

Moore-Penrose's Pseudoinverse and the Cramer's Rule

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Abstract

As a different alternative to the well established theory for solving systems of rectangular equations, where the inverse matrix given by *Moore – Penrose* is used, in this work we employ the generalized cross product in a peculiar way in order to construct complete systems of equations $n \times n$ from systems of linear equations of $(n - 1) \times n$. In these new systems $n \times n$ we apply the *Cramer* rule, and furthermore

we are able to show that these solutions correspond exactly to the same ones obtained when the *Moore – Penrose* pseudo-inverse matrix is applied. Illustrative examples are given and the conclusions of the work are presented at the end.

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1 Introduction

Systems of linear equations have been solved by various methods, in fact one of the most direct and elegant methods is *Cramer’s Rule*, developed in 1750, by *Gabriel Cramer* (1704-1752) and it was published in his *Introduction á l’analyse des lignes courbes algébriques*, however it should be noted that *Colin Maclaurin* had already published it in his *Treatise of Geometry* of 1748, and it is said that *Maclaurin had already known this result since 1729*.

Formal solutions to systems of non-quadratic linear equations appeared later with the mathematician *Karl Frederick Gauss* (1777-1855); since he proposed a systematic mechanism to solve the most general systems, which include $m \times n$ with $m < n$; *Gauss* was the first to say that this type of systems admit infinite solutions or that otherwise there is no solution to such a system, but not unique solutions, which was the traditional way.

Subsequently to the *Cramer Rule* appeared other alternatives to give solution to the systems of quadratic equations $n \times n$, for example, through the calculation of the inverse matrix; since once calculated, it is relatively easy to find the solutions of the system. However, this method of calculating the inverse matrix could not be applied to matrices of the form $m \times n$ with $m < n$, this made no sense, due to the very nature of the system.

It was not until 1920 when *E.H. Moore* [1], proposed a method to find the inverse for systems of equations $m \times n$ for both $m < n$ and $m > n$, in fact it is called the *Pseudo – Inverse* matrix, later in 1951 *Bjerhammar* [2], as well as in 1955 [3] *Roger Penrose*, obtained very similar results, although it is important to say that the term *Pseudoinverse* of a matrix had already been mentioned before by *Fredholm*, who introduced the concept of the pseudoinverse of the integral operator in 1903.

There are other works where solutions to rectangular equations are found, in which projection methods are applied, where iterative methods are proposed to be able to solve indeterminate or rectangular systems of equations [4], [5]. In this work, starting from the generalized cross product [6], [7], [8], [9], [10], we define it in such a way to be able to apply it to rectangular systems of equations $(n - 1) \times n$, and achieve the construction of new systems of equations, where the rectangular system is completed to quadratics. This new system of equations has the solutions of applying Cramer's rule, which correspond to the same solutions of the original system. The interesting thing is that these solutions also correspond exactly to those obtained by applying the *Pseudo - Inverse* matrix method of *Moore - Penrose* to solve this rectangular system.

The rigorous demonstration of these equivalences on how to achieve exactly the same solutions, between our technique and that of the *Pseudo - Inverse*, we only do it for rectangular systems of 2×3 through a theorem, and the reason consists in that only when considering this simple case, the calculations expand in fact for the case of the system of 3×4 where it is so dispersed, that it does not make sense to capture it in this work. However, the generalization is very clearly fulfilled, and we present a program in *MATLAB* where, through an illustrative example, we give the evidence that our predictions are fulfilled.

2 Background

In order to apply the method of the *Pseudo - Inverse* and therefore to solve systems of non-quadratic equations, there is a way where it is summarized how to apply this mechanism, as it was done in [5] where other results were obtained; and it will be described as follows.

We define the *Moore - Penrose pseudoinverse* matrix as follows

Let $A \in M_{m \times n}(\mathbb{R})$, such that there exists a matrix $R \in M_{m \times n}(\mathbb{R})$, where is true that

$$A \cdot R = I_n$$

we can prove that $R = A^t(AA^t)^{-1}$, two conditions must be fulfilled

- 1.- $range(A) = m$
- 2.- and that $(AA^t)^{-1}$ exists

Then it is said that R , corresponds to the matrix *Pseudoinverse* by the

Right.

If there exists a matrix $L \in M_{m \times n}(\mathbb{R})$, which satisfies that

$$L \cdot A = I_n$$

in the same way we can prove that $L = (A^t A)^{-1} A^t$, the next must be fulfilled

- 1.- $\text{range}(A) = n$
- 2.- and that $(A^t A)^{-1}$ exists

Then it is said that L , corresponds to the matrix *Pseudoinverse* by the **Left**.

Now, let's solve the following system of equations of 3×4 .

$$\begin{aligned} x - y + 2w &= 2 \\ 2x + y + z - w &= -3 \\ 2x + y - z &= 1 \end{aligned} \tag{1}$$

To solve this system of equations through the pseudoinverse matrix of *Moore – Penrose*, we must calculate the pseudoinverse associated with the system of equations, i.e.

