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An Approximate Representation of the Value of $Sin(1^\circ)$

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Abstract

The problem of finding the exact value of $sin(1°)$ remains open to date; solving it would be a major step toward the exact representation of $Sin(\theta)$ for θ within a wide range of values. For this reason, in this work we propose an alternative representation which, although approximate, differs from the classical one given by the $Taylor$ series expansion of the value of $sin(1°)$, as well as from other techniques reported in the literature. We then present some applications and, finally, our conclusions.

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Keywords: Sine function, Approximate values

1 Introduction

The study of trigonometric functions, as well as their evaluation to obtain exact values, is a problem that dates back to the very origins of trigonometry. It goes back to the era of the *Babylonians*, and much of its development is owed to the *Greeks* [1-4]; for example, the *sine* function appears in the $Sulba - Sutras$, written in India between the seventh and sixth centuries BC. Trigonometric functions were studied extensively by great mathematicians such as $Hipparchus\ of\ Nicaea$ (180-125 BC), *Aryabhata* (456-550 BC), $Varahamihira$, $Brahmagupta$, $Omar\ Khayyam$, $Bhaskara - II$ (1464).

On the other hand, in his work $Introductio\ in\ analysin\ infinitorum$ (1748) [5], *Leonhard Euler* established the analytical treatment of trigonometric functions in Europe by expressing them as infinite series, also known as $Euler\ Formulas$.

There are works such as that of *Shailesh A.Shirali* [1], who model the behavior of the functions $Sin(\theta)$ and $Cos(\theta)$ through the use of $Polynomials$, as well as that of *Ranjita Naik* and *Riyazahammad* [2], who employ the $CORDIC\ Algorithm$ to compute values of the *Sine* and *Cosine* functions. Since the exact values of $Sin(3°)$ and $Cos(3°)$ are known in the literature, it is natural to think that the exact value of $Sin(1°)$ would also be known; however, no such result exists. That is, it cannot be obtained a priori in analytical form; in fact, it can be considered an open problem. It is worth noting that there are even entire books that address this particular problem, explaining how, over the centuries, different cultures around the world have tried to solve it [6]. What we do in this work is provide a mechanism capable of yielding an approximate value of $sin(1°)$ accurate to 12 digits. This is achieved by using

an isosceles right triangle and relying on the law of cosines; we then analyze some implications of this result and, finally, present our conclusions.

2 The exact value of $sin(3^\circ)$

In the literature, the evaluation of trigonometric functions reduces to very well-known expressions such as: $sin(\theta)$, with $\theta = \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{6}, \frac{5\pi}{12}$, and by applying angle addition and subtraction, or even solving certain trigonometric identities that yield third- or fifth-degree equations, it is possible to compute trigonometric functions at more sophisticated values such as:

$$Sin(\frac{13\pi}{32}) \quad , Sin(\frac{\pi}{12}), \quad Sin(\frac{\pi}{10}), \quad Sin(\frac{\pi}{5}), \quad Sin(\frac{2\pi}{17})$$

Now, combining $sin(\frac{\pi}{10}) = \frac{\sqrt{5}-1}{4}, sin(\frac{\pi}{12}) = \frac{-\sqrt{2}+\sqrt{6}}{4}$, which are the solutions of interest obtained by solving algebraic equations of $Third$ and $Fifth$ degree, respectively. Likewise, the values of $cos(\frac{\pi}{10}), cos(\frac{\pi}{12})$ can be found. We thus obtain one of the most important results:

$$sin(\frac{\pi}{60}) = sin(\frac{\pi}{10} - \frac{\pi}{12}) = sin(\frac{\pi}{10})cos(\frac{\pi}{12}) - sin(\frac{\pi}{12})cos(\frac{\pi}{10})$$

$$= (\frac{\sqrt{5}-1}{4})\frac{1}{4}(\sqrt{2}+\sqrt{6}) - (\frac{1}{4})(-\sqrt{2}+\sqrt{6})\frac{1}{2}(\sqrt{\frac{5+\sqrt{5}}{2}})$$

that is,

$$sin(3^\circ) = \frac{1}{16}(-2(-1+\sqrt{3})\sqrt{5+\sqrt{5}} + (-1+\sqrt{5})(\sqrt{2}+\sqrt{6})) \qquad (1)$$

Although this result is interesting, it unfortunately corresponds to the smallest value of the *sine* function (or of any other trigonometric function) for an angle that can be obtained in *Exact* form; see [7].

