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# Weakly 2-Nil Primary Ideals of Commutative Rings

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**Abstract**

We study the effect of restricting the 2-nil primary condition to nonzero triple products and introduce the resulting class of weakly 2-nil primary ideals. The distinction between the weak and non-weak notions is characterized by 2-nil primary zero triples. We also examine the behavior of these ideals under homomorphisms, quotient rings, localization, and direct products.

In integral domains, an ideal $I \subsetneq R$ satisfies the weakly 2-nil primary condition exactly when $\sqrt{I}$ is prime. Hence, in this case, such ideals are quasi-primary, and the weak and non-weak 2-nil primary conditions agree. This also yields a complete description in principal ideal domains.

We further determine the weakly 2-nil primary ideals of $\mathbb{Z}_n$. If $n = p_1^{k_1} \cdots p_s^{k_s}$, then for $s \leq 2$, all proper ideals of $\mathbb{Z}_n$ satisfy the weakly 2-nil primary condition. For $s \geq 3$, the weakly 2-nil primary ideals are exactly

$$(0) \quad \text{and} \quad (p_i^t), \qquad 1 \leq i \leq s, \quad 1 \leq t \leq k_i.$$

Hence, in this case, their number is $1 + \sum_{i=1}^{s} k_i$.

**Mathematics Subject Classification:** 13A15, 13C05

**Keywords:** weakly 2-nil primary ideal; 2-nil primary ideal; nilradical; quasi-primary ideal; localization; $\mathbb{Z}_n$

# 1 Introduction

Several extensions of prime and primary ideals have been introduced by considering conditions on products of elements. One of the basic notions in this direction is the 2-absorbing ideal, introduced by Badawi [1]. An ideal $I \subsetneq R$ is called 2-absorbing if $abc \in I$ implies that at least one of $ab$, $ac$, or $bc$ belongs to $I$. A weak form of this notion was later studied by Badawi and Yousefian Darani [2], where the condition is required only when $abc \neq 0$. This restriction is particularly relevant in rings with zero divisors.

Several related classes have since been investigated, including 2-absorbing primary ideals and their weak versions [3, 4], 2-absorbing quasi-primary ideals [5], $n$-ideals [6], and $(2, n)$-ideals [7].

The nilradical appears explicitly in another related class. Yetkin Çelikel introduced 2-nil ideals [8], and Qaralleh, Al-Kaseasbeh and Yetkin Çelikel later studied 2-nil primary ideals [9]. An ideal $I \subsetneq R$ is 2-nil primary if $abc \in I$ implies

$$ab \in \sqrt{(0)} \quad \text{or} \quad ac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I}.$$

In this paper, we consider the corresponding weak condition. An ideal $I \subsetneq R$ is called *weakly 2-nil primary* if $0 \neq abc \in I$ implies

$$ab \in \sqrt{(0)} \quad \text{or} \quad ac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I}.$$

The difference from the 2-nil primary condition appears only when the triple product is zero. This leads to the notion of a 2-nil primary zero triple. We prove that the existence of such a triple is exactly what prevents a weakly 2-nil primary ideal from being 2-nil primary.

We also examine the behavior of the weakly 2-nil primary condition under ring homomorphisms, quotient rings, localization, and direct products.

For integral domains, we show that an ideal $I \subsetneq R$ is weakly 2-nil primary exactly when $\sqrt{I}$ is prime. Hence these ideals are precisely the quasi-primary ideals, and the weak and non-weak 2-nil primary conditions agree. For a principal ideal domain, this gives

$$(0) \quad \text{and} \quad (p^m), \qquad m \geq 1,$$

where $p$ is a prime element.

We finally investigate the case of $\mathbb{Z}_n$. Writing $n = p_1^{k_1} \cdots p_s^{k_s}$, we show that when $s \leq 2$, every proper ideal satisfies the weakly 2-nil primary condition, whereas for $s \geq 3$ the ideals in this class are exactly

$$(0) \quad \text{and} \quad (p_i^t), \qquad 1 \leq i \leq s, \quad 1 \leq t \leq k_i.$$

This also gives an explicit formula for their number.

