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Some *abc*-Properties of the Generalized Pell's Equations $x^2 - D \cdot y^2 = \pm N$ and $x^2 - D \cdot y^2 = N^2$

Janis Kuzmanis

Riga, Latvia
yanisku@gmail.com

Abstract

Primary *abc*-triples, formed by the set of roots for the generalized Pell's equations $x^2 - D \cdot y^2 = \pm N$ (with $N > 2$), induce formation of secondary *abc*-triples in the set of roots for equations $x^2 - D \cdot y^2 = N^2$.

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1 Introduction

The proposed article is planned as continuation of [1], where necessary background knowledge concerning radicals of numbers, *abc*-conjecture, continuants and continued fractions was summarized. Mentioned [1] proves *abc*-triple induction in Pell's equations $x^2 - D \cdot y^2 = N$ with $N = \pm 1$ and ± 2, but [2] shows similar rules for $N = \pm 4$.

The main result of the given article is the proof of formation of *abc*-triples in the set of roots for the generalized Pell's equations $x^2 - D \cdot y^2 = N^2$ with odd $N > 2$, which is induced by corresponding *abc*-triples from roots of the equations $x^2 - D \cdot y^2 = \pm N$. This induction partially preserves also in the situations with even N values, where reduction to coprime terms become necessary for obtaining of correct *abc*-equations.

Throughout this article we preserve notations, introduced in [1] and [2]: π for palindromic components in continuants and continued fractions; shortening of longer repeating continuant expressions accordingly to their number of

π units, so $K(a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi, 2a_0, \pi)$ become $K(a_0, 5\pi)$; manipulation with symbols $K(\rho, \omega)/K(\omega)$ for fundamental roots of generalized Pell's equations.

2 Conjugation

2.1 Fundamental roots

If we have an ordinary negative Pell's equation $x^2 - D \cdot y^2 = -1$ with fundamental roots $K(a_0, \pi)/K(\pi)$ and two generalized Pell's equations

- $x^2 - D \cdot y^2 = -N$ with fundamental roots $\pm K(\rho, \omega)/K(\omega)$, and
- $x^2 - D \cdot y^2 = +N$ with fundamental roots $\pm K(\rho', \omega')/K(\omega')$,

then for all natural $N > 2$ the following relations exist:

$$\begin{cases} K(\rho', \omega') = -K(\rho, \omega) \cdot K(a_0, \pi) + D \cdot K(\omega) \cdot K(\pi) \\ K(\omega') = -K(\rho, \omega) \cdot K(\pi) + K(\omega) \cdot K(a_0, \pi), \end{cases} \quad (1)$$

$$\begin{cases} K(\rho, \omega) = K(\rho', \omega') \cdot K(a_0, \pi) - D \cdot K(\omega') \cdot K(\pi) \\ K(\omega) = K(\rho', \omega') \cdot K(\pi) - K(\omega') \cdot K(a_0, \pi). \end{cases} \quad (2)$$

From (1):

$$\begin{aligned} & K^2(\rho', \omega') - D \cdot K^2(\omega') \\ & = K^2(\rho, \omega) \cdot K^2(a_0, \pi) - 2D \cdot K(\rho, \omega) \cdot K(\omega) \cdot K(a_0, \pi) \cdot K(\pi) \\ & \quad + D^2 \cdot K^2(\omega) \cdot K^2(\pi) - D \cdot [K^2(\rho, \omega) \cdot K^2(\pi) \\ & \quad\quad - 2K(\rho, \omega) \cdot K(\omega) \cdot K(a_0, \pi) \cdot K(\pi) + K^2(\omega) \cdot K^2(a_0, \pi)] \\ & = K^2(\rho, \omega) \cdot [K^2(a_0, \pi) - D \cdot K^2(\pi)] - D \cdot K^2(\omega) \cdot [K^2(a_0, \pi) - D \cdot K^2(\pi)] \\ & = [K^2(a_0, \pi) - D \cdot K^2(\pi)] \cdot [K^2(\rho, \omega) - D \cdot K^2(\omega)] = (-1) \cdot (-N) = +N. \end{aligned}$$

Analogously from (2) we get $K^2(\rho, \omega) - D \cdot K^2(\omega) = ... = -N$.

Thus, if there exists solution for the ordinary negative Pell's equation with given particular D value, then fundamental solutions of the corresponding generalized Pell's equations $x^2 - D \cdot y^2 = -N$ and $x^2 - D \cdot y^2 = +N$ are conjugated. Henceforth we will call such system of three Pell's equations as corrresponding to Property A. Situation with $N = 2$ is different, see [1].

2.2 Higher analogues

Under Property A conditions for longer palindromic sequences the following expressions are valid ($n = 1, 2, 3, ...$).

$$\begin{cases} K(a_0, n\pi, a_0 + \rho, \omega) = K(\rho, \omega) \cdot K(a_0, n\pi) + D \cdot K(\omega) \cdot K(n\pi) \\ K(n\pi, a_0 + \rho, \omega) = K(\rho, \omega) \cdot K(n\pi) + K(\omega) \cdot K(a_0, n\pi). \end{cases} \quad (3)$$

$$\begin{cases} K(a_0, n\pi, a_0+\rho', \omega') &= K(\rho',\omega')\cdot K(a_0,n\pi) + D\cdot K(\omega')\cdot K(n\pi) \\ K(n\pi, a_0+\rho', \omega') &= K(\rho',\omega')\cdot K(n\pi) + K(\omega')\cdot K(a_0,n\pi). \end{cases} \tag{4}$$

$$\begin{cases} K(a_0, n\pi, a_0+\rho, \omega) &= K(\rho',\omega')\cdot K(a_0,(n+1)\pi) \\ & \quad -D\cdot K(\omega')\cdot K((n+1)\pi) \\ K(n\pi, a_0+\rho, \omega) &= K(\rho',\omega')\cdot K((n+1)\pi) \\ & \quad -K(\omega')\cdot K(a_0,(n+1)\pi). \end{cases} \tag{5}$$

$$\begin{cases} K(a_0, n\pi, a_0+\rho', \omega') &= -K(\rho,\omega)\cdot K(a_0,(n+1)\pi) \\ & \quad +D\cdot K(\omega)\cdot K((n+1)\pi) \\ K(n\pi, a_0+\rho', \omega') &= -K(\rho,\omega)\cdot K((n+1)\pi) \\ & \quad +K(\omega)\cdot K(a_0,(n+1)\pi). \end{cases} \tag{6}$$

