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Finite Groups with Consecutive Character Codegrees

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Abstract

Let G be a finite group and $\mathrm{Irr}(G)$ be the set of irreducible characters of G. The codegree of an irreducible character χ of the group G is defined as $\mathrm{cod}(\chi) = |G : \ker(\chi)|/\chi(1)$. In this paper, we consider the finite group G whose all irreducible character codegrees are consecutive integers $1, 2, 3, ..., k-1, k$. We prove that $k \leq 3$ and G is an elementary abelian group or a Frobenius group.

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1 Introduction and Preliminaries

For a finite group G, let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G and $\mathrm{cd}(G) = \{\chi(1) | \chi \in \mathrm{Irr}(G)\}$. The finite groups whose all nonlinear irreducible character degrees are consecutive integers were firstly investigated by Huppert in section 32 of [1] and he proved that "Suppose that

$\mathrm{cd}(G) = \{1, 2, 3, ..., k-1, k\}$. If $k > 4$, then $k = 6$ and $G = HZ(G)$ where $H \cong \mathrm{SL}(2, 5)$". After then, Qian considered general finite groups with above conditions in [6].

For an irreducible character $\chi \in \mathrm{Irr}(G)$, the codegree of character χ is defined as $\mathrm{cod}(\chi) = |G : \ker(\chi)|/\chi(1)$ in [5]. Set $\mathrm{cod}(G) = \{\mathrm{cod}(\chi) | \chi \in \mathrm{Irr}(G)\}$. In this paper, we consider the finite groups such that all irreducible character codegrees are consecutive integers and prove the following result.

Theorem 1.1. *Let G be a finite group. If $\mathrm{cod}(G) = \{1, 2, 3, ..., k-1, k\}$ ($k > 1$), then $k = 2$ or 3. If $k = 2$, G is an elementary abelian 2-group; if $k = 3$ and G is a Frobenius group such that (1) the Frobenius kernel is an elementary abelian 3-group, (2) Frobenius complement has order 2.*

The notation in this paper is standard, see [3]. Let G be a finite group and N be a normal subgroup of G. For any $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$, $\chi|_N$ is the restriction of χ to subgroup N and θ^G is the induced character of θ to G. For any integer n, denote by $\pi(n)$ the set of all prime divisors of n.

2 Main Results

These are the main results of the paper.

For the set $\mathrm{cod}(G)$, the codegree graph $\Gamma(G)$ is defined as follows: The vertex set $V(G)$ of $\Gamma(G)$ consists of all primes dividing some integer in $\mathrm{cod}(G)$. There is an edge between distinct primes $p, q \in V(G)$ if pq divides some integer in $\mathrm{cod}(G)$.

Lemma 2.1. *Let G be a finite group with codegree graph $\Gamma(G)$. Then $\Gamma(G)$ has at most two connected components.*

Proof. The Lemma can be obtained from the Theorem E of [7]. □

The relationship between element order and character codegree is studied in [4] and [8].

Lemma 2.2. *Let G be a finite group. For any element $g \in G$, there exists $\chi \in \mathrm{Irr}(G)$ such that $\pi(|g|) \subseteq \pi(\mathrm{cod}(\chi))$.*

The following result is very useful in the proof of the main result of this paper.

Lemma 2.3. *Let G be a finite group and N be a minimal normal subgroup of G such that N is abelian. For any nontrivial $\lambda \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G|\lambda)$, then $|N| \mid \mathrm{cod}(\chi)$.*

Proof. For any $1 \neq \lambda \in \mathrm{Irr}(N)$ and any $\chi \in \mathrm{Irr}(G|\lambda)$, we have that $\chi \in \mathrm{Irr}(G/\ker(\chi))$ and $N \cap \ker(\chi) = 1$ since N is a minimal normal subgroup of G. Because $N\ker(\chi)/\ker(\chi) \cong N/N \cap \ker(\chi) \cong N$, λ can be viewed as a character of $N\ker(\chi)/\ker(\chi)$ and so $\chi(1) = \frac{\chi(1)}{\lambda(1)} \mid |G : N\ker(\chi)|$. Thus $\mathrm{cod}(\chi) = \frac{|G:\ker(\chi)|}{\chi(1)} = \frac{|G:N\ker(\chi)|}{\chi(1)} \frac{|N\ker(\chi)|}{|\ker(\chi)|}$ is divided by $|N|$. □

Next we show several sets of some integers can not be codegree sets of solvable groups.