The paper is organized as follows. Section 2 collects the notation and standard facts needed later. Section 3 introduces the weakly 2-nil primary condition and the zero-triple criterion. Section 4 discusses homomorphisms and quotient rings, Section 5 deals with localization, and Section 6 considers direct products. Section 7 treats integral domains and principal ideal domains. The classification for $\mathbb{Z}_n$ is given in Section 8, and Section 9 compares the new notion with weakly 2-absorbing primary ideals. Section 10 contains the concluding remarks.

# 2 Preliminaries

Throughout this paper, all rings are assumed to be commutative with identity $1 \neq 0$, and all ring homomorphisms preserve the identity.

For an ideal $I$ of a ring $R$, its radical is

$$\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n \geq 1\}.$$

We write $(0_R)$ for the zero ideal of $R$, and $\sqrt{(0_R)} = \mathrm{Nil}(R)$ for the nilradical of $R$. When the ring is clear from the context, we write simply $\sqrt{(0)}$. A ring $R$ is reduced if $\mathrm{Nil}(R) = (0)$. An ideal $I \subsetneq R$ is called quasi-primary if $\sqrt{I}$ is prime.

For a positive integer $n$, we write $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. If $d \mid n$, then $(d)$ denotes the ideal of $\mathbb{Z}_n$ generated by the residue class of $d$.

Let $S \subseteq R$ be a multiplicatively closed set. We denote by $S^{-1}R$ the localization of $R$ at $S$, and by $S^{-1}I$ the extension of $I$ to $S^{-1}R$. We use

$$\sqrt{S^{-1}I} = S^{-1}\sqrt{I}.$$

For ideals $I_i \subseteq R_i$, we use

$$\sqrt{I_1 \times \cdots \times I_m} = \sqrt{I_1} \times \cdots \times \sqrt{I_m}, \qquad \mathrm{Nil}(R_1 \times \cdots \times R_m) = \mathrm{Nil}(R_1) \times \cdots \times \mathrm{Nil}(R_m).$$

Finally, we use the standard correspondence between ideals of finite direct products and products of ideals, together with the Chinese Remainder Theorem in the form

$$\mathbb{Z}_n \cong \prod_{i=1}^{s} \mathbb{Z}_{p_i^{k_i}}, \qquad n = \prod_{i=1}^{s} p_i^{k_i}.$$

# 3 Basic Properties and Zero Triples

We now introduce the main notion of the paper.

**Definition 3.1.** Let $I \subsetneq R$ be an ideal. We say that $I$ is *weakly 2-nil primary* if, for all $a, b, c \in R$, $0 \neq abc \in I$ implies

$$ab \in \sqrt{(0)} \quad \text{or} \quad ac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I}.$$

Recall that an ideal $I \subsetneq R$ is 2-nil primary [9] if $abc \in I$ implies

$$ab \in \sqrt{(0)} \quad \text{or} \quad ac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I}.$$

Hence every 2-nil primary ideal satisfies the weakly 2-nil primary condition. The reverse implication does not hold in general. Also, the zero ideal of any ring is weakly 2-nil primary, since there are no elements $a, b, c \in R$ for which $0 \neq abc \in (0)$.

The weak and non-weak conditions can differ even for finite reduced rings.

**Example 3.2.** Let $R = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ and $I = (0)$. Consider

$$a = (1,1,0), \qquad b = (1,0,1), \qquad c = (0,1,1).$$

Then $abc = (0,0,0)$, while

$$ab = (1,0,0), \qquad ac = (0,1,0), \qquad bc = (0,0,1).$$

Since $R$ is reduced and $I = (0)$, we have $\sqrt{(0)} = (0) = \sqrt{I}$. Therefore

$$ab \notin \sqrt{(0)}, \qquad ac \notin \sqrt{I}, \qquad bc \notin \sqrt{I}.$$

Hence $I$ is weakly 2-nil primary but not 2-nil primary.