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & -1 \\ 2 & 1 & -1 & 0 \end{pmatrix}$$

then according to the theory, as $\text{range}(A) = 3$, then the Pseudoinverse is ensured on the right hand side and we compute

$$\begin{aligned} R &= \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & -1 \\ 2 & 1 & -1 & 0 \end{bmatrix}^t \left(\begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & -1 \\ 2 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & -1 \\ 2 & 1 & -1 & 0 \end{bmatrix}^t \right)^{-1} = \\ &= \begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & -1 \\ 2 & 1 & -1 & 0 \end{pmatrix}^t \begin{pmatrix} 6 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & 4 & 6 \end{pmatrix}^{-1} \\ R &= \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{26}{135} & \frac{2}{27} & -\frac{11}{135} \\ \frac{2}{27} & \frac{2}{27} & -\frac{5}{27} \\ -\frac{11}{135} & -\frac{5}{27} & \frac{41}{135} \end{pmatrix} = \begin{pmatrix} \frac{8}{45} & \frac{2}{9} & \frac{7}{45} \\ -\frac{1}{5} & 0 & \frac{1}{5} \\ \frac{7}{45} & \frac{4}{9} & -\frac{22}{45} \\ \frac{14}{45} & -\frac{1}{9} & \frac{1}{45} \end{pmatrix} \end{aligned}$$

i.e. the pseudoinverse matrix, is given by

$$R = \begin{pmatrix} \frac{8}{45} & \frac{2}{9} & \frac{7}{45} \\ -\frac{1}{5} & 0 & \frac{1}{5} \\ \frac{7}{45} & \frac{4}{9} & -\frac{22}{45} \\ \frac{14}{45} & -\frac{1}{9} & \frac{1}{45} \end{pmatrix} \quad (2)$$

Now the system of equations (2.1) can be expressed as

$$\begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & -1 \\ 2 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \Rightarrow AX = b \quad (3)$$

Now, after applying the least squares criterion, the approximation can be carried out as follows

$$X^* \approx Rb \quad (4)$$

with R , given by (2.2), then it follows that

$$X^* \approx \begin{pmatrix} \frac{8}{45} & \frac{2}{9} & \frac{7}{45} \\ -\frac{1}{5} & 0 & \frac{1}{5} \\ \frac{7}{45} & \frac{4}{9} & -\frac{22}{45} \\ \frac{14}{45} & -\frac{1}{9} & \frac{1}{45} \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{45} \\ -\frac{1}{5} \\ -\frac{68}{45} \\ \frac{44}{45} \end{pmatrix} \quad (5)$$

i.e. $X^* \approx \begin{pmatrix} -\frac{7}{45} \\ -\frac{1}{5} \\ -\frac{68}{45} \\ \frac{44}{45} \end{pmatrix}$ correspond to the solutions of the system of equations (2.1)

3 The cross product in R^n

3.1 The cross product in R^3

To clearly understand how to achieve the generalization of the cross product, we will perform the construction of this for the best known case as it was done in [11], and it is precisely for the case of R^3 .

Let be the vectors R^3 : $\vec{A} = (a_1, a_2, a_3)$, and $\vec{B} = (b_1, b_2, b_3)$.

Now, if we define the following expression

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} (1, 0, 0) & (0, 1, 0) & (0, 0, 1) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$\begin{aligned}
&= \left(\begin{vmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, 0, 0 \right) + \left(0, \begin{vmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, 0 \right) + \left(0, 0, \begin{vmatrix} 0 & 0 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \right) = \\
&= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, 0, 0 \right) + \left(0, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, 0 \right) + \left(0, 0, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) = \\
&= (1, 0, 0) \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - (0, 1, 0) \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + (0, 0, 1) \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \\
&= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{aligned}$$

i.e. we have that

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad (6)$$

It is curious to note that this last expression (2.1) is rarely justified in literature books.

Well, the vector $\vec{C}$, as defined, allows us to assure that it will be a perpendicular vector to the vectors $\vec{A}$ and $\vec{B}$, as well as to any linear combination of these vectors. In other words this vector $\vec{C}$ would be perpendicular to the whole plane that contains the vectors $\vec{A}$ and $\vec{B}$; which means that if we make the dot product to the linear combination of $\vec{A}$ and $\vec{B}$ it would give as result $\vec{0}$, this can be easily seen in the following way

$$\begin{aligned}
&\vec{C} \cdot (\alpha \vec{A} + \beta \vec{B}) = \alpha \vec{C} \cdot \vec{A} + \beta \vec{C} \cdot \vec{B} \\
&\alpha \left(\hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot (\hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3) + \\
&+ \beta \left(\hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot (\hat{i}b_1 + \hat{j}b_2 + \hat{k}b_3) = \\
&= \alpha \left(a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) + \\
&+ \beta \left(b_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \\
&= \alpha \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \beta \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \alpha(0) + \beta(0) = 0
\end{aligned}$$

and this is due to the properties of the determinants, as lines are repeated for both determinants; in other words, it follows that

$$\vec{C} \cdot (\alpha \vec{A} + \beta \vec{B}) = 0$$

i.e. the vector $\vec{C}$, is perpendicular to the entire plane containing the vectors $\vec{A}$ and $\vec{B}$.