As an interesting fact, there is also the *Heptadecagon*; this "*Exact*" calculation was first obtained by the *Legendary* Mathematician *Carl Friedrich Gauss* in 1796 [8]

$$sin(\frac{2\pi}{17}) = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}+\sqrt{170+38\sqrt{17}}}}{16}$$

from this result, a whole family of *sines* and *cosines* can be constructed for angles of the form $\frac{k2^n\pi}{17}, k, n \in Z$

On the other hand, the construction of $sin(\frac{\pi}{180}) = sin(1^\circ)$ is impossible, since attempting to obtain it exactly in terms of the previous results cannot be done, given that they must be *Rational Multiples* of π; that is, as

$$sin(\frac{\pi}{180}) = sin(\frac{\pi}{2^2 \times 5 \times 3^3})$$

has a repeated factor of 3 in the denominator; therefore $Sin(1^\circ)$ cannot be constructed using square roots in its final expression. In fact, there is the alternative of trying to find this value by solving the trigonometric identity (which maps to a *Cubic* equation)

$$sin(3\theta) = -4sin^3(\theta) + 3sin(\theta) \tag{2}$$

setting $\theta = 1$; however, none of the roots found yields explicit solutions in terms of expressions with radicals for $sin(1^\circ)$, and these solutions are said to be irreducible.

Another way of looking at these solutions to (2) (see [7]) is

$$sin(1^\circ) = \frac{1}{2i}(\sqrt[3]{cos(3^\circ) + isin(3^\circ)} - \sqrt[3]{cos(3^\circ) - isin(3^\circ)})$$

In the next section we propose an original mechanism for obtaining a very close approximation to the value of $sin(1^\circ)$

3 Main result: Approximate representation of $sin(1^\circ)$

3.1 Construction of $sin(1^\circ)$

The objective of this work is to find an approximate representation of $Sin(1^\circ)$. From our review of the various works that address the subject, none reports an exact and real representation of this expression; that is, in the articles we have analyzed, the representations involve exact real expressions that nevertheless invariably appear accompanied by imaginary parts, in some cases simple expressions and in others more complicated ones, but in no case completely real and exact representations. It is worth noting that the concrete calculation of $Sin(1^\circ)$ is a very interesting subject, so much so that books have even been written about it, such as [6].

Now, to achieve our objective, we begin with an approximate representation of what we seek to find, starting as follows:

According to Figure 1

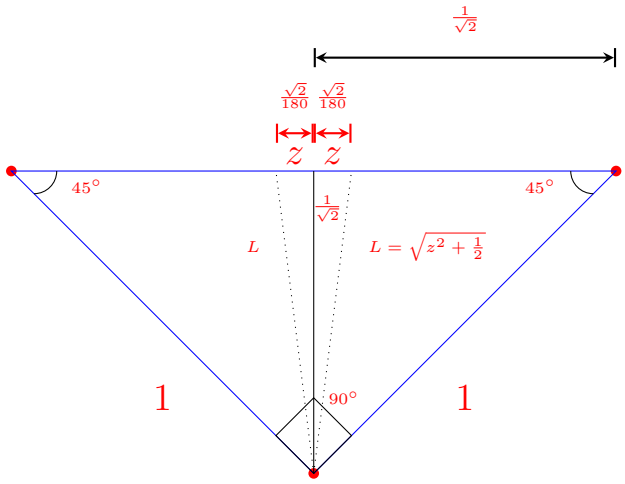


Figure 1: Here we can see how the construction is first proposed for the calculation of $Cos(1^\circ)$

$$y_1 = \sqrt{1^2 - (\frac{\sqrt{2}}{2})^2} = \frac{1}{\sqrt{2}}$$

furthermore,

$$L_1 = \sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{\sqrt{2}}{180})^2} = \sqrt{\frac{8101}{16200}}$$