For a weakly 2-nil primary ideal, the 2-nil primary condition can fail only when the triple product is zero. This motivates the following definition.

**Definition 3.3.** Let $I$ be a weakly 2-nil primary ideal of $R$. A triple $(a, b, c) \in R^3$ is called a *2-nil primary zero triple of $I$* if

$$abc = 0, \qquad ab \notin \sqrt{(0)}, \qquad ac \notin \sqrt{I}, \qquad bc \notin \sqrt{I}.$$

**Theorem 3.4.** *Let $I$ be a weakly 2-nil primary ideal of $R$. Then $I$ is 2-nil primary if and only if $I$ has no 2-nil primary zero triple.*

*Proof.* Assume first that $I$ is 2-nil primary. If $(a, b, c)$ were a 2-nil primary zero triple of $I$, then $abc = 0 \in I$, and the 2-nil primary condition would give

$$ab \in \sqrt{(0)} \quad \text{or} \quad ac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I},$$

a contradiction.

Conversely, suppose that $I$ has no 2-nil primary zero triple and let $abc \in I$. If $abc \neq 0$, the weakly 2-nil primary condition gives the required alternatives. If $abc = 0$, then $(a, b, c)$ cannot be a 2-nil primary zero triple, so the same alternatives must hold. Therefore $I$ is 2-nil primary. $\square$

# 4 Homomorphisms and Quotient Rings

We begin by considering the weakly 2-nil primary condition under ring homomorphisms.

**Proposition 4.1.** *Consider a ring homomorphism $h : R \longrightarrow S$.*

(1) *Suppose that $h$ is injective and $L \subsetneq S$ satisfies the weakly 2-nil primary condition. Then the inverse image $h^{-1}(L)$ has the same property in $R$.*

(2) *Suppose that $h$ is surjective and $K \subsetneq R$ is weakly 2-nil primary. If $\ker h \subseteq K$, then $h(K)$ is weakly 2-nil primary in $S$.*

*Proof.* For (1), $h^{-1}(L)$ is proper since $L$ is proper. Let $0 \neq abc \in h^{-1}(L)$. Then

$$0 \neq h(abc) = h(a)h(b)h(c) \in L,$$

where the nonzero condition follows from the injectivity of $h$. Hence

$$h(ab) \in \sqrt{(0_S)} \quad \text{or} \quad h(ac) \in \sqrt{L} \quad \text{or} \quad h(bc) \in \sqrt{L}.$$

If $h(ab) \in \sqrt{(0_S)}$, then $h((ab)^n) = 0$ for some $n \geq 1$. Injectivity gives $(ab)^n = 0$, and therefore $ab \in \sqrt{(0_R)}$. If $h(ac) \in \sqrt{L}$, then $h((ac)^m) \in L$ for some $m \geq 1$, so $ac \in \sqrt{h^{-1}(L)}$. The case of $bc$ is analogous.

For (2), $h(K)$ is proper. Indeed, if $1_S = h(u)$ for some $u \in K$, then $u - 1_R \in \ker h \subseteq K$, which gives $1_R \in K$, a contradiction.

Now let $0 \neq xyz \in h(K)$. Choose $a, b, c \in R$ such that

$$h(a) = x, \qquad h(b) = y, \qquad h(c) = z.$$

Since $xyz \in h(K)$, there exists $u \in K$ such that $h(abc) = h(u)$. Thus $abc - u \in \ker h \subseteq K$, and hence $abc \in K$. Moreover, $abc \neq 0$ because $h(abc) = xyz \neq 0$. Therefore

$$ab \in \sqrt{(0_R)} \quad \text{or} \quad ac \in \sqrt{K} \quad \text{or} \quad bc \in \sqrt{K}.$$

Applying $h$ gives

$$xy \in \sqrt{(0_S)} \quad \text{or} \quad xz \in \sqrt{h(K)} \quad \text{or} \quad yz \in \sqrt{h(K)}.$$

Thus $h(K)$ is weakly 2-nil primary. □

The preceding result also shows that the condition is invariant under ring isomorphisms.