3.2 Generalized cross product for R^n

As can be seen, expanding these results and thus being able to generalize it to R^n is something we can do quite naturally, see [11], since basically the developed algorithm can be extended without any problem for n -dimensions. Then, for the case of R^n , the vector $\vec{W}$ to consider will be now, one that is constructed from $n - 1$ vectors, linearly independent $\vec{A}_1, \vec{A}_2, \vec{A}_3, \dots, \vec{A}_{n-1}$, that is to say that from the next structure

$$\vec{W} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \dots & \hat{e}_j & \dots & \hat{e}_n \\ a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1,n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,j} & \dots & a_{n-1,n} \end{vmatrix} = \sum_{k=1}^{n-1} \hat{e}_k (-1)^{1+k} |W_k|$$

where each $|W_k|$ corresponds to the determinant of the minor matrix associated with each value of $\hat{e}_k$, as developed for the cases of R^3 and R^4 ; and of course that the vector $\vec{W}$ will be a perpendicular vector to each of the vectors $\vec{A}_1, \vec{A}_2, \vec{A}_3, \dots, \vec{A}_{n-1}$, as well as to any linear combination of these, i.e. if

$$\vec{v} = \alpha_1 \vec{A}_1 + \alpha_2 \vec{A}_2 + \dots + \alpha_{n-1} \vec{A}_{n-1} = \sum_{k=1}^{n-1} \alpha_k \vec{A}_k$$

therefore, it follows that

$$\vec{W} \cdot \sum_{k=1}^{n-1} \alpha_k \vec{A}_k = 0$$

this is clear since

$$\begin{aligned} \vec{W} \cdot \sum_{k=1}^{n-1} \alpha_k \vec{A}_k &= \sum_{k=1}^{n-1} \alpha_k \vec{W} \cdot \vec{A}_k = \sum_{k=1}^{n-1} \sum_{j=1}^n \alpha_k \hat{e}_j (-1)^{1+j} |W_k| \cdot \vec{A}_k = \sum_{k=1}^{n-1} \sum_{j=1}^n (\alpha_k) a_{kj} (-1)^{1+j} |M_k| = \\ &= \alpha_1 a_{11} |M_1| - \alpha_2 a_{12} |M_2| + \alpha_3 a_{13} |M_3| - \dots - \alpha_{n-1} a_{1,n-1} (-1)^{1+n-1} |M_{n-1}| = \end{aligned}$$

$$= \alpha_1 a_{11}(0) - \alpha_2 a_{12}(0) + \alpha_3 a_{13}(0) + \dots + \alpha_n a_{1,n}(-1)^{1+k}(0) = 0$$

where M_k corresponds to a matrix of size $n-1 \times n-1$, in which some row is repeated for each value of k in the development, according to what we showed for the cases of R^3 and R^4 ; and therefore the determinants $|M_k|$ are annulled according to the properties of the determinants, so then

$$\vec{W} \cdot \sum_{k=1}^{n-1} \alpha_k \vec{A}_k = 0$$

i.e. the vector

$$\vec{W} = \sum_{k=1}^{n-1} \hat{e}_k |\vec{W}_k|$$

will be perpendicular to any of the set $\vec{A}_1, \vec{A}_2, \vec{A}_3, \dots, \vec{A}_{n-1}$, as to the space generated (or hyperplane) by this set of vectors.

4 Rectangular Equations Systems

4.1 Let's analyze the case of 2×3 systems

Theorem

Let be the next equations system

$$\begin{aligned} a_1 x + a_2 y + a_3 z &= k_1 \\ b_1 x + b_2 y + b_3 z &= k_2 \end{aligned} \tag{7}$$

then the solutions to such a system, according to the expression

$$x^* \approx \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \tag{8}$$

where

$$R = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}^t \left[\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}^t \right]^{-1} \tag{9}$$

is the *Pseudoinverse Moore – Penrose* matrix.

