each ideals J, K and L of R, $J \cap K \cap L \subseteq I$ implies that $J \cap K \subseteq I$ or $J \cap L \subseteq I$ or $K \cap L \subseteq I$ and treated consequences correlated to these definitions.

Remark 2.4. It is well-known that if R is Artinian ring then, it is Noetherian ring but its converse doesn't hold. In [7.7 Example, 6] Sharp showed its example as Z. It is PID thus, Z is Noetherian but due to strictly descending chain of ideals of Z as$2Z \supset 2^2Z \supset \cdots\cdots \supset 2^iZ \supset 2^{i+1}Z \supset \cdots\cdots$we cannot conclude that Z is Artinian ([7.7 Example, 6]).

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