Proofs of (3) and (4) are by simple splitting, but for (5) and (6) – analogously to proof of (1).

2.3 Ambiguity and conjugation

Fundamental solutions $K(\rho,\omega) + K(\omega)\cdot\sqrt{D}$ and $K(\rho',\omega') + K(\omega')\cdot\sqrt{D}$ are defined as the least non-negative values of the given associativity class, satisfying the corresponding generalized Pell's equations [4]. For ambiguous classes zero fundamental solutions can occur. We deduce for the Property A system:

- for all positive $N = k^2$ we have ambiguous fundamental solutions in the form $(k, 0)$ or $K(\rho',\omega') = k$, $K(\omega') = 0$, where $k = 2, 3, 4, ...$ As $K(\omega') = 0$, conjugation relations (2) give ambiguous fundamental solutions

$$\begin{cases} K(\rho,\omega) &= K(\rho',\omega')\cdot K(a_0,\pi) \\ K(\omega) &= K(\rho',\omega')\cdot K(\pi). \end{cases} \tag{7}$$

- for all negative $N = k^2\cdot D$ we have ambiguous fundamental solutions in the form $(0, k)$ or $K(\rho,\omega) = 0$, $K(\omega) = k$, where $k = 1, 2, 3, ...$ As $K(\rho,\omega) = 0$, conjugation relations (1) give ambiguous fundamental solutions

$$\begin{cases} K(\rho',\omega') &= D\cdot K(\pi)\cdot K(\omega) \\ K(\omega') &= K(a_0,\pi)\cdot K(\omega). \end{cases} \tag{8}$$

Example 2.1. *Table 1 shows ambiguous fundamental solutions for Pell's equations $x^2 - 13y^2 = \pm N$ with different N values. For $D = 13$ we have* $K(a_0,\pi) = 18$, $K(\pi) = 5$, *so the reader can test the relations (7) and (8). A lot of non-ambiguous fundamental solutions for the same equations and different N values are not shown in Table 1.*

Table 1.

N	$K(\rho,\omega)/K(\omega)$	$K(\rho',\omega')/K(\omega')$
4	36/10	2/0
9	54/15	3/0
13	0/1	65/18
16	72/20	4/0
25	90/25	5/0
36	108/30	6/0
49	126/35	7/0
52	0/2	130/36
...	...	...

3 Squaring: from N to N^2

3.1 Overview and 10 versions

We have $\pm K(\rho,\omega)/K(\omega)$ and $\pm K(\rho',\omega')/K(\omega')$ as fundamental solutions for generalized Pell's equations $x^2 - D\cdot y^2 = -N$ and $x^2 - D\cdot y^2 = +N$, corresponding to Property A system. Then fundamental solutions $\pm K(\sigma,\tau)/K(\tau)$ and $\pm K(\sigma',\tau')/K(\tau')$ of generalized Pell's equations $x^2 - D\cdot y^2 = \pm N^2$ can be obtained by multiplication. In total there are 10 possible versions.

1. $$[+K(\rho',\omega') + \sqrt{D}\cdot K(\omega')]\cdot[+K(\rho',\omega') + \sqrt{D}\cdot K(\omega')]$$
$$= K^2(\rho',\omega') + D\cdot K^2(\omega') + \sqrt{D}\cdot 2K(\rho',\omega')\cdot K(\omega').$$

2. $$[-K(\rho',\omega') + \sqrt{D}\cdot K(\omega')]\cdot[+K(\rho',\omega') + \sqrt{D}\cdot K(\omega')]$$
$$= -K^2(\rho',\omega') + D\cdot K^2(\omega') = -N.$$

3. $$[-K(\rho',\omega') + \sqrt{D}\cdot K(\omega')]\cdot[-K(\rho',\omega') + \sqrt{D}\cdot K(\omega')]$$
$$= K^2(\rho',\omega') + D\cdot K^2(\omega') - \sqrt{D}\cdot 2K(\rho',\omega')\cdot K(\omega').$$

4. $$[+K(\rho',\omega') + \sqrt{D}\cdot K(\omega')]\cdot[+K(\rho,\omega) + \sqrt{D}\cdot K(\omega)]$$
$$= K(\rho',\omega')\cdot K(\rho,\omega) + D\cdot K(\omega')\cdot K(\omega) + \sqrt{D}\cdot[K(\rho,\omega)\cdot K(\omega') + K(\rho',\omega')\cdot K(\omega)].$$

5. $$[+K(\rho',\omega') + \sqrt{D}\cdot K(\omega')]\cdot[-K(\rho,\omega) + \sqrt{D}\cdot K(\omega)]$$
$$= -K(\rho',\omega')\cdot K(\rho,\omega) + D\cdot K(\omega')\cdot K(\omega) + \sqrt{D}\cdot[K(\rho',\omega')\cdot K(\omega) - K(\rho,\omega)\cdot K(\omega')].$$