In addition to these solutions, these can be obtained in terms of the cramer's rule, when applied to a new system of equations

$$\begin{array}{rcl} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} x & - & \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} y & + & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} z & = & 0 \\ a_1 x & & + a_2 y & & + a_3 z & = & k_1 \\ b_1 x & & + b_2 y & & + b_3 z & = & k_2 \end{array} \quad (10)$$

By directly applying Cramer's rule

$$x = \frac{\begin{vmatrix} 0 & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ k_1 & a_2 & a_3 \\ k_2 & b_2 & b_3 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}, y = \frac{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & 0 & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ a_1 & k_1 & a_3 \\ b_1 & k_2 & b_3 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}, z = \frac{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & 0 \\ a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

PROOF

from (2.3)

$$R = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}^t \left[\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}^t \right]^{-1} \quad (11)$$

$$R = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}^t \begin{pmatrix} a_1^2 + a_2^2 + a_3^2 & a_1 b_1 + a_2 b_2 + a_3 b_3 \\ a_1 b_1 + a_2 b_2 + a_3 b_3 & b_1^2 + b_2^2 + b_3^2 \end{pmatrix}^{-1} \quad (12)$$

i.e.

$$R = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}^t \begin{pmatrix} \frac{b_1^2 + b_2^2 + b_3^2}{-(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} & \frac{-a_1 b_1 - a_2 b_2 - a_3 b_3}{-(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 + (a_1^2 + a_2^2 + a_3^2)} \\ \frac{-a_1 b_1 - a_2 b_2 - a_3 b_3}{-(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} & \frac{a_1^2 + a_2^2 + a_3^2}{-(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 + (a_1^2 + a_2^2 + a_3^2)} \end{pmatrix} \quad (13)$$

then

$$R = \begin{pmatrix} \frac{b_1(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{a_1(b_1^2+b_2^2+b_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}, \\ \frac{b_1(a_1^2+a_2^2+a_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{a_1(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}, \\ \\ \frac{b_2(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{a_2(b_1^2+b_2^2+b_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}, \\ \frac{b_2(a_1^2+a_2^2+a_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{a_2(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}, \\ \\ \frac{b_3(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{a_3(b_1^2+b_2^2+b_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}, \\ \frac{b_3(a_1^2+a_2^2+a_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{a_3(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} \end{pmatrix}$$

and according to the expression (4.2)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{k_1b_1(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{k_1a_1(b_1^2+b_2^2+b_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \\ \frac{k_2b_1(a_1^2+a_2^2+a_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{k_2a_1(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} \\ \\ \frac{k_1b_2(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{k_1a_2(b_1^2+b_2^2+b_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \\ \frac{k_2b_2(a_1^2+a_2^2+a_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{k_2a_2(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} \\ \\ \frac{k_1b_3(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{k_1a_3(b_1^2+b_2^2+b_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \\ \frac{k_2b_3(a_1^2+a_2^2+a_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \frac{k_2a_3(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} \end{pmatrix} \quad (14)$$

from which the solutions are immediately derived.

For x we have that

$$\begin{aligned} x = & \frac{k_1b_1(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \\ & + \frac{k_1a_1(b_1^2+b_2^2+b_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \\ & + \frac{k_2b_1(a_1^2+a_2^2+a_3^2)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} + \\ & + \frac{k_2a_1(-a_1b_1-a_2b_2-a_3b_3)}{-(a_1b_1+a_2b_2+a_3b_3)^2+(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} \end{aligned}$$

after some simplification it is found that for x

$$x = \frac{-a_2b_1b_2k_1 - a_3b_2^2k_1 - a_3b_1b_3k_1 + a_1b_3^2k_1 + a_2^2b_1k_2 + a_3^2b_1k_2 - a_1a_2b_2k_2 - a_1a_3b_2k_2}{a_3^2(b_1^2 + b_2^2) - 2a_1a_3b_1b_3 - 2a_2b_2(a_1b_1 + a_3b_3) + a_2^2(b_1^2 + b_3^2) + a_1^2(b_2^2 + b_3^2)} \quad (15)$$

in the same way we compute for y

$$\begin{aligned} y = & \frac{k_1b_2(-a_1b_1 - a_2b_2 - a_3b_3)}{-(a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} + \\ & \frac{k_1a_2(b_1^2 + b_2^2 + b_3^2)}{-(a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} + \\ & + \frac{k_2b_2(a_1^2 + a_2^2 + a_3^2)}{-(a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} + \\ & \frac{k_2a_2(-a_1b_1 - a_2b_2 - a_3b_3)}{-(a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} \end{aligned}$$

in the same way when simplifying we obtain for y

$$y = \frac{a_2b_1^2k_1 - a_1b_1b_2k_1 - a_3b_2b_3k_1 + a_2b_3^2k_1 - a_1a_2b_1k_2 + a_1^2b_2k_2 + a_3^2b_2k_2 - a_2a_3b_3k_2}{a_3^2(b_1^2 + b_2^2) - 2a_1a_3b_1b_3 - 2a_2b_2(a_1b_1 + a_3b_3) + a_2^2(b_1^2 + b_3^2) + a_1^2(b_2^2 + b_3^2)} \quad (16)$$