**Corollary 4.2.** *Let $h : R \longrightarrow S$ be a ring isomorphism, and let $K \subsetneq R$. Then $K$ is weakly 2-nil primary in $R$ exactly when $h(K)$ is weakly 2-nil primary in $S$.*

*Proof.* This follows from Proposition 4.1, applied to $h$ and $h^{-1}$. $\square$

For quotient rings, let $L \subseteq K \subsetneq R$ and suppose that $K$ is weakly 2-nil primary. For the canonical projection $\pi : R \longrightarrow R/L$, we have $\ker \pi = L \subseteq K$. Hence Proposition 4.1(2) shows that $K/L$ is weakly 2-nil primary in $R/L$.

# 5 Localization

We now study the weakly 2-nil primary condition under localization. A nonzero product in $R$ may become zero in $S^{-1}R$, so this case requires some care. We use the standard identities

$$\sqrt{S^{-1}I} = S^{-1}\sqrt{I}, \qquad \sqrt{(0_{S^{-1}R})} = S^{-1}\sqrt{(0_R)}.$$

**Theorem 5.1.** *Let $I \subsetneq R$ be a weakly 2-nil primary ideal, and let $S \subseteq R$ be a multiplicatively closed set such that $S^{-1}I \neq S^{-1}R$. Then $S^{-1}I$ is weakly 2-nil primary in $S^{-1}R$.*

*Proof.* Let

$$0 \neq \frac{abc}{s_1 s_2 s_3} \in S^{-1}I.$$

Then there exists $u \in S$ such that $uabc \in I$. Moreover, $uabc \neq 0$; otherwise $\frac{abc}{s_1 s_2 s_3} = 0$ in $S^{-1}R$, a contradiction. Hence $0 \neq (ua)bc \in I$. Since $I$ is weakly 2-nil primary,

$$uab \in \sqrt{(0_R)} \quad \text{or} \quad uac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I}.$$

If $uab \in \sqrt{(0_R)}$, then

$$\frac{ab}{s_1 s_2} = \frac{uab}{u s_1 s_2} \in S^{-1}\sqrt{(0_R)} = \sqrt{(0_{S^{-1}R})}.$$

If $uac \in \sqrt{I}$, then

$$\frac{ac}{s_1 s_3} \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I},$$

and if $bc \in \sqrt{I}$, then

$$\frac{bc}{s_2 s_3} \in \sqrt{S^{-1}I}.$$

Thus $S^{-1}I$ is weakly 2-nil primary. $\square$

# 6 Direct Product Rings

We now turn to direct product rings. For ideals $I_i \subseteq R_i$, we use the standard identity

$$\sqrt{I_1 \times \cdots \times I_m} = \sqrt{I_1} \times \cdots \times \sqrt{I_m}.$$

We consider product ideals with only one proper component.

**Proposition 6.1.** *Let $R_1$ and $R_2$ be rings, and let $I_1 \subsetneq R_1$. Then $I_1 \times R_2$ satisfies the weakly 2-nil primary condition in $R_1 \times R_2$ exactly when $\sqrt{I_1}$ is prime in $R_1$.*

*Proof.* Suppose that $I_1 \times R_2$ satisfies the weakly 2-nil primary condition. Let $x, y \in R_1$ satisfy $xy \in \sqrt{I_1}$. Then there exists $n \geq 1$ such that $x^n y^n \in I_1$. Set

$$A = (x^n, 1), \qquad B = (y^n, 1), \qquad C = (1, 1).$$

Then

$$0 \neq ABC = (x^n y^n, 1) \in I_1 \times R_2.$$

Hence

$$AB \in \sqrt{(0)} \quad \text{or} \quad AC \in \sqrt{I_1 \times R_2} \quad \text{or} \quad BC \in \sqrt{I_1 \times R_2}.$$

The first alternative is impossible because the second component of $AB$ is 1. Therefore

$$x^n \in \sqrt{I_1} \quad \text{or} \quad y^n \in \sqrt{I_1},$$

and hence $x \in \sqrt{I_1}$ or $y \in \sqrt{I_1}$. Thus $\sqrt{I_1}$ is prime in $R_1$.