6. $[-K(\rho',\omega')+\sqrt{D}\cdot K(\omega')]\cdot[+K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]$
$= -K(\rho',\omega')\cdot K(\rho,\omega)+D\cdot K(\omega')\cdot K(\omega)+\sqrt{D}\cdot[K(\rho,\omega)\cdot K(\omega')-K(\rho',\omega')\cdot K(\omega)].$

7. $[-K(\rho',\omega')+\sqrt{D}\cdot K(\omega')]\cdot[-K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]$
$= K(\rho',\omega')\cdot K(\rho,\omega)+D\cdot K(\omega')\cdot K(\omega)-\sqrt{D}\cdot[K(\rho',\omega')\cdot K(\omega)+K(\rho,\omega)\cdot K(\omega')].$

8. $[+K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]\cdot[+K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]$
$= K^2(\rho,\omega)+D\cdot K^2(\omega)+\sqrt{D}\cdot 2K(\rho,\omega)\cdot K(\omega).$

9. $[-K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]\cdot[+K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]$
$= -K^2(\rho,\omega)+D\cdot K^2(\omega)=N.$

10. $[-K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]\cdot[-K(\rho,\omega)+\sqrt{D}\cdot K(\omega)]$
$= K^2(\rho,\omega)+D\cdot K^2(\omega)-\sqrt{D}\cdot 2K(\rho,\omega)\cdot K(\omega).$

Versions **2** and **9** do not give summands, including $\sqrt{D}$, they are ambiguous roots of type $(k,0)$. Both factors in the left side of equation for version **9** are associated to $-N$, so their multiple is associated to $\pm K(\sigma',\tau')/K(\tau')$ and is ambiguous: $K(\sigma',\tau')=\pm N$, $K(\tau')=0$. Similarly for version **2.**

Versions **4** and **7** have summands, including $\sqrt{D}$. Factors in the left side of equations for these versions are associated to $-N$ and $+N$, so their multiple is associated to $\pm K(\sigma,\tau)/K(\tau)$. We have for both versions:

$$\begin{aligned}K(\tau) &= K(\rho,\omega)\cdot K(\omega')+K(\rho',\omega')\cdot K(\omega)\\ &= -K^2(\rho,\omega)\cdot K(\pi)+K(a_0,\pi)\cdot K(\rho,\omega)\cdot K(\omega)\\ &\quad -K(a_0,\pi)\cdot K(\rho,\omega)\cdot K(\omega)+D\cdot K^2(\omega)\cdot K(\pi)\\ &= -K(\pi)\cdot[K^2(\rho,\omega)-D\cdot K^2(\omega)]=N\cdot K(\pi).\end{aligned}$$

$$\begin{aligned}K(\sigma,\tau) &= K(\rho,\omega)\cdot K(\rho',\omega')+D\cdot K(\omega')\cdot K(\omega)\\ &= -K^2(\rho,\omega)\cdot K(a_0,\pi)+D\cdot K(\pi)\cdot K(\rho,\omega)\cdot K(\omega)\\ &\quad -D\cdot K(\pi)\cdot K(\rho,\omega)\cdot K(\omega)+D\cdot K^2(\omega)\cdot K(a_0,\pi)\\ &= -K(a_0,\pi)\cdot[K^2(\rho,\omega)-D\cdot K^2(\omega)]=N\cdot K(a_0,\pi).\end{aligned}$$

Clearly they are ambiguous conjugates, corresponding to relations (7). Significantly, so obtained $K(\sigma,\tau)$ and $K(\tau)$ are not coprime.

Factors in the left side of equations for versions **5** and **6** are associated to $-N$ and $+N$, so their multiple is associated to $\pm K(\sigma,\tau)/K(\tau)$. But then for roots $\pm K(\sigma',\tau')/K(\tau')$ we have uncertainty – versions **1** and **3,** or versions **8** and **10.** Suppose that roots, obtained from version **5,** are positive:

$$\left\{\begin{array}{rcl} K(\sigma,\tau) & = & -K(\rho',\omega')\cdot K(\rho,\omega)+D\cdot K(\omega')\cdot K(\omega)>0 \\ K(\tau) & = & K(\rho',\omega')\cdot K(\omega)-K(\rho,\omega)\cdot K(\omega')>0. \end{array}\right. \quad (9)$$

Then conjugation relations (1) gives us the following:

$$\begin{aligned} K(\tau') &= -K(\sigma,\tau)\cdot K(\pi)+K(\tau)\cdot K(a_0,\pi) \\ &= K(\rho',\omega')\cdot K(\rho,\omega)\cdot K(\pi)-D\cdot K(\omega')\cdot K(\omega)\cdot K(\pi) \\ &\quad + K(\rho',\omega')\cdot K(\omega)\cdot K(a_0,\pi)-K(\rho,\omega)\cdot K(\omega')\cdot K(a_0,\pi) \\ &= K(\rho,\omega)\cdot \underbrace{[-K(\omega')\cdot K(a_0,\pi)+K(\rho',\omega')\cdot K(\pi)]}_{K(\omega)} \\ &\quad + K(\omega)\cdot \underbrace{[K(\rho',\omega')\cdot K(a_0,\pi)-D\cdot K(\omega')\cdot K(\pi)]}_{K(\rho,\omega)} = 2K(\rho,\omega)\cdot K(\omega). \end{aligned}$$

$$\begin{aligned} K(\sigma',\tau') &= -K(\sigma,\tau)\cdot K(a_0,\pi)+D\cdot K(\tau)\cdot K(\pi) \\ &= K(\rho',\omega')\cdot K(\rho,\omega)\cdot K(a_0,\pi)-D\cdot K(\omega')\cdot K(\omega)\cdot K(a_0,\pi) \\ &\quad + D\cdot K(\rho',\omega')\cdot K(\omega)\cdot K(\pi)-D\cdot K(\rho,\omega)\cdot K(\omega')\cdot K(\pi) \\ &= K(\rho,\omega)\cdot \underbrace{[K(\rho',\omega')\cdot K(a_0,\pi)-D\cdot K(\omega')\cdot K(\pi)]}_{K(\rho,\omega)} \\ &\quad + D\cdot K(\omega)\cdot \underbrace{[K(\rho',\omega')\cdot K(\pi)-K(\omega')\cdot K(a_0,\pi)]}_{K(\omega)} \\ &= K^2(\rho,\omega)+D\cdot K^2(\omega). \end{aligned}$$

So versions **8** and **10** are valid in this case.