and for z

$$\begin{aligned} z = & \frac{k_1b_3(-a_1b_1 - a_2b_2 - a_3b_3)}{-(a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} + \\ & \frac{k_1a_3(b_1^2 + b_2^2 + b_3^2)}{-(a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} + \\ & + \frac{k_2b_3(a_1^2 + a_2^2 + a_3^2)}{-(a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} + \\ & \frac{k_2a_3(-a_1b_1 - a_2b_2 - a_3b_3)}{-(a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} \end{aligned}$$

simplifying z

$$z = \frac{-a_3b_1^2k_1 + a_3b_2^2k_1 + a_1b_1b_3k_1 - a_2b_2b_3k_1 + a_1a_3b_1k_2 - a_2a_3b_2k_2 - a_1^2b_3k_2 + a_2^2b_3k_2}{a_3^2(b_1^2 + b_2^2) - 2a_1a_3b_1b_3 - 2a_2b_2(a_1b_1 + a_3b_3) + a_2^2(b_1^2 + b_3^2) + a_1^2(b_2^2 + b_3^2)} \quad (17)$$

On the other hand, applying the *Cramer's rule* directly to the expression (4.4), it is obtained that

$$x = \frac{\begin{vmatrix} 0 & - & \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ k_1 & & a_2 & a_3 \\ k_2 & & b_2 & b_3 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ a_1 & & a_2 & a_3 \\ b_1 & & b_2 & b_3 \end{vmatrix}} =$$

$$= \frac{-a_2b_1b_2k_1 - a_3b_2^2k_1 - a_3b_1b_3k_1 + a_1b_3^2k_1 + a_2^2b_1k_2 + a_3^2b_1k_2 - a_1a_2b_2k_2 - a_1a_3b_2k_2}{a_3^2(b_1^2 + b_2^2) - 2a_1a_3b_1b_3 - 2a_2b_2(a_1b_1 + a_3b_3) + a_2^2(b_1^2 + b_3^2) + a_1^2(b_2^2 + b_3^2)} \quad (18)$$

$$y = \frac{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & 0 & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ a_1 & k_1 & a_3 \\ b_1 & k_2 & b_3 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ a_1 & & a_2 & a_3 \\ b_1 & & b_2 & b_3 \end{vmatrix}} =$$

$$= \frac{a_2b_1^2k_1 - a_1b_1b_2k_1 - a_3b_2b_3k_1 + a_2b_3^2k_1 - a_1a_2b_1k_2 + a_1^2b_2k_2 + a_3^2b_2k_2 - a_2a_3b_3k_2}{a_3^2(b_1^2 + b_2^2) - 2a_1a_3b_1b_3 - 2a_2b_2(a_1b_1 + a_3b_3) + a_2^2(b_1^2 + b_3^2) + a_1^2(b_2^2 + b_3^2)} \quad (19)$$

$$\begin{aligned}
z &= \frac{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & 0 \\ a_1 & & a_2 & k_1 \\ b_1 & & b_2 & k_2 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ a_1 & & a_2 & a_3 \\ b_1 & & b_2 & b_3 \end{vmatrix}} = \\
&= \frac{-a_3b_1^2k_1 + a_3b_2^2k_1 + a_1b_1b_3k_1 - a_2b_2b_3k_1 + a_1a_3b_1k_2 - a_2a_3b_2k_2 - a_1^2b_3k_2 + a_2^2b_3k_2}{a_3^2(b_1^2 + b_2^2) - 2a_1a_3b_1b_3 - 2a_2b_2(a_1b_1 + a_3b_3) + a_2^2(b_1^2 + b_3^2) + a_1^2(b_2^2 + b_3^2)} \quad (20)
\end{aligned}$$

then by comparing equations (9), (10) and (11) with (12), (13) and (14), it is clear that the following proof is satisfied

It is important to make some observations in this regard:

1.- The *first* equation of the system (4.4) was obtained by applying the cross product (which corresponds to the simplest case of generalized cross product) to the first two vectors of the system of equations $((a_1, a_2, a_3), (b_1, b_2, b_3))$.

2.- Geometrically, the vector is perpendicular to the plane that contains the vectors $((a_1, a_2, a_3), (b_1, b_2, b_3))$, and therefore if these vectors are linearly independent a unique solution is assured. (See figure 1).