Conversely, assume that $\sqrt{I_1}$ is prime and let $0 \neq ABC \in I_1 \times R_2$, where

$$A = (a_1, a_2), \qquad B = (b_1, b_2), \qquad C = (c_1, c_2).$$

Then $a_1 b_1 c_1 \in I_1 \subseteq \sqrt{I_1}$. Since $\sqrt{I_1}$ is prime, at least one of $a_1, b_1, c_1$ belongs to $\sqrt{I_1}$. If $a_1 \in \sqrt{I_1}$ or $c_1 \in \sqrt{I_1}$, then

$$AC \in \sqrt{I_1 \times R_2}.$$

If $b_1 \in \sqrt{I_1}$, then

$$BC \in \sqrt{I_1 \times R_2}.$$

Hence $I_1 \times R_2$ is weakly 2-nil primary. $\square$

# 7 Integral Domains and Principal Ideal Domains

For integral domains, the weakly 2-nil primary condition admits a simple description in terms of the radical.

**Theorem 7.1.** *Let $R$ be an integral domain and let $I \subsetneq R$ be an ideal. Then the following conditions are equivalent:*

(1) *$I$ is weakly 2-nil primary;*

(2) *$\sqrt{I}$ is prime;*

(3) *$I$ is quasi-primary.*

*Proof.* By definition, conditions (2) and (3) are equivalent.

Suppose that $I$ is weakly 2-nil primary and let $ab \in \sqrt{I}$. Then $a^n b^n \in I$ for some $n \geq 1$.

If $ab = 0$, then, since $R$ is a domain, $a = 0$ or $b = 0$, and hence

$$a \in \sqrt{I} \quad \text{or} \quad b \in \sqrt{I}.$$

If $ab \neq 0$, then $0 \neq a^n b^n \in I$. Applying the weakly 2-nil primary condition to $a^n, b^n, 1$, we get

$$a^n b^n \in \sqrt{(0)} \quad \text{or} \quad a^n \in \sqrt{I} \quad \text{or} \quad b^n \in \sqrt{I}.$$

Since $R$ is a domain, $\sqrt{(0)} = (0)$, so

$$a \in \sqrt{I} \quad \text{or} \quad b \in \sqrt{I}.$$

Thus $\sqrt{I}$ is prime.

Conversely, suppose that $\sqrt{I}$ is prime and let $0 \neq abc \in I$. Since $abc \in \sqrt{I}$, at least one of $a, b, c$ belongs to $\sqrt{I}$. Hence

$$ac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I},$$

and $I$ is weakly 2-nil primary. □

An immediate consequence is that the weak and non-weak conditions agree in integral domains.

**Corollary 7.2.** *Let $R$ be an integral domain and let $I \subsetneq R$ be an ideal. Then $I$ is weakly 2-nil primary exactly when it is 2-nil primary.*

*Proof.* If $I$ is weakly 2-nil primary, then Theorem 7.1 gives that $\sqrt{I}$ is prime. Thus, for $abc \in I$, at least one of $a, b, c$ belongs to $\sqrt{I}$, and hence

$$ac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I}.$$

Therefore $I$ is 2-nil primary. The reverse implication follows directly from the definitions. □

For principal ideal domains, the preceding characterization gives an explicit list.

**Corollary 7.3.** *Let $R$ be a principal ideal domain. The weakly 2-nil primary ideals of $R$ are*

$$(0) \quad \textit{and} \quad (p^m), \qquad m \geq 1,$$

*where $p$ is a prime element of $R$.*

*Proof.* The zero ideal is weakly 2-nil primary by Theorem 7.1. Let $I = (a) \neq (0)$ be proper. By Theorem 7.1, $I$ is weakly 2-nil primary exactly when $\sqrt{(a)}$ is prime.