If $K(\sigma,\tau)$ value from version **5** in (9) is negative, we take it's opposite:

$$\left\{\begin{array}{rcl} K(\sigma,\tau) & = & K(\rho',\omega')\cdot K(\rho,\omega)-D\cdot K(\omega')\cdot K(\omega)>0 \\ K(\tau) & = & K(\rho',\omega')\cdot K(\omega)-K(\rho,\omega)\cdot K(\omega'). \end{array}\right.$$

Then, by analogous calculations, we get $K(\tau')=2K(\rho',\omega')\cdot K(\omega')$ and $K(\sigma',\tau')=K^2(\rho',\omega')+D\cdot K^2(\omega')$. Now versions **1** and **3** are valid.

Obtained result can be formulated as separate

3.2 Criterion

Under Property A conditions the following relations exist:

- If $D\cdot K(\omega')\cdot K(\omega)>K(\rho',\omega')\cdot K(\rho,\omega)$, then

$$\left\{\begin{array}{rcl} K(\sigma',\tau') & = & K^2(\rho,\omega)+D\cdot K^2(\omega) \\ K(\tau') & = & 2K(\rho,\omega)\cdot K(\omega). \end{array}\right. \quad (10)$$

- If $D \cdot K(\omega') \cdot K(\omega) < K(\rho', \omega') \cdot K(\rho, \omega)$, then

$$\begin{cases} K(\sigma', \tau') &= K^2(\rho', \omega') + D \cdot K^2(\omega') \\ K(\tau') &= 2K(\rho', \omega') \cdot K(\omega'). \end{cases} \tag{11}$$

Thus fundamental non-ambiguous solutions $\pm K(\sigma', \tau')/K(\tau')$ of generalized Pell's equations $x^2 - D \cdot y^2 = +N^2$ can be obtained. Corresponding ambiguous solutions $K(\sigma', \tau') = k$, $K(\tau') = 0$ of these equations were already mentioned.

Example 3.1. *For $D = 13$ we have $K(a_0, \pi) = 18$, $K(\pi) = 5$. If $N = 3$, we have fundamental roots $\pm K(\rho, \omega) = 7$, $K(\omega) = 2$ and $\pm K(\rho', \omega') = 4$, $K(\omega') = 1$. Criterion $13 \cdot 2 \cdot 1 < 7 \cdot 4$, therefore fundamental roots $K(\sigma', \tau')/K(\tau')$ of generalized Pell's equation $x^2 - D \cdot y^2 = 9$ are formed from $K(\rho', \omega')/K(\omega')$. We have $K(\sigma', \tau') = 4^2 + 13 \cdot 1^2 = 29$ and $K(\tau') = 2 \cdot 4 \cdot 1 = 8$. Conjugation relations or versions* ***5/6*** *give roots $K(\sigma, \tau) = 2$, $K(\tau) = 1$. Calculations of ambiguous roots are trivial.*

Thus we have for generalized Pell's equation with $D = 13$ and $N = -9$ fundamental non-ambiguous roots $\pm 2, 1$ and ambiguous $54, 15$, but for generalized Pell's equation with $D = 13$ and $N = 9$ fundamental non-ambiguous roots are $\pm 29, 8$ and ambiguous $3, 0$.

Remark. *With composite N values more than one pair of $\pm K(\rho, \omega)/K(\omega)$ and $\pm K(\rho', \omega')/K(\omega')$ fundamental roots can occur. Calculations, based on versions* ***1–10*** *and criterion, must be done with conjugation pairs, connected by relations (1) and (2).*

Example 3.2. *Again $D = 13$ with $K(a_0, \pi) = 18$, $K(\pi) = 5$. If $N = 51$, we have two pairs of conjugates. Testing of their conjugation and calculations of new roots are left to concerned reader.*

1. $\pm K(\rho, \omega) = 1$, $K(\omega) = 2$ *and* $\pm K(\rho', \omega') = 112$, $K(\omega') = 31$,
2. $\pm K(\rho, \omega) = 79$, $K(\omega) = 22$ *and* $\pm K(\rho', \omega') = 8$, $K(\omega') = 1$.