We consider the division of $\sqrt{2}$ into 90 parts, since the opening of $cos(90)$ is assumed to be precisely $\sqrt{2}$; we therefore take $\frac{\sqrt{2}}{90}$ in the hope of obtaining $Cos(1^\circ)$, since the analysis is carried out on the central part of the angle. Applying the law of cosines to the triangle $\triangle\, abc$

$$L_1^2 + L_1^2 - 2L_1L_1Cos(\theta) = (2z_1)^2$$

implies that

$$cos(\theta) = \frac{2L_1^2 - (2z_1)^2}{2L_1} = \frac{2(\sqrt{\frac{8101}{16200}})^2 - (\frac{2\sqrt{2}}{180})^2}{2(\sqrt{\frac{8101}{16200}})^2}$$

then, solving for θ

$$\theta = ArcCos(\frac{2(\sqrt{\frac{8101}{16200}})^2 - (\frac{2\sqrt{2}}{180})^2}{2(\sqrt{\frac{8101}{16200}})^2}) = 1.273^\circ$$

that is, the angle is $\theta \neq 1°$. This leads us to the following reformulation: is there a way to find a value of a such that $\sqrt{2-a}$ is reduced in such a way that the opening fraction $\frac{2\sqrt{2-a}}{180}$ yields $cos(1°)$? This idea is not unreasonable, since it arises from the fact that the value of $\frac{\sqrt{2}}{90}$ was larger than expected: $cos(1.273°) \neq cos(1°)$.

After performing several calculations by repeatedly applying the previous procedure, we managed to find a first interesting value of a that gives a good approximation to θ; this value is $a* = \frac{142169}{300000}$. Then, following the procedure used to obtain $\theta = 1.273°$ (see Fig. 2), we have that:

$$y_2 = \frac{\sqrt{\frac{742169}{30}}}{200}$$

furthermore,

$$L_2 = \sqrt{(\frac{\frac{\sqrt{\frac{457831}{30}}}{100}}{180})^2 + (\frac{\sqrt{\frac{742169}{30}}}{200})^2} = \frac{\sqrt{\frac{6012026731}{30}}}{18000}$$

$$z_2 = \frac{\frac{\sqrt{\frac{457831}{30}}}{100}}{180}$$

Then, applying the law of cosines to $\triangle\, a'b'c'$, we have

$$L_2^2 + L_2^2 - 2L_2L_2cos(\theta) = (2z_2)^2$$

which implies that

$$Cos(\theta) = \frac{2L_2^2 - (2z_2)^2}{2L_2} = \frac{2(\frac{\sqrt{\frac{6012026731}{30}}}{18000})^2 - (2\frac{\sqrt{\frac{457831}{30}}}{\frac{100}{180}})^2}{2(\frac{\sqrt{\frac{6012026731}{30}}}{18000})^2} = \frac{6011111069}{6012026731}$$

in other words,

$$cos(\theta) = \frac{6011111069}{6012026731} = 0.9998476949553... \tag{3}$$

One interesting point to highlight is that this partial result (expression (3)) yields a value very close to $Cos(1°)$. This contrasts with what is known

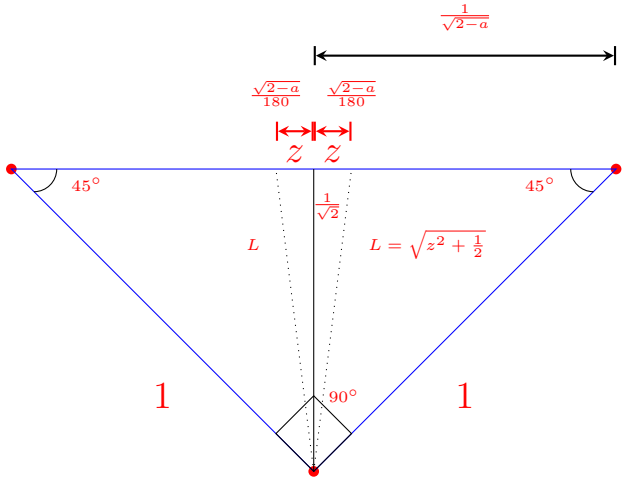


Figure 2: Here our analysis is strengthened by introducing the parameter a, in order to weight our calculations

in the literature through the *Taylor* series for the function $Cos(x), x = 1^\circ$, where the exact value of the expression is guaranteed, but whose mathematical representation nevertheless requires an infinite series. As can be seen in our partial result, although it is only an approximation, the representation we obtain is a compact rational expression for the value of $Cos(1^\circ)$; this important result will be emphasized later on. Now, if we solve (3) for the angle θ

$$\theta = ArcCos(\frac{6011111069}{6012026731}) = 1.000000660175... \tag{4}$$

that is,

$$\theta = 1.000000660175... \approx 1^\circ$$

By continuing to apply our technique, we constructed the following table for various values of a, where the approach to $\theta = 1^\circ$ can be appreciated for values before and after $a*$