3.- Moreover, an infinite number of solutions to the system given by (4.4) can be constructed, which must correspond to what is reported in the traditional theory when applying the *Gauus – Jordan* method to (4.1), see figure 2 (the one that mentions the free parameters), and these solutions can be obtained by adding the parameters α, β to the Cramer's rule solutions given above, as shown below

$$\begin{aligned}
x &= \frac{\begin{vmatrix} 0 & -\beta \\ k_1 & a_2 \\ k_2 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}}{\begin{vmatrix} \alpha & a_2 & a_3 \\ b_2 & b_3 & -\beta \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}, y = \frac{\begin{vmatrix} \alpha & a_2 & a_3 \\ b_2 & b_3 & 0 \\ a_1 & k_1 & a_3 \\ b_1 & k_2 & b_3 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}{\begin{vmatrix} \alpha & a_2 & a_3 \\ b_2 & b_3 & -\beta \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}, \\
z &= \frac{\begin{vmatrix} \alpha & a_2 & a_3 \\ b_2 & b_3 & -\beta \\ a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \end{vmatrix}}{\begin{vmatrix} \alpha & a_2 & a_3 \\ b_2 & b_3 & -\beta \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}
\end{aligned}$$

4.- It is clear that even for this case of the solution to the system of 2×3 it is easier to handle the system of equations extended to 3×3 , that is, to make use of the cross product to add the missing equation, and solve it later by applying the *Cramer's rule* than going by the method of the *Moore – Penrose's Pseudoinverse*.

On the other hand, the demonstration of the following case, that is the system of 3×4 , although basically it is the same mechanism to follow that for the case of 2×3 , the necessary calculations to justify its development grow very considerably, reason why it is not included in this article.

4.2 The case of the 3×4 system

The results obtained in the previous section, can be extended without any problem to systems of linear equations of higher dimensions of the form $(n - 1) \times n$, for example systems of equations of 3×4 . It should be noted that for this case we will not carry out the demonstration as mentioned in the previous section, because although the calculations follow the same pattern of behavior, they are too extensive.

Consider the system of equations

$$\begin{array}{cccccc} a_1x & +a_2y & +a_3z & +a_4w & = & k_1 \\ b_1x & +b_2y & +b_3z & +b_4w & = & k_2 \\ c_1x & +c_2y & +c_3z & +c_4w & = & k_3 \end{array} \quad (21)$$

then the solutions to the system according to the *Moore – Penrose* Pseudoinverse method would be given by

$$x* \approx \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = R \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

where

$$R = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}^t \left[\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}^t \right]^{-1}$$

is the *Moore – Penrose's Pseudoinverse* matrix.

Now, these same solutions can be obtained in terms of the *Cramer's* rule, when applied to a new system of equations where it is emphasized that the first equation is obtained by applying the generalized cross product to the vectors

$$(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)$$

so that by construction it is perpendicular to these vectors and to all the space they generate, that is to say, to all R^3 .

The new system of equations of 4×4 is given as follows:

$$\begin{array}{ccccccccc} \left| \begin{array}{ccc} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{array} \right| x & - & \left| \begin{array}{ccc} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{array} \right| y & + & \left| \begin{array}{ccc} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{array} \right| z & - & \left| \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right| w & = & 0 \\ a_1x & & +a_2y & & +a_3z & & +a_4w & & = k_1 \\ b_1x & & +b_2y & & +b_3z & & +b_4w & & = k_2 \\ c_1x & & +c_2y & & +c_3z & & +c_4w & & = k_3 \end{array} \quad (22)$$

where

$$\left(\left| \begin{array}{ccc} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{array} \right|, - \left| \begin{array}{ccc} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{array} \right|, \left| \begin{array}{ccc} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{array} \right|, - \left| \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right| \right)$$

is the vector that corresponds to the generalized cross product for R^4 . Now, applying Cramer's rule to the expression (4.16), we obtain the values for the variables x, y, z, w .

$$x = \frac{\begin{vmatrix} 0 & - & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ k_1 & & a_2 & & a_3 & & a_4 \\ k_2 & & b_2 & & b_3 & & b_4 \\ k_3 & & c_2 & & c_3 & & c_4 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ a_1 & & a_2 & & a_3 & & a_4 \\ b_1 & & b_2 & & b_3 & & b_4 \\ c_1 & & c_2 & & c_3 & & c_4 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & 0 & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ a_1 & & k_1 & & a_3 & & a_4 \\ b_1 & & k_2 & & b_3 & & b_4 \\ c_1 & & k_3 & & c_3 & & c_4 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ a_1 & & a_2 & & a_3 & & a_4 \\ b_1 & & b_2 & & b_3 & & b_4 \\ c_1 & & c_2 & & c_3 & & c_4 \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & 0 & - & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ a_1 & & a_2 & & k_1 & & a_4 \\ b_1 & & b_2 & & k_2 & & b_4 \\ c_1 & & c_2 & & k_3 & & c_4 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ a_1 & & a_2 & & a_3 & & a_4 \\ b_1 & & b_2 & & b_3 & & b_4 \\ c_1 & & c_2 & & c_3 & & c_4 \end{vmatrix}}$$

$$w = \frac{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & 0 \\ a_1 & a_2 & a_3 & k_1 \\ b_1 & b_2 & b_3 & k_2 \\ c_1 & c_2 & c_3 & k_3 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} & - & \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}}$$

