International Journal of Algebra, Vol. 20, 2026, no. 1, 1 - 7
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2026.92001

A Note on Third-Power Associative Absolute Valued Real Algebras

Alassane Diouf, Mbayang Amar, Mohamed Traoré and Kandé Diaby

Département de Mathématiques et Informatique
Faculté des Sciences et Techniques
Université Cheikh Anta Diop, 5005 Dakar, Sénégal

Abstract

We prove that $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$, $\mathbb{H}$, $\overset{\star}{\mathbb{H}}$, $\mathbb{O}$ and $\overset{\star}{\mathbb{O}}$ are the only third-power associative absolute valued real algebras with a nonzero weak central element. We show also that if A is a third-power associative absolute valued real algebra with a nonzero alternative element, then A is power-associative, and isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

Keywords: Absolute Valued algebras, weakly central element, alternative element, third-power associative.
Mathematics Subject Classification: 17A30, 17A35, 17A60, 17A80.

1 Introduction

By definition, an *absolute-valued algebra* is an algebra A over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ endowed with an absolute value, i.e. a norm $\|\cdot\|$ on the vector space of A satisfying $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$. The classical algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (Hamilton's quaternions) and $\mathbb{O}$ (Cayley's octonions) are the only absolute valued unital real algebras [8].

Let $\mathbb{A} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. We recall that ${}^*\mathbb{A}$, $\overset{\star}{\mathbb{A}}$ and $\mathbb{A}^*$ are obtained by endowing the normed space $\mathbb{A}$ with the products $x \cdot y = x^*y$, $x \cdot y = x^*y^*$ and $x \cdot y = xy^*$, respectively, where $x \mapsto x^*$ means the standard involution.

El-Mallah and Micali show that if A is a flexible absolute valued real algebra, then A is finite-dimensional [6]. In 1983, El-Mallah proves that if A is a third-power associative pre-Hilbert absolute valued real algebra, then A is finite-dimensional [4]. In 1990, El-Mallah shows that if A is a finite-dimensional third-power associative absolute valued real algebra, then A is flexible, and isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$, $\mathbb{H}$, $\overset{\star}{\mathbb{H}}$, $\mathbb{O}$, $\overset{\star}{\mathbb{O}}$ or $\mathbb{P}$ [5]. We can also find in [1] a classification of absolute valued algebras satisfy the identity $(x, x, x) = 0$ and contain a nonzero flexible algebraic element [1]. Cuenca shows that $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$, $\mathbb{H}$, $\overset{\star}{\mathbb{H}}$, $\mathbb{O}$ and $\overset{\star}{\mathbb{O}}$ are the only third-power associative absolute valued real algebras with one idempotent commuting with all the idempotents [2, Theorem 4.1]. Kandé and Rochdi show that if A is a third-power associative absolute valued real algebra with nonzero central element, then A is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$, $\mathbb{H}$, $\overset{\star}{\mathbb{H}}$, $\mathbb{O}$ or $\overset{\star}{\mathbb{O}}$ [3, Theorem 1].

In this work, we show that $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$, $\mathbb{H}$, $\overset{\star}{\mathbb{H}}$, $\mathbb{O}$ and $\overset{\star}{\mathbb{O}}$ are the only third-power associative absolute valued real algebras containing a nonzero weakly central element (Theorem 3.2) and $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only third-power associative absolute valued real algebras containing a nonzero alternative element (Theorem 3.6).

2 Notations and Preliminary

Let A be an algebra over any field of zero characteristic. Given elements a, b, c in A, we set $(a, b, c) := (ab)c - a(bc)$ for the associator of a, b and c; $[a, b] := ab - ba$ for the commutator of a and b; and $a \bullet b := \frac{1}{2}(ab + ba)$ for the symmetrized product of a and b.

Recall that A is said to be *third-power associative* if it satisfies the identity $(x, x, x) = 0$, which can also be rewritten as $[x^2, x] = 0$. Linearizing we get the identity

$$(y, x, x) + (x, y, x) + (x, x, y) = 0, \tag{2.1}$$

which can also be rewritten as

$$[x^2, y] + [2x \bullet y, x] = 0 \tag{2.2}$$

The algebra A is said to be *(121)-power associative* whenever A satisfies the identity $(x, x^2, x) = 0$. Every third-power associative algebra is (121)-power associative.

Recall also that an element e in A is called central (respectively, flexible) if $[e, x] = 0$ (respectively, $(e, x, e) = 0$) for every x in A, and e is called square root of central element if $R_e^2 = L_e^2$. An element e in A is called *weak central element* if e is flexible and square root of central element. It is clear that any central element is a weak central element.