Value of a	$\cos(\theta)$	Value of θ
$\frac{14}{30}$	0.9998465250	1.00383°
$\frac{142}{300}$	0.9998476040	1.00383°
$\frac{1421}{3000}$	0.999847657840	1.00029°
$\frac{14216}{30000}$	0.999847690115	1.000016°
$\frac{142169}{300000}*$	0.9998476949555	1.00000066°
$\frac{1421693}{3000000}$	0.999847695116653	1.00000013°
$\frac{142169373}{300000000}$	0.999847695155915	1.000000001°
$\frac{14216937388}{30000000000}$	0.999847695156389	1.000000000007°
$\frac{1421693738844}{3000000000000}$	0.999847695156390937	1.0000000000009°

Observations:

1.- It is important to clarify that the value we adopt for $cos(\overline{1^\circ})$ corresponds to the last value found in this table, which could be taken as exact from a computational standpoint, since if its equivalent is expanded in decimals

$$cos(\overline{1^\circ}) = \frac{1669753916065979}{1670008266413821} = 0.\underline{9998476951563}9093666... \qquad (5)$$

It turns out that when the inverse function of $cos(\overline{1^\circ})$ is computed, starting from the 14^{th} digit, an exact value for 1 degree is obtained up to 12 digits of precision. However, since the calculation ultimately stems from an approximation, we denote it as $\overline{1^\circ}$, once a computing package such as MATHEMATICA

is applied; that is,

$$ArcCos(0.\underline{9998476951563}90937) = 1.0000000000009 = \overline{1^\circ}$$

put another way

$$\overline{1^\circ} = 1.0000000000009 \approx 1^\circ$$

2.- This is to be expected, since the first digits of $cos(\frac{\pi}{180}) = cos(1^\circ)$, according to the *Taylor* expansion, yield the following

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

$$cos(\frac{\pi}{180}) = cos(1^\circ) \approx 0.\underline{99984769515639}123916... \tag{6}$$

And as can be observed, from the 14^{th} digit onward our result obtained in (5) deviates from what the *Taylor* expansion gives in (6). This means that, for some computing packages such as MATHEMATICA, it could be sufficient to use only the first 14 digits to accept the value of 1° as good, depending of course on the context in which the calculation is to be used. This is the reason why, for practical purposes, our result obtained in (5) can be considered a good approximation.

3.- Moreover, as previously mentioned, expression (5) is a compact representation of the value of $Cos(1^\circ)$. This result is among the most important of this work; it is not reported in the literature and it breaks with the structures reported in various published articles such as [7], where it is reported in terms of real expressions always accompanied by imaginary parts.

Now, using (5) we have that

$$sin(1^\circ) \approx \sqrt{1 - Cos^2(\overline{1^\circ})} = \sqrt{1 - (\frac{1667539160659799}{1670008266413821})^2} =$$

$$= \frac{60\sqrt{2359637980064706133718 31}}{1670008266413821}$$

that is,

$$sin(1^\circ) \approx \frac{60\sqrt{2359637980064706133718 31}}{1670008266413821}$$

And so we finally achieve the construction of the compact representation of the value of $Sin(1^\circ)$

4 Some applications

Once the representation of the value of $sin(1^\circ)$ has been achieved, we will be able to represent some other trigonometric functions, for example: $sin(2^\circ), cos(2^\circ), tan(2^\circ)$

$$sin(2^\circ) \approx sin(2(\overline{1^\circ})) = 2Sin(\overline{1^\circ})Cos(\overline{1^\circ}) =$$

$$\Rightarrow sin(2^\circ) \approx 2\frac{60\sqrt{2359637980064706133718 31}(1669753916065979)}{(1670008266413821)^2}$$

similarly

$$cos(2^\circ) \approx \frac{90306208926042717561772046766 49}{90361254560841593325856005414 49}$$

$$tan(2^\circ) \approx \frac{200370469927917480\sqrt{2359637980064706133718 31}}{27872286705448491490437426368 41}$$