4.3 Example

Let us apply this result to the system of equations of 3×4 , given in equation (2.1) of section 2

$$\begin{aligned} x - y + 2w &= 2 \\ 2x + y + z - w &= -3 \\ 2x + y - z &= 1 \end{aligned}$$

i.e.

$$x = \frac{\begin{vmatrix} 0 & - & \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 2 & 1 & 0 \end{vmatrix} & - & \begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} \\ 2 & -1 & 0 & 2 \\ -3 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} \begin{vmatrix} -1 & 0 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} & - & \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 2 & -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 2 & 1 & 0 \end{vmatrix} & - & \begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} \\ 1 & 0 & 2 & 2 \\ 2 & 1 & -1 & -1 \\ 2 & 1 & 0 & 0 \end{vmatrix}} = -\frac{7}{45}$$

$$\begin{aligned}
y &= \frac{\left| \begin{array}{c|ccc|ccc|ccc|ccc} -1 & 0 & 2 & & & & 1 & -1 & 2 & & & & 1 & -1 & 0 & & & \\ & 1 & 1 & -1 & & 0 & & 2 & 1 & -1 & & - & & 2 & 1 & 1 & & & \\ & 1 & -1 & 0 & & & & 2 & 1 & 0 & & & & 2 & 1 & -1 & & & \\ & & 1 & & & 2 & & & 0 & & & & & 2 & & & & \\ & & 2 & & & -3 & & & 1 & & & & & -1 & & & & \\ & & 2 & & & 1 & & & -1 & & & & & 0 & & & & \end{array} \right|}{\left| \begin{array}{c|ccc|ccc|ccc|ccc} -1 & 0 & 2 & & & & 1 & 0 & 2 & & & & 1 & -1 & 2 & & & \\ & 1 & 1 & -1 & & - & & 2 & 1 & -1 & & & & 2 & 1 & -1 & & - & \\ & 1 & -1 & 0 & & & & 2 & -1 & 0 & & & & 2 & 1 & 0 & & - & \\ & & 1 & & & & & -1 & & & & & & 0 & & & & 2 & \\ & & 2 & & & & & 1 & & & & & & 1 & & & & -1 & \\ & & 2 & & & & & 1 & & & & & & -1 & & & & 0 & \end{array} \right|} = -\frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
z &= \frac{\left| \begin{array}{c|ccc|ccc|ccc|ccc} -1 & 0 & 2 & & & & 1 & 0 & 2 & & & & 1 & -1 & 0 & & & \\ & 1 & 1 & -1 & & - & & 2 & 1 & -1 & & 0 & - & & 2 & 1 & 1 & & \\ & 1 & -1 & 0 & & & & 2 & -1 & 0 & & & & 2 & 1 & -1 & & & \\ & & 1 & & & & & -1 & & & & & & 2 & & & & 2 & \\ & & 2 & & & & & 1 & & & & & & -3 & & & & -1 & \\ & & 2 & & & & & 1 & & & & & & 1 & & & & 0 & \end{array} \right|}{\left| \begin{array}{c|ccc|ccc|ccc|ccc} -1 & 0 & 2 & & & & 1 & 0 & 2 & & & & 1 & -1 & 2 & & & \\ & 1 & 1 & -1 & & - & & 2 & 1 & -1 & & & & 2 & 1 & -1 & & - & \\ & 1 & -1 & 0 & & & & 2 & -1 & 0 & & & & 2 & 1 & 0 & & - & \\ & & 1 & & & & & -1 & & & & & & 0 & & & & 2 & \\ & & 2 & & & & & 1 & & & & & & 1 & & & & -1 & \\ & & 2 & & & & & 1 & & & & & & -1 & & & & 0 & \end{array} \right|} = -\frac{68}{45}
\end{aligned}$$

$$\begin{aligned}
w &= \frac{\left| \begin{array}{c|ccc|ccc|ccc|ccc} -1 & 0 & 2 & & & & 1 & 0 & 2 & & & & 1 & -1 & 2 & & & \\ & 1 & 1 & -1 & & - & & 2 & 1 & -1 & & & & 2 & 1 & -1 & & & 0 & \\ & 1 & -1 & 0 & & & & 2 & -1 & 0 & & & & 2 & 1 & 0 & & & 2 & \\ & & 1 & & & & & -1 & & & & & & 0 & & & & -3 & \\ & & 2 & & & & & 1 & & & & & & 1 & & & & 1 & \\ & & 2 & & & & & 1 & & & & & & -1 & & & & 1 & \end{array} \right|}{\left| \begin{array}{c|ccc|ccc|ccc|ccc} -1 & 0 & 2 & & & & 1 & 0 & 2 & & & & 1 & -1 & 2 & & & \\ & 1 & 1 & -1 & & - & & 2 & 1 & -1 & & & & 2 & 1 & -1 & & - & \\ & 1 & -1 & 0 & & & & 2 & -1 & 0 & & & & 2 & 1 & 0 & & - & \\ & & 1 & & & & & -1 & & & & & & 0 & & & & 2 & \\ & & 2 & & & & & 1 & & & & & & 1 & & & & -1 & \\ & & 2 & & & & & 1 & & & & & & -1 & & & & 0 & \end{array} \right|} = \frac{44}{45}
\end{aligned}$$