The first pair has criterion $>$, so fundamental roots $\pm K(\sigma', \tau')/K(\tau')$ of the generalized Pell's equation $x^2 - D \cdot y^2 = 51^2$ will be formed from $K(\rho, \omega)/K(\omega)$ values, we get $K(\sigma', \tau') = 53$, $K(\tau') = 4$. Conjugation relations or versions ***5/6*** *give fundamental roots of the generalized Pell's equation $x^2 - D \cdot y^2 = -51^2$ as $K(\sigma, \tau) = 694$, $K(\tau) = 193$. Calculations of ambiguous roots are trivial.*

The second pair has opposite criterion, therefore roots $\pm K(\sigma', \tau')/K(\tau')$ for this pair will be formed from $K(\rho', \omega')/K(\omega')$ values. Now $K(\sigma', \tau') = 77$, $K(\tau') = 16$. Conjugation relations or versions ***5/6*** *give roots $K(\sigma, \tau) = 346$, $K(\tau) = 97$. Calculations of ambiguous roots again are trivial.*

As initial values $K(\rho, \omega)/K(\omega)$ and $K(\rho', \omega')/K(\omega')$ were coprime, we also get coprime roots $\pm K(\sigma', \tau')/K(\tau')$ and $K(\sigma, \tau)/K(\tau)$; this comes from the relations (1) and (2), as well as from (10) and (11). There are additional non-coprime fundamental roots for this Pell's equation with $D = 13$ and $N = \pm 51^2$, *whose formation involves mixing of both pairs, but their investigation is not the goal of this article. Here we restrict our interests with formation of coprime roots, because our final aim is hereditary properties of abc-triples.*

3.3 Criterion – higher roots

Relations, analogous to versions **1–10,** can be written for roots with longer palindromic sequences, giving similar uncertainty and need for criterion. We denote $D \cdot K(\omega') \cdot K(\omega) = A$ and $K(\rho', \omega') \cdot K(\rho, \omega) = B$ in our criterion expression. Then by increment π we have:

$$\begin{aligned} A &= D \cdot K(\pi, a_0 + \rho', \omega') \cdot K(\pi, a_0 + \rho, \omega) = ... \\ &= D \cdot K^2(a_0, \pi) \cdot K(\omega') \cdot K(\omega) \\ &\quad + D \cdot K(a_0, \pi) \cdot K(\pi) \cdot [K(\rho', \omega') \cdot K(\omega) + K(\omega') \cdot K(\rho, \omega)] \\ &\quad\quad + D \cdot K^2(\pi) \cdot K(\rho', \omega') \cdot K(\rho, \omega). \end{aligned}$$

$$\begin{aligned} B &= K(a_0, \pi, a_0 + \rho', \omega') \cdot K(a_0, \pi, a_0 + \rho, \omega) = ... \\ &= D^2 \cdot K^2(\pi) \cdot K(\omega') \cdot K(\omega) \\ &\quad + D \cdot K(a_0, \pi) \cdot K(\pi) \cdot [K(\rho', \omega') \cdot K(\omega) + K(\omega') \cdot K(\rho, \omega)] \\ &\quad\quad + K^2(a_0, \pi) \cdot K(\rho', \omega') \cdot K(\rho, \omega). \end{aligned}$$

Now the difference of the first summands in new A and B expressions becomes

$$D \cdot \underbrace{[K^2(a_0, \pi) - D \cdot K^2(\pi)]}_{=-1} \cdot K(\omega') \cdot K(\omega) = -D \cdot K(\omega') \cdot K(\omega). \tag{12}$$

The difference of the second summands is zero, but the difference of the third summands is

$$[D \cdot K^2(\pi) - K^2(a_0, \pi)] \cdot K(\rho', \omega') \cdot K(\rho, \omega) = K(\rho', \omega') \cdot K(\rho, \omega). \tag{13}$$

That means $A - B = K(\rho', \omega') \cdot K(\rho, \omega) - D \cdot K(\omega') \cdot K(\omega)$ – our criterion changed it's direction. Each increment by π will change the signs in the differences of the first and the third summands, given by expressions in square brackets for (12) and (13). As the result – criterion preserves it's direction for even number of π increments, but changes direction for odd number of π increments.

3.4 Divisibility relations of higher roots

Here we will produce a set of divisibility relations between higher roots for generalized Pell's equations $x^2 - D \cdot y^2 = \pm N$ and $x^2 - D \cdot y^2 = \pm N^2$.

1. Suppose that criterion is >, so relations (10) are valid. Then from (3):

$$\begin{aligned}
&2K(a_0,\pi,a_0+\rho,\omega)\cdot K(\pi,a_0+\rho,\omega)=\\
&=2[K(\rho,\omega)\cdot K(a_0,\pi)+D\cdot K(\omega)\cdot K(\pi)]\cdot[K(\rho,\omega)\cdot K(\pi)+K(\omega)\cdot K(a_0,\pi)]\\
&=2[K^2(\rho,\omega)\cdot K(a_0,\pi)\cdot K(\pi)+D\cdot K^2(\omega)\cdot K(a_0,\pi)\cdot K(\pi)\\
&\qquad+D\cdot K^2(\pi)\cdot K(\rho,\omega)\cdot K(\omega)+K^2(a_0,\pi)\cdot K(\rho,\omega)\cdot K(\omega)]\\
&=2K(a_0,\pi)\cdot K(\pi)\cdot[K^2(\rho,\omega)+D\cdot K^2(\omega)]\\
&\qquad+2K(\rho,\omega)\cdot K(\omega)\cdot[K^2(a_0,\pi)+D\cdot K^2(\pi)]\\
&=K(2\pi)\cdot K(\sigma',\tau')+K(\tau')\cdot K(a_0,2\pi)=K(2\pi,a_0+\sigma',\tau').
\end{aligned}$$