Let $\mathbb{A} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. We precise that 1 is a nonzero weak central element in ${}^*\mathbb{A}$ and $\mathbb{A}^*$, but 1 is not central in ${}^*\mathbb{A}$ and $\mathbb{A}^*$. We also specify that the algebras ${}^*\mathbb{A}$ and $\mathbb{A}^*$ do not contain a nonzero central element.

An element e in A is called *alternative* if $e(ex) = e^2x$ and $(xe)e = xe^2$ for every $x \in A$.

If an absolute valued algebra A contains a nonzero element e such that e is central and alternative, then A has a unit element [8, Theorem 2].

If A contains a nonzero flexible idempotent e, then e is a nonzero central idempotent in $B := (A, \cdot)$ where $x \cdot y = (ex)(ye)$ [1].

Lemma 2.1. *Let A be a (121)-power associative real algebra containing a nonzero alternative element e. If e is flexible, then the set $\{e, e^2, e^3\}$ is commutative.*

Proof. It is clear that $[e^2, e] = 0$ and $[e^3, e] = (e, e^2, e) = 0$. We have also $e^2e^3 = e(ee^3) = e(e^3e) = ee^3e$ and $e^3e^2 = (e^3e)e = (ee^3)e = ee^3e$, and hence $e^2e^3 = e^3e^2$. So the set $\{e, e^2, e^3\}$ is commutative. □

Lemma 2.2. *Let A be a (121)-power associative real algebra containing a nonzero alternative element e. If e is flexible, then $[e^2e^3, e] = 0$.*

Proof. Linearizing $(x, x^2, x) = 0$, we obtain

$$(x, x^2, y) + (x, xy + yx, x) + (y, x^2, x) = 0. \tag{2.3}$$

Taking $x = e$ and $y = e^3$ in (2.3) and keeping in mind that e is flexible and e^2 commutes with e^3, we have

$$\begin{aligned} 0 &= (e, e^2, e^3) + (e, ee^3 + e^3e, e) + (e^3, e^2, e) \\ &= (e, e^2, e^3) + (e^3, e^2, e) \\ &= (e^3)^2 - e(e^2e^3) + (e^3e^2)e - (e^3)^2 \\ &= (e^3e^2)e - e(e^2e^3) \\ &= (e^2e^3)e - e(e^2e^3). \end{aligned}$$

□

Lemma 2.3. *Let A be a (121)-power associative real algebra containing a nonzero alternative element e. If e is flexible, then the equalities $[e^2, e^3e] = 0$ and $[e^2, e^2e^3] = 0$ holds.*

Proof. Keeping in mind Lemma 2.1 and 2.2, we have

$$e^2(e^2e^3) = e\big(e(e^2e^3)\big) = e\big((e^2e^3)e\big) = \big(e(e^2e^3)\big)e = \big((e^2e^3)e\big)e = (e^2e^3)e^2,$$

and

$$e^2(e^3e) = e\big(e(e^3e)\big) = e\big((ee^3)e\big) = e\big((e^3e)e\big) = \big(e(e^3e)\big)e = \big((ee^3)e\big)e = (ee^3)e^2 = (e^3e)e^2.$$

□

3 Main results

Proposition 3.1. *Let A be a third-power associative absolute valued real algebra containing a nonzero flexible idempotent e. Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$, $\mathbb{H}$, $\overset{\star}{\mathbb{H}}$, $\mathbb{O}$, $\overset{\star}{\mathbb{O}}$ or $\mathbb{P}$.*

Proof. The normed space of A becomes an absolute valued algebra with nonzero central idempotent e under the product $x \cdot y = (ex)(ye)$. Then the absolute value of A derives from an inner product [5, Theorem 3.6] and A is finite-dimensional [4, Theorem 2.13]. The result follows from [5, Theorem 4.1]. □

Theorem 3.2. *Let A be a third-power associative absolute valued real algebra containing a nonzero weak central element e. Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$, $\mathbb{H}$, $\overset{\star}{\mathbb{H}}$, $\mathbb{O}$ or $\overset{\star}{\mathbb{O}}$.*

Proof. Taking $x = e$ and $y = x$ in (2.2) we obtain

$$\begin{aligned} 0 &= [e^2, x] + [2e \bullet x, e] \\ &= [e^2, x] + [ex + xe, e] \\ &= [e^2, x] + (ex)e + (xe)e - e(ex) - e(xe) \\ &= [e^2, x] + (ex)e - e(xe) \\ &= [e^2, x], \end{aligned}$$

and so e^2 is central. Therefore, by [3, Theorem 1], A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\overset{\star}{\mathbb{C}}$, $\mathbb{H}$, $\overset{\star}{\mathbb{H}}$, $\mathbb{O}$ or $\overset{\star}{\mathbb{O}}$. □

Lemma 3.4 is proven in [1]. Nevertheless, for the sake of completeness, we give here a proof.