As can be seen, these solutions are the same as those obtained by applying the *Moore – Penrose's Pseudo – inverse* matrix method in section 2 and presented in eq. (2.5). Confirming that the solutions obtained by the system “extended” $n \times n$, which was obtained from one of $(n - 1) \times n$ when applying

the generalized cross product, matches exactly with those provided by the *Moore – Penrose's Pseudo – Inverse* technique.

5 Generalization of the method

In this section it should be noted that, since the calculations are becoming more and more complex, we proceeded to perform a program in *MATLAB*, applying the algorithm that allows us to match and verify that the solutions to a system of equations of the form $(n - 1) \times n$, can be achieved by applying the *Moore – Penrose's Pseudoinverse* method, as well as the extended matrix method, which relies on the generalized cross product. [style=Matlab-editor] MATLAB code...

Overview

This document presents the comparison between solving an underdetermined system of linear equations of size $N \times (N + 1)$ using:

1. The **Moore-Penrose Pseudoinverse** ($A^\dagger = A^T(AA^T)^{-1}$).
2. An **Extended Matrix Method** by constructing an orthogonal vector to complete an $M \times M$ system (where $M = N + 1$).

MATLAB Script

```
[style=matlabstyle, language=Matlab] clearvars; clc; close all
N = 3; M = N + 1; A = randi(4, N, M); F = (1:N)'; Sistema = [A F]
R = A' / (A * A'); fprintf("solution with the pseudoinverse:"); XP = (R *
F)'
AA = zeros(1, M); for i = 1:(M - N) for k = 1:M B = A; B(:, k) = [];
AA(k) = (-1)^(k + 1) * detRR(B); end A = [AA; A]; F = [0; F]; end
fprintf("solution with the extended matrix:"); XS = (A F)' error = abs((XP
- XS) ./ XP)
```

Execution Output & Example

Below is the execution result for an example system:

[colback=white,colframe=headerblue,title=**Example Matrix System**]

$$\begin{bmatrix} 2 & 1 & 2 & 4 \\ 4 & 3 & 4 & 2 \\ 1 & 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

[style=outputstyle] solution with the pseudoinverse: XP = -0.3021 0.8489
0.0947 0.1415

solution with the extended matrix: XS = -0.3021 0.8489 0.0947 0.1415

error = 1.0e-14 * 0.0551 0 0.1173 0.0392

Example of program execution for N=2, the two methods give the same results

Original system

$$7x + 6y + 3z = 1$$

$$2x + 6y + 2z = 2$$

Solution obtained by applying the Pseudoinverse method: XP = -0.2120
0.3840 0.0600

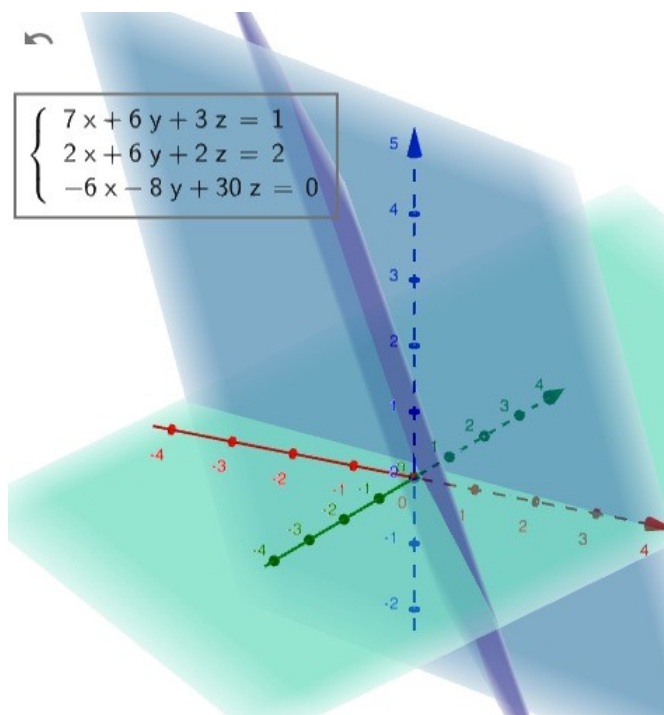
Solution by adding a plane that passes through the origin and is perpendicular to the other two

$$-6x - 8y + 30z = 0$$

$$7x + 6y + 3z = 1$$

$$2x + 6y + 2z = 2$$

XS = -0.2120 0.3840 0.0600



Graphic of the drawings associated with the system

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