Next we show a table with various approximate values of the representation of the function $sin(\theta)$, for various values of θ

Value of θ	$Sin(\theta)$
5°	$\frac{9030620892601271756177204676619\left(-2(-1+\sqrt{3})\sqrt{5+\sqrt{5}}+(-1+\sqrt{5})(\sqrt{2}+\sqrt{6})\right)}{1445780072973465493213696086631 84}+$ $+\frac{50092617481979370\sqrt{4719275960129412267436 62(4+\sqrt{7+\sqrt{5}+\sqrt{6(5+\sqrt{5})}})}}{27889276098904957374600198200 41}$
11°	$\frac{1669753916065979\left(-(2+\sqrt{3})(-1+\sqrt{5})+\sqrt{2(5+\sqrt{5})}+2(-1+\sqrt{3})\sqrt{4+\sqrt{7+\sqrt{5}+\sqrt{6(5+\sqrt{5})}}}\right)}{25720132262621136}$ - $\frac{60\sqrt{2359637980064706133718 31}\sqrt{(1-\frac{1}{256}\left(-(2+\sqrt{3})(-1+\sqrt{5})+\sqrt{2(5+\sqrt{5})}+2(-1+\sqrt{3})\sqrt{4+\sqrt{7+\sqrt{5}+\sqrt{6(5+\sqrt{5})}}}\right)^2)}}{1670008266413821}$
16°	$\frac{1669753916065979(-1+\sqrt{3})+60\sqrt{2359637980064706133718 31}(1+\sqrt{3})}{3340016532827642\sqrt{2}}$
31°	$\frac{1669753916065979+180\sqrt{78654599335490204457277}}{3340016532827642}$
44°	$\frac{1669753916065979\sqrt{2}-60\sqrt{47192759601294122674366 2}}{3340016532827642}$
59°	$\frac{\sqrt{3}(169753916065979-60\sqrt{7865459933549020445727 7})}{3340016532827642}$
77°	$\frac{100185234963958740\sqrt{2359637980064705133718 31(2-\sqrt{3})}}{27889276098904957374600198200 41}+\frac{9030620892601271756177204676619(\sqrt{2}+\sqrt{6})}{3614450182433663733034240216 5796}$

It is very important to note that the results reported in the preceding table are novel analytical expressions for the various representations of approximate values of $Sin(\theta)$ with $\theta = 5^\circ, 11^\circ, 16^\circ, 31^\circ, 44^\circ, 59^\circ, 77^\circ$

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References

[1] Shailesh A. Shirali, The Bhrskara-Aryabhata Approximation to the Sine function, *Mathematics Magazine*, **84** (2011), No. 2, 98-107. https://doi.org/10.4169/math.mag.84.2.098

[2] Ranjita Naik, Riyazahammad Nadaf., Sine-Cosine Computation Using CORDIC Algorithm, *International Journal of Advance Research in Computer and Communications Engineering*, **4** (2015), Issue 9, September.

[3] Boyer, Carl Benjamin, *History of Mathematics* (2nd. edition), John Wiley and Sons, Inc., 1991.

[4] O'Connor, John J., Robertson, Edmund F., *Trigonometric functions*, MacTutor History of Mathematics archive, Universidad de Saint Andrews, 1996.

[5] Leonhard Euler, J. D. Blanton, *Introduction to analysis of the infinite*, Book 1, Springer, 1988. https://doi.org/10.1007/978-1-4612-1021-4

[6] Clemence Montelle, Kim Plofker and Glen Van Blummelem, *From Samarqand to Jaipur, Sources and Studies in the History of Mathematics and Physical Sciences*, Springer (electronic), 2025.
https://doi.org/10.1007/978-3-031-89087-1

[7] Travis Kowalski, The Sine of a single Degree, *The College Mathematical Journal*, **47** (2016), No. 5, November.
https://doi.org/10.4169/college.math.j.47.5.322

[8] Callagy, James J., El ángulo central del polígono regular de 17 Lados, *Mathematical Gazette*, **67** (1983), 290-292.
https://doi.org/10.2307/3617271

[9] Katz, Victor J., *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Princeton: Princeton University Press, 2007.
https://doi.org/10.1515/9780691235394

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