Remark 3.3. *If A is an absolute valued real algebra containing a norm-one central algebraic element, then A is a pre-Hilbert space. An effect, as $A(a)$ is finite-dimensional, so there exists (norm-one) $b \in A(a)$ such that $a = R_{a^2}(b) = ba^2$. Since normed space of A becomes an absolute valued real algebra with nonzero central idempotent element a under the product $x \cdot y = b(xy)$, then A is a pre-Hilbert space [5, Theorem 3.6].*

Lemma 3.4. *Let A be an absolute valued real algebra containing a nonzero flexible element e such that $A(e)$ is isomorphic to $\mathbb{C}$. Then A is a pre-Hilbert space.*

Proof. We can assume, without lost of generality that $\|e\| = 1$. It is clear that there exists $a \in A(e)$ such that $ae = ea = e$. The normed space of A becomes an absolute valued real algebra B with nonzero central element a under the product $x \cdot y = (ex)(ye)$. Since the subalgebra $B(a)$ of B generated by a is contained in $A(e)$, so $B(a)$ is finite-dimensional, and hence a is a central algebraic element in B. The result follows from Remark 3.3. $\square$

Remark 3.5. *Let A be a third-power associative real algebra containing a nonzero alternative element e. Taking $x = e$ and $y = x$ in (2.1) and keeping in mind that e is alternative, we have $0 = (e,e,x) + (e,x,e) + (x,e,e) = (e,x,e)$, so e is a nonzero flexible element. Since e is alternative, we have $ee^2 = e(ee) = e^2e$, $(e^2e)e = (e^2)^2$ and $e(ee^2) = (e^2)^2$, and so $ee^3 = e^3e = (e^2)^2$. Taking $x = e^2$ and $y = e^3$ in (2.2), we obtain $0 = [(e^2)^2, e^3] + 2[e^2e^3, e^2]$, and hence $[(e^2)^2, e^3] = 0$ because of Lemmas 2.1 and 2.3. Taking also $x = e^3$ and $y = e$ in (2.2) and keeping in mind that $[e^3, e] = 0$, we get*

$$\begin{aligned} 0 &= [(e^3)^2, e] + 2[e^3e, e^3] \\ &= [(e^3)^2, e] + 2[(e^2)^2, e^3] \\ &= [(e^3)^2, e], \end{aligned}$$

and taking $x = e$ and $y = (e^3)^2$ in (2.1) and keeping in mind that e is flexible and $[(e^3)^2, e] = 0$, we obtain

$$\begin{aligned} 0 &= \big(e,e,(e^3)^2\big) + \big(e,(e^3)^2,e\big) + \big((e^3)^2,e,e\big) \\ &= \big(e,e,(e^3)^2\big) + \big((e^3)^2,e,e\big) \\ &= e^2(e^3)^2 - e\big(e(e^3)^2\big) + \big((e^3)^2e\big)e - (e^3)^2e^2 \\ &= e^2(e^3)^2 - e\big((e^3)^2e\big) + \big(e(e^3)^2\big)e - (e^3)^2e^2 \\ &= e^2(e^3)^2 - (e^3)^2e^2 \\ &= [e^2, (e^3)^2]. \end{aligned}$$

The following result is an extension of [8, Theorem 2].

Theorem 3.6. *Let A be a third-power associative absolute valued real algebra containing a nonzero alternative element e. Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.*

Proof. We can assume, without lost of generality that $\|e\| = 1$. We will distinguish the following cases:

First case. If e is collinear to e^2. Then $e^2 = \lambda e$, where $\lambda \in \mathbb{R}\backslash\{0\}$. By putting, $e_0 = \lambda^{-1}e$, we have $e_0^2 = e_0$. Since e is a nonzero alternative element, so e_0 is an alternative nonzero idempotent, and hence $e_0(e_0x) = e_0x$ and $(xe_0)e_0 = xe_0$ for all x in A. So $e_0x = xe_0 = x$ for every x in A, because A has no nonzero divisor of zero. The result follows from [8, Theorem 1].