$$\begin{aligned}
&K^2(a_0,\pi,a_0+\rho,\omega)+D\cdot K^2(\pi,a_0+\rho,\omega)=\\
&=[D\cdot K(\pi)\cdot K(\omega)+K(a_0,\pi)\cdot K(\rho,\omega)]^2\\
&\qquad+D\cdot[K(a_0,\pi)\cdot K(\omega)+K(\pi)\cdot K(\rho,\omega)]^2\\
&=D^2\cdot K^2(\pi)\cdot K^2(\omega)+2D\cdot K(\pi)\cdot K(a_0,\pi)\cdot K(\omega)\cdot K(\rho,\omega)\\
&\qquad+K^2(a_0,\pi)\cdot K^2(\rho,\omega)+D\cdot K^2(a_0,\pi)\cdot K^2(\omega)\\
&\qquad+2D\cdot K(\pi)\cdot K(a_0,\pi)\cdot K(\omega)\cdot K(\rho,\omega)+D\cdot K^2(\pi)\cdot K^2(\rho,\omega)\\
&=4D\cdot K(\pi)\cdot K(a_0,\pi)\cdot K(\omega)\cdot K(\rho,\omega)+\\
&\qquad[K^2(a_0,\pi)+D\cdot K^2(\pi)]\cdot[K^2(\rho,\omega)+D\cdot K^2(\omega)]\\
&=2D\cdot K(2\pi)\cdot K(\rho,\omega)\cdot K(\omega)+K(a_0,2\pi)[K^2(\rho,\omega)+D\cdot K^2(\omega)]\\
&=D\cdot K(2\pi)\cdot K(\tau')+K(a_0,2\pi)\cdot K(\sigma',\tau')\\
&=K(a_0,2\pi,a_0)\cdot K(\tau')+K(a_0,2\pi)\cdot K(\sigma',\tau')=K(a_0,2\pi,a_0+\sigma',\tau').
\end{aligned}$$

Analogous transformations with longer palindromic sequences result in general formula:

$$\left\{\begin{array}{rcl}
K(a_0,2n\pi,a_0+\sigma',\tau') &=& K^2(a_0,n\pi,a_0+\rho,\omega)\\
&& \quad +D\cdot K^2(n\pi,a_0+\rho,\omega),\\
K(2n\pi,a_0+\sigma',\tau') &=& 2K(a_0,n\pi,a_0+\rho,\omega)\cdot K(n\pi,a_0+\rho,\omega).
\end{array}\right. \tag{14}$$

Here $n=1,2,3,\dots.$ Of course, Property A conditions and criterion > are obligatory.

Now similar set of relations.

$$\begin{aligned}
&K(\pi,a_0+\sigma,\tau)=K(\sigma,\tau)\cdot K(\pi)+K(\tau)\cdot K(a_0,\pi)\\
&=[K(\sigma',\tau')\cdot K(a_0,\pi)-D\cdot K(\tau')\cdot K(\pi)]\cdot K(\pi)\\
&\qquad+[K(\sigma',\tau')\cdot K(\pi)-K(\tau')\cdot K(\pi)]\cdot K(a_0,\pi)\\
&=2K(\sigma',\tau')\cdot K(\pi)\cdot K(a_0,\pi)-K(\tau')\cdot D\cdot K^2(\pi)-K(\tau')\cdot K^2(a_0,\pi)\\
&=K(\sigma',\tau')\cdot 2K(a_0,\pi)\cdot K(\pi)-K(\tau')\cdot[K^2(a_0,\pi)+D\cdot K^2(\pi)]\\
&=[K^2(\rho,\omega)+D\cdot K^2(\omega)]\cdot 2K(a_0,\pi)\cdot K(\pi)\\
&\qquad-2K(\rho,\omega)\cdot K(\omega)\cdot[K^2(a_0,\pi)+D\cdot K^2(\pi)]
\end{aligned}$$

$$
\begin{aligned}
&= 2K^2(\rho,\omega)\cdot K(a_0,\pi)\cdot K(\pi) + D\cdot K^2(\omega)\cdot 2K(a_0,\pi)\cdot K(\pi)\\
&\qquad - 2K^2(a_0,\pi)\cdot K(\rho,\omega)\cdot K(\omega) - 2D\cdot K^2(\pi)\cdot K(\rho,\omega)\cdot K(\omega)\\
&= 2[K(\rho,\omega)\cdot K(\pi)\cdot [K(\rho,\omega)\cdot K(a_0,\pi) - D\cdot K(\omega)\cdot K(\pi)]\\
&\qquad + K(\omega)\cdot K(a_0,\pi)\cdot [D\cdot K(\omega)\cdot K(\pi) - K(a_0,\pi)\cdot K(\rho,\omega)]]\\
&= 2K(\rho',\omega')\cdot [K(\omega)\cdot K(a_0,\pi) - K(\rho,\omega)\cdot K(\pi)] = 2K(\rho',\omega')\cdot K(\omega').
\end{aligned}
$$

$$
\begin{aligned}
&K(a_0,\pi,a_0+\sigma,\tau) = K(a_0,\pi,a_0)\cdot K(\tau) + K(a_0,\pi)\cdot K(\sigma,\tau)\\
&= D\cdot K(\pi)\cdot [K(\sigma',\tau')\cdot K(\pi) - K(\tau')\cdot K(a_0,\pi)]\\
&\qquad + K(a_0,\pi)\cdot [K(\sigma',\tau')\cdot K(a_0,\pi) - D\cdot K(\tau')\cdot K(\pi)]\\
&= D\cdot K^2(\pi)\cdot K(\sigma',\tau') - D\cdot K(a_0,\pi)K(\pi)\cdot K(\tau')\\
&\qquad + K^2(a_0,\pi)\cdot K(\sigma',\tau') - D\cdot K(a_0,\pi)\cdot K(\pi)\cdot K(\tau')\\
&= K(\sigma',\tau')\cdot [K^2(a_0,\pi) + D\cdot K^2(\pi)] - 2D\cdot K(a_0,\pi)\cdot K(\pi)\cdot K(\tau')\\
&= [K^2(\rho,\omega) + D\cdot K^2(\omega)]\cdot [K^2(a_0,\pi) + D\cdot K^2(\pi)]\\
&\qquad - 2D\cdot K(a_0,\pi)\cdot K(\pi)\cdot 2K(\rho,\omega)\cdot K(\omega)\\
&= K^2(\rho,\omega)\cdot K^2(a_0,\pi) + D\cdot K^2(\omega)\cdot K^2(a_0,\pi) + D\cdot K^2(\rho,\omega)\cdot K^2(\pi)\\
&\qquad + D^2\cdot K^2(\omega)\cdot K^2(\pi) - 2D\cdot K(a_0,\pi)\cdot K(\pi)\cdot 2K(\rho,\omega)\cdot K(\omega)\\
&= K^2(\rho,\omega)\cdot K^2(a_0,\pi) - 2D\cdot K(a_0,\pi)\cdot K(\pi)\cdot K(\rho,\omega)\cdot K(\omega)\\
&\qquad\quad + D^2\cdot K^2(\omega)\cdot K^2(\pi) + D\cdot [K^2(\rho,\omega)\cdot K^2(\pi)\\
&\qquad\quad - 2K(a_0,\pi)\cdot K(\pi)\cdot K(\rho,\omega)\cdot K(\omega) + K^2(\omega)\cdot K^2(a_0,\pi)]\\
&= K^2(\rho',\omega') + D\cdot K^2(\omega').
\end{aligned}
$$