Lemma 2.4. *The set* $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ *can not be codegree set of solvable group.*

Proof. Assume the Lemma is not true and let G be a counterexample. Choose a minimal normal p-subgroup M of G. If $p = 5$ or 7, then $|M| = p$ by Lemma 2.3. Since $|Aut(M)| = p - 1$, we have that $35 \mid |C_G(M)|$ and there exists an element with order 35. So 35 divides a codegree in $\mathrm{cod}(G)$ by Lemma 2.2. A contradiction. If $p = 3$, we have that $|M| = 3$ or 9 by Lemma 2.3. Since $7 \nmid |Aut(M)|$, we have that $7 \mid |C_G(M)|$ and there exists an element with order 21. So 21 divides a codegree in $\mathrm{cod}(G)$ by Lemma 2.2. A contradiction. If $p = 2$, we have that $|M| = 2, 4$ or 8 by Lemma 2.3. If $|M| = 2$ or 4, we have that $7 \mid |C_G(M)|$ for $7 \nmid |Aut(M)|$ and there exists an element with order 14. So 14 divides a codegree in $\mathrm{cod}(G)$ by Lemma 2.2. A contradiction. If $|M| = 8$, then $\mathrm{cod}(\chi) = 8$ for any $\chi \in \mathrm{Irr}(G|M)$ by Lemma 2.3. Let P be a Sylow 3-subgroup of G. So P acts Frobeniusly on M by Theorem A of [7] and it implies $|P| \mid |M| - 1 = 7$, a contradiction. □

Lemma 2.5. *The set* $\{1, 2, 3, 4, 5, 6\}$ *can not be codegree set of solvable group.*

Proof. Assume the Lemma is not true and let G be a counterexample. Choose a minimal normal p-subgroup M of G. If $p = 5$, then $|M| = 5$ by Lemma 2.3. Since $|Aut(M)| = 4$, we have that $3 \mid |C_G(M)|$ and there exists an element with order 15. So 15 divides a codegree in $\mathrm{cod}(G)$ by Lemma 2.2. A contradiction. If $p = 3$, we have that $|M| = 3$ by Lemma 2.3. Since $5 \nmid |Aut(M)|$, we have that $5 \mid |C_G(M)|$ and there exists an element with order 15. So 15 divides a codegree in $\mathrm{cod}(G)$ by Lemma 2.2. A contradiction. If $p = 2$, we have that $|M| = 2$ or 4. Since $5 \nmid |Aut(M)|$, we have that $5 \mid |C_G(M)|$ and there exists an element with order 10. So 10 divides a codegree in $\mathrm{cod}(G)$. A contradiction. □

Lemma 2.6. *The set* $\{1, 2, 3, 4\}$ *can not be codegree set of solvable group.*

Proof. Assume the Lemma is not true and let G be a minimal counterexample. Choose a minimal normal p-subgroup M of G. If $p = 3$, then $|M| = 3$. Since $|Aut(M)| = 2$ and $4 \mid |G|$, we have that $2 \mid |C_G(M)|$ and there exists an element with order 6. So 6 divides a codegree in $\mathrm{cod}(G)$. A contradiction. If

$p = 2$, we have that $|M| = 2$ or 4. If $3 \mid |C_G(M)|$ and there exists element with order 6. So 6 divides one codegree in $\mathrm{cod}(G)$. A contradiction. Hence $C_G(M)$ is a 2-subgroup. Furthermore $|M| = 4$ and $Aut(M) \cong S_3$. If $|G/C_G(M)| = 3$, then $3 \mid |G/G'|$. Choose $\chi \in \mathrm{Irr}(G)$ with $\mathrm{cod}(\chi) = 2$. Then $\chi(1) = 1$ for $\chi(1) < \mathrm{cod}(\chi)$ and $2 \mid |G/G'|$. So $6 \mid |G/G'|$ and $6 \in \mathrm{cod}(G)$, a contradiction. So $G/C_G(M) \cong S_3$ and $\{1, 2, 3\} \subseteq \mathrm{cod}(G/M)$. If $M < C_G(M)$, then G/M has a normal 2-subgroup $C_G(M)/M$ and G/M can not be a Frobenius group, which means $\mathrm{cod}(G/M) = \{1, 2, 3, 4\}$ by Theorem 3.4 of [2], a contradiction with that G is a minimal counterexample. So $M = C_G(M)$ which implies M is the unique minimal normal subgroup of G. Choose $1 \neq \lambda \in \mathrm{Irr}(M)$ and $\chi \in \mathrm{Irr}(G|\lambda)$. Then $\chi(1) \mid |G/M| = 6$ and hence $\chi(1) = 2$ or 3. Because of $ker(\chi) = 1$, we have $\mathrm{cod}(\chi) = 12$ or 8, a contradiction. □

Now we give the proof of Theorem 1.1.

Proof. If $k \geq 11$, then there are two different primes p, q such that $k/2 < p, q \leq k$ by [9]. So codegree graph $\Gamma(G)$ has at least three connected components, a contradiction with Lemma 2.1. Hence $k \leq 10$ and G is solvable by Theorem 1 of [10]. If $k = 5, 7, 8$ or 9, the codegree graph $\Gamma(G)$ has three connected components, a contradiction. By Lemmas 2.4, 2.5 and 2.6, k cannot be $4, 6, 10$. So $k = 2$ or 3, and the Theorem is true from Theorems 3.1, 3.4 and 3.5 of [2]. □

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