Write

$$a = up_1^{e_1} \cdots p_t^{e_t},$$

where $u$ is a unit, the $p_i$ are pairwise nonassociate prime elements, and $e_i \geq 1$. Then

$$\sqrt{(a)} = (p_1 \cdots p_t).$$

This radical is prime exactly when $t = 1$. Hence $I = (p^m)$ for some prime element $p$ and some $m \geq 1$. □

# 8 Weakly 2-Nil Primary Ideals of $\mathbb{Z}_n$

Let $n \geq 2$, and write $n = p_1^{k_1} \cdots p_s^{k_s}$, where $p_1, \dots, p_s$ are distinct primes and $k_i \geq 1$. Every ideal of $\mathbb{Z}_n$ is generated by a divisor

$$d = p_1^{t_1} \cdots p_s^{t_s}, \qquad 0 \leq t_i \leq k_i.$$

By the Chinese Remainder Theorem,

$$\mathbb{Z}_n \cong \prod_{i=1}^{s} \mathbb{Z}_{p_i^{k_i}}.$$

Set $R_i = \mathbb{Z}_{p_i^{k_i}}$. Under this decomposition, the ideal $(d)$ corresponds to $J_1 \times \cdots \times J_s$, where

$$J_i = \begin{cases} R_i, & t_i = 0, \\ (p_i^{t_i}), & 1 \leq t_i \leq k_i. \end{cases}$$

**Lemma 8.1.** *Let $R = \mathbb{Z}_{p^k}$, where $p$ is prime and $k \geq 1$. Then*

$$\mathrm{Nil}(R) = \sqrt{(0)} = (p), \qquad \sqrt{(p^t)} = (p) \quad (1 \leq t \leq k).$$

*Thus every proper ideal of $R$ has radical $(p)$.*

*Proof.* An element $\overline{x} \in \mathbb{Z}_{p^k}$ is nilpotent exactly when $p \mid x$, and hence $\mathrm{Nil}(R) = \sqrt{(0)} = (p)$. If $x \in \sqrt{(p^t)}$, then $x^m \in (p^t)$ for some $m \geq 1$, so $p \mid x$. Conversely, $p^t \in (p^t)$, so $p \in \sqrt{(p^t)}$. Therefore $\sqrt{(p^t)} = (p)$. Finally, $R/(p) \cong \mathbb{Z}_p$, so $(p)$ is prime. □

**Lemma 8.2.** *Let $R = \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^\ell}$, where $p$ and $q$ are distinct primes. Then every proper ideal of $R$ is weakly 2-nil primary.*

*Proof.* Write $R = R_1 \times R_2$ and $I = I_1 \times I_2$. If exactly one component of $I$ is proper, the result follows from Proposition 6.1 and Lemma 8.1.

Suppose that both $I_1$ and $I_2$ are proper. Then

$$\sqrt{I} = \mathrm{Nil}(R) = (p) \times (q).$$

Let $0 \neq ABC \in I$, where

$$A = (a_1, a_2), \qquad B = (b_1, b_2), \qquad C = (c_1, c_2).$$

If $AB \in \mathrm{Nil}(R)$, there is nothing to prove. Otherwise, without loss of generality, $a_1 b_1 \notin (p)$. Since $(p)$ is prime, $a_1, b_1 \notin (p)$, and $a_1 b_1 c_1 \in I_1 \subseteq (p)$ gives $c_1 \in (p)$. Hence $a_1 c_1, b_1 c_1 \in (p)$.