That means:

$$
\begin{cases} K(a_0,\pi,a_0+\sigma,\tau) &= K^2(\rho',\omega') + D\cdot K^2(\omega'),\\ K(\pi,a_0+\sigma,\tau) &= 2K(\rho',\omega')\cdot K(\omega'). \end{cases} \tag{15}
$$

From (5):

$$
\begin{aligned}
&K(3\pi,a_0+\sigma,\tau) = K(\sigma',\tau')\cdot K(4\pi) - K(\tau')\cdot K(a_0,4\pi)\\
&= [K^2(\rho,\omega) + D\cdot K^2(\omega)]\cdot 2K(2\pi)\cdot K(a_0,2\pi)\\
&\qquad - 2K(\rho,\omega)\cdot K(\omega)\cdot [K^2(a_0,2\pi) + D\cdot K^2(2\pi)]\\
&= K^2(\rho,\omega)\cdot 2K(2\pi)\cdot K(a_0,2\pi) + D\cdot K^2(\omega)\cdot 2K(2\pi)\cdot K(a_0,2\pi)\\
&\qquad - 2K(\rho,\omega)\cdot K(\omega)\cdot K^2(a_0,2\pi) - 2D\cdot K(\rho,\omega)\cdot K(\omega)\cdot K^2(2\pi).
\end{aligned} \tag{16}
$$

From (6):

$$
\begin{aligned}
&2K(a_0,\pi,a_0+\rho',\omega')\cdot K(\pi,a_0+\rho',\omega')\\
&= 2[-K(\rho,\omega)\cdot K(a_0,2\pi) + D\cdot K(\omega)\cdot K(2\pi)]\\
&\qquad \cdot [-K(\rho,\omega)\cdot K(2\pi) + K(\omega)\cdot K(a_0,2\pi)]\\
&= 2[K^2(\rho,\omega)\cdot K(a_0,2\pi)\cdot K(2\pi) - K^2(a_0,2\pi)\cdot K(\rho,\omega)\cdot K(\omega)\\
&\qquad + D\cdot K^2(\omega)\cdot K(2\pi)\cdot K(a_0,2\pi) - D\cdot K^2(2\pi)\cdot K(\omega)\cdot K(\rho,\omega)].
\end{aligned} \tag{17}
$$

Obtained (16) and (17) gives the same, this confirms relation:

$$K(3\pi, a_0 + \sigma, \tau) = 2K(a_0, \pi, a_0 + \rho', \omega') \cdot K(\pi, a_0 + \rho', \omega').$$

Verification of the relation

$$K(a_0, 3\pi, a_0 + \sigma, \tau) = K^2(a_0, \pi, a_0 + \rho', \omega') + D \cdot K^2(\pi, a_0 + \rho', \omega')$$

can be made similarly, it is left for concerned reader.

Again analogous transformations with longer palindromic sequences and generally:

$$\begin{cases} K(a_0, (2n+1)\pi, a_0 + \sigma, \tau) &= K^2(a_0, n\pi, a_0 + \rho', \omega') \\ & \quad +D \cdot K^2(n\pi, a_0 + \rho', \omega'), \\ K((2n+1)\pi, a_0 + \sigma, \tau) &= 2K(a_0, n\pi, a_0 + \rho', \omega') \\ & \quad \cdot K(n\pi, a_0 + \rho', \omega'). \end{cases} \tag{18}$$

Here $n = 1, 2, 3, ...$, Property A conditions and criterion $>$ are obligatory.
2. Now suppose that criterion is $<$ and relations (11) are valid. Then:

$$\begin{cases} K(a_0, 2n\pi, a_0 + \sigma', \tau') &= K^2(a_0, n\pi, a_0 + \rho', \omega') \\ & \quad +D \cdot K^2(n\pi, a_0 + \rho', \omega'), \\ K(2n\pi, a_0 + \sigma', \tau') &= 2K(a_0, n\pi, a_0 + \rho', \omega') \\ & \quad \cdot K(n\pi, a_0 + \rho', \omega'). \end{cases} \tag{19}$$

Here $n = 1, 2, 3, ...$, Property A conditions and criterion $<$ are obligatory.

$$\begin{cases} K(a_0, \pi, a_0 + \sigma, \tau) &= K^2(\rho, \omega) + D \cdot K^2(\omega), \\ K(\pi, a_0 + \sigma, \tau) &= 2K(\rho, \omega) \cdot K(\omega). \end{cases} \tag{20}$$

$$\begin{cases} K(a_0, (2n+1)\pi, a_0 + \sigma, \tau) &= K^2(a_0, n\pi, a_0 + \rho, \omega) \\ & \quad +D \cdot K^2(n\pi, a_0 + \rho, \omega), \\ K((2n+1)\pi, a_0 + \sigma, \tau) &= 2K(a_0, n\pi, a_0 + \rho, \omega) \\ & \quad \cdot K(n\pi, a_0 + \rho, \omega). \end{cases} \tag{21}$$

Here $n = 1, 2, 3, ...$, Property A conditions and criterion $<$ are obligatory.