Second case. Suppose that e is not collinear to e^2 and keeping in mind the preliminary lemmas and Remark 3.5. Since the set $\{e, e^2, e^3\}$ is commutative, so the vector subspace $\mathbb{R}e + \mathbb{R}e^2 + \mathbb{R}e^3$ is an inner-product space [8, Lemma 1]. Suppose that $dim(\mathbb{R}e + \mathbb{R}e^2 + \mathbb{R}e^3) = 3$. There exists a norm-one $e_0 \in \mathbb{R}e + \mathbb{R}e^2 + \mathbb{R}e^3$ orthogonal to e and e^2, we obtain

$$\|e_0^2 - e^2\| = \|(e_0 - e)(e_0 + e)\| \;=\; \|e_0 - e\|\|e_0 + e\| = 2,$$

and $e^2 e_0^2 = e_0^2 e^2$, and hence $e_0^2 + e^2 = 0$ [8, Lemma 3]. Furthermore, since

$$\|e_0^2 - (e^2)^2\| = \|(e_0 - e^2)(e_0 + e^2)\| \;=\; \|e_0 - e^2\|\|e_0 + e^2\| = 2,$$

and $(e^2)^2 e_0^2 = e_0^2 (e^2)^2$, so $e_0^2 + (e^2)^2 = 0$ by [8, Lemma 3]. We deduce that $(e^2)^2 - e^2 = (e^2 - e)(e^2 + e) = 0$ and keeping in mind that A has no nonzero divisors of zero, we have $e^2 = e$ or $e^2 = -e$, absurd because e is not collinear to e^2. We realize that $dim(\mathbb{R}e + \mathbb{R}e^2 + \mathbb{R}e^3) = 2$, so $e^3 \in \mathbb{R}e + \mathbb{R}e^2$. Since $[e^2, e^3] = 0$, by [2, Corollary 2.3], we obtain $(e^2)^2 \in \mathbb{R}e^2 + \mathbb{R}e^3$, and hence $(e^2)^2 \in \mathbb{R}e + \mathbb{R}e^2$ because $\mathbb{R}e^2 + \mathbb{R}e^3 \subset \mathbb{R}e + \mathbb{R}e^2$. We obtain $A(e) = \mathbb{R}e + \mathbb{R}e^2$, so the commutative two-dimensional absolute valued real algebra $A(e)$ is isomorphic to $\mathbb{C}$ or $\overset{\star}{\mathbb{C}}$ [8, Theorem 3], and consequently $A(e)$ is isomorphic to $\mathbb{C}$, because $\overset{\star}{\mathbb{C}}$ not contains a nonzero alternative element.

This implies that A is a pre-Hilbert space because of Lemma 3.4, so A is finite-dimensional [4, Theorem 2.13] and hence A is flexible [5, Corollary 4.2]. Since A contains $\mathbb{C}$, therefore by [6, Théorème 3.3], A is isomorphic to $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. □

References

[1] A. Chandid and A. Rochdi, Mutations of absolute valued algebras, *Int. J. Algebra*, **2** (2008), 357-368.

[2] J. A. Cuenca, Third-power associative absolute valued algebras with a nonzero idempotent commuting with all idempotents, *Publ. Mat.*, **58** (2014), 469-484. https://doi.org/10.5565/publmat_58214_23

[3] K. Diaby and A. Rochdi, A note on absolute valued algebras satisfying $(xx^2)x = x(x^2x)$, *J. Al. Appl.*, (2022) 2250172, (8 pages). https://doi.org/10.1142/s0219498822501729

[4] M. L. El-Mallah, Sur algèbres absolument valuées qui vérifient $(x, x, x) = 0$, *J. Algebra*, **80** (1983), 314-322. https://doi.org/10.1016/0021-8693(83)90004-2

[5] M. L. El-Mallah, Absolute valued algebras containing a central idempotent, *J. Algebra,* **128** (1990), 180-187. https://doi.org/10.1016/0021-8693(90)90048-s

[6] M. L. El-Mallah and A. Micali, Sur les dimensions des algèbres absolument valuées, *J. Algebra* **68** (1981), 237-246. https://doi.org/10.1016/0021-8693(81)90262-3

[7] E. Idnarour and A. Rochdi, Algebraic extensions of some results by Yang, *Comm. Algebra*, **52** (2024), 1230-1236. https://doi.org/10.1080/00927872.2023.2259474

[8] K. Urbanik and F. B. Wright, Absolute valued algebras, *Proc. Amer. Math. Soc.*, **11** (1960), 861-866. https://doi.org/10.1090/s0002-9939-1960-0120264-6

Received: December 30, 2025; Published: January 19, 2026