Also, $a_2 b_2 c_2 \in I_2 \subseteq (q)$. Since $(q)$ is prime, one of $a_2, b_2, c_2$ belongs to $(q)$. If $a_2 \in (q)$ or $c_2 \in (q)$, then $AC \in \sqrt{I}$, while if $b_2 \in (q)$, then $BC \in \sqrt{I}$. Thus $I$ is weakly 2-nil primary. □

**Theorem 8.3.** *Suppose that $n = p_1^{k_1} \cdots p_s^{k_s}$, where $p_1, \ldots, p_s$ are distinct primes. If $s \leq 2$, every proper ideal of $\mathbb{Z}_n$ satisfies the weakly 2-nil primary condition.*

*If $s \geq 3$, the proper ideals of $\mathbb{Z}_n$ having this property are exactly*

$$(0) \quad \text{and} \quad (p_i^t), \qquad 1 \leq i \leq s, \quad 1 \leq t \leq k_i.$$

*Proof.* By Corollary 4.2, the weakly 2-nil primary condition is preserved under ring isomorphisms.

Suppose first that $s = 1$. Let $I \subsetneq \mathbb{Z}_n$ and $0 \neq abc \in I$. By Lemma 8.1,

$$I \subseteq \sqrt{I} = (p_1) = \sqrt{(0)}.$$

Since $(p_1)$ is prime, at least one of $a, b, c$ belongs to $(p_1)$. If $a \in (p_1)$ or $b \in (p_1)$, then $ab \in \sqrt{(0)}$, while if $c \in (p_1)$, then $ac \in \sqrt{I}$. Thus every proper ideal satisfies the weak condition.

For $s = 2$, the result follows from Lemma 8.2.

Now assume that $s \geq 3$. By the Chinese Remainder Theorem,

$$\mathbb{Z}_n \cong R_1 \times \cdots \times R_s, \qquad R_i = \mathbb{Z}_{p_i^{k_i}}.$$

Write $I = J_1 \times \cdots \times J_s$, where each $J_i$ is either $R_i$ or a proper ideal of $R_i$.

The zero ideal is weakly 2-nil primary since the defining condition is vacuously satisfied. If exactly one component, say $J_j$, is proper, then $\sqrt{J_j} = (p_j)$ by Lemma 8.1. Since $(p_j)$ is prime, Proposition 6.1 shows that $I$ is weakly 2-nil primary.

Suppose next that at least two, but not all, of the components are proper. Choose distinct $i, j$ with $J_i \subsetneq R_i$, $J_j \subsetneq R_j$, and $k$ with $J_k = R_k$. Define $A, B, C$ coordinatewise by

$$(A_i, B_i, C_i) = (1, 0, 1), \qquad (A_j, B_j, C_j) = (0, 1, 1), \qquad (A_k, B_k, C_k) = (1, 1, 1),$$

with all remaining coordinates equal to zero. Then $0 \neq ABC \in I$, but

$$AB \notin \mathrm{Nil}(R), \qquad AC \notin \sqrt{I}, \qquad BC \notin \sqrt{I},$$

a contradiction.

Finally, suppose that every $J_i$ is proper and $I \neq (0)$. Choose $k$ such that $J_k \neq (0)$, take $0 \neq u \in J_k$, and choose distinct $i, j \neq k$. Define

$$(A_i, B_i, C_i) = (1, 0, 1), \qquad (A_j, B_j, C_j) = (0, 1, 1), \qquad (A_k, B_k, C_k) = (1, 1, u),$$

with all remaining coordinates equal to zero. Again $0 \neq ABC \in I$, while

$$AB \notin \mathrm{Nil}(R), \qquad AC \notin \sqrt{I}, \qquad BC \notin \sqrt{I},$$

which is impossible.

Thus, for $s \geq 3$, every nonzero weakly 2-nil primary ideal has exactly one proper component. Under the Chinese Remainder isomorphism, these ideals correspond to

$$(p_i^t), \qquad 1 \leq i \leq s, \quad 1 \leq t \leq k_i.$$

Together with the zero ideal, this proves the classification. □

**Corollary 8.4.** *Let $n = p_1^{k_1} \cdots p_s^{k_s}$, where $s \geq 3$. Then the number of weakly 2-nil primary ideals of $\mathbb{Z}_n$ is*

$$1 + \sum_{i=1}^{s} k_i.$$

*Proof.* For each $i$, Theorem 8.3 gives $(p_i), (p_i^2), \ldots, (p_i^{k_i})$, giving $\sum_{i=1}^{s} k_i$ distinct nonzero ideals. Adding the zero ideal gives the result. □