Proofs of the relations (19)–(21) are fully analogous to that for (14)–(18).

So criterion $>$ means relations (10), (14), (15) and (18), while criterion $<$ gives relations (11), (19), (20) and (21).

4 Squaring and *abc*-triples

Experimental calculations revealed that *abc*-triples, formed by components of generalized Pell's equations $x^2 - D \cdot y^2 = \pm N$, according to some rules induced formation of *abc*-triples by components of generalized Pell's equations $x^2 - D \cdot y^2 = N^2$, where N is odd.

4.1 General considerations

We have an *abc*-equation $x^2 - D \cdot y^2 = \pm N$, where N is odd or even natural number, equation components are coprime and, according to [3], $D \equiv 1, 2 \pmod 4$.

For even N values *abc*-coprimality means $D \equiv 1 \pmod 4$, with further $x^2 - D \cdot y^2 \equiv 0 \pmod 4$, so only $4|N$ are possible. But then both fundamental roots $K(\sigma', \tau')$ and $K(\tau')$ for the equation $K^2(\sigma', \tau') - D \cdot K^2(\tau') = N^2$ will be even numbers (see relations (10) and (11)) and we do not get an *abc*-equation. About this situation see subsection 4.3, but at first our initial *abc*-equation has odd N value.

4.2 Odd N values

Theorem 4.1. *Under Property A conditions and with criterion > the following relations exist (for odd N values).*
1. *If fundamental roots $K(\rho, \omega)/K(\omega)$ of the generalized Pell's equation $x^2 - D \cdot y^2 = -N$ produce an abc-triple, then abc-triple is also produced by fundamental roots $K(\sigma', \tau')/K(\tau')$ of the generalized Pell's equation $x^2 - D \cdot y^2 = N^2$.*
2. *If fundamental roots $K(\rho', \omega')/K(\omega')$ of the generalized Pell's equation $x^2 - D \cdot y^2 = N$ produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation $x^2 - D \cdot y^2 = N^2$, having one palindromic unit π (roots $K(a_0, \pi, a_0 + \sigma, \tau)$ and $K(\pi, a_0 + \sigma, \tau)$).*
3. *If roots $K(a_0, n\pi, a_0 + \rho, \omega)$ and $K(n\pi, a_0 + \rho, \omega)$ of the generalized Pell's equation $x^2 - D \cdot y^2 = \pm N$, having $n > 0$ palindromic units π, produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation $x^2 - D \cdot y^2 = N^2$, having $2n$ palindromic units π.*
4. *If roots $K(a_0, n\pi, a_0 + \rho', \omega')$ and $K(n\pi, a_0 + \rho', \omega')$ of the generalized Pell's equation $x^2 - D \cdot y^2 = \pm N$, having $n > 0$ palindromic units π, produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation $x^2 - D \cdot y^2 = N^2$, having $2n + 1$ palindromic units π.*

Proof. **1.** So our initial equation $K^2(\rho, \omega) - D \cdot K^2(\omega) = -N$ already is an *abc*-triple, which means

$$R[N] \cdot R[K(\rho, \omega)] \cdot R[D \cdot K(\omega)] < D \cdot K^2(\omega). \tag{22}$$

We must prove that equation $K^2(\sigma', \tau') - D \cdot K^2(\tau') = N^2$ also is an *abc*-triple, so

$$R[K(\sigma', \tau') \cdot N \cdot D \cdot K(\tau')] < K^2(\sigma', \tau'). \tag{23}$$

At first – do we have formation of *abc*-equation? As $K(\rho, \omega) \perp D \cdot K(\omega)$ and N is odd, relations (10) give $K(\sigma', \tau') \perp K(\tau')$. Together with $D \perp N$ that means formation of *abc*-equation. Then instead of (23) we must prove

$$R[K(\sigma', \tau')] \cdot R[N \cdot D \cdot K(\tau')] < K^2(\sigma', \tau'). \tag{24}$$

From (10) and (22):

$$\begin{aligned} R[N \cdot D \cdot K(\tau')] &= R[N \cdot D \cdot 2K(\rho,\omega) \cdot K(\omega)] \\ &= R[N \cdot D \cdot K(\rho,\omega) \cdot K(\omega)] < D \cdot K^2(\omega) < K(\sigma',\tau'). \end{aligned}$$

Factor 2 can be omitted, because N is odd, so one of remaining summands in initial equation must be even. Obtained inequality we multiply with

$$R[K^2(\sigma',\tau')] = R[K(\sigma',\tau')] \leq K(\sigma',\tau')$$

and get necessary (24).

2. Now initial equation $K^2(\rho',\omega') - D \cdot K^2(\omega') = N$ is an *abc*-triple, therefore

$$R[N] \cdot R[K(\rho',\omega')] \cdot R[D \cdot K(\omega')] < K^2(\rho',\omega'). \tag{25}$$

We must show that equation $K^2(a_0,\pi,a_0+\sigma,\tau) - D \cdot K^2(\pi,a_0+\sigma,\tau) = N^2$ also is an *abc*-triple, which means

$$R[K(a_0,\pi,a_0+\sigma,\tau) \cdot N \