**Example 8.5.** The ideal (6) of $\mathbb{Z}_{60}$ does not satisfy the weakly 2-nil primary condition. Take

$$a = \overline{21}, \qquad b = \overline{16}, \qquad c = \overline{1}.$$

Then $abc = \overline{36} \neq \overline{0}$ and $\overline{36} \in (6)$. Moreover,

$$ab = \overline{36} \notin (30) = \mathrm{Nil}(\mathbb{Z}_{60}),$$

while

$$\sqrt{(6)} = (6), \qquad ac = \overline{21} \notin (6), \qquad bc = \overline{16} \notin (6).$$

Therefore none of the alternatives in Definition 3.1 holds.

# 9 Comparison with Related Notions

We now compare the weakly 2-nil primary condition with the closely related notion of weakly 2-absorbing primary ideals.

The connection with 2-nil primary ideals [9] was discussed in Section 3. In particular, Theorem 3.4 shows that a weakly 2-nil primary ideal fails to be 2-nil primary precisely when a 2-nil primary zero triple exists.

Recall that an ideal $I \subsetneq R$ is weakly 2-absorbing primary [4] if $0 \neq abc \in I$ implies

$$ab \in I \quad \text{or} \quad ac \in \sqrt{I} \quad \text{or} \quad bc \in \sqrt{I}.$$

Thus the two conditions differ only in the first alternative: weakly 2-absorbing primary ideals require $ab \in I$, whereas the weakly 2-nil primary condition requires $ab \in \sqrt{(0)}$.

**Proposition 9.1.** *Let $I \subsetneq R$ be an ideal.*

(1) *If $\sqrt{(0)} \subseteq I$ and $I$ is weakly 2-nil primary, then $I$ is weakly 2-absorbing primary.*

(2) *If $I \subseteq \sqrt{(0)}$ and $I$ is weakly 2-absorbing primary, then $I$ is weakly 2-nil primary.*

*Proof.* For (1), let $0 \neq abc \in I$. If $ab \in \sqrt{(0)}$, then $ab \in I$ because $\sqrt{(0)} \subseteq I$. The other two alternatives are already the same as those in the weakly 2-absorbing primary condition.

For (2), let $0 \neq abc \in I$. If $ab \in I$, then $ab \in \sqrt{(0)}$ because $I \subseteq \sqrt{(0)}$. Again, the other two alternatives are unchanged. Thus $I$ is weakly 2-nil primary. □

In particular, if $I = \sqrt{(0)}$, then the weakly 2-nil primary and weakly 2-absorbing primary conditions coincide.

# 10 Conclusion

In this paper, we introduced and studied weakly 2-nil primary ideals, obtained by imposing the 2-nil primary condition only on nonzero triple products. The role of zero products was clarified through 2-nil primary zero triples, which provide an exact criterion for when the weak and non-weak conditions differ.

We studied the behavior of this class under ring homomorphisms, quotient rings, localization, and direct products. For an integral domain, an ideal $I \subsetneq R$ is weakly 2-nil primary exactly when $\sqrt{I}$ is prime. Hence these ideals coincide with quasi-primary ideals, and the weak and non-weak 2-nil primary conditions agree. For a principal ideal domain, the complete list is (0) and $(p^m)$, $m \geq 1$, where $p$ is a prime element.

We also obtained a complete classification in $\mathbb{Z}_n$. Writing $n = p_1^{k_1} \cdots p_s^{k_s}$, every proper ideal satisfies the weakly 2-nil primary condition when $s \leq 2$. If $s \geq 3$, the ideals in this class are exactly

$$(0) \quad \text{and} \quad (p_i^t), \qquad 1 \leq i \leq s, \quad 1 \leq t \leq k_i,$$

and their number is $1 + \sum_{i=1}^{s} k_i$.

Finally, the weakly 2-nil primary condition was compared with the closely related notion of weakly 2-absorbing primary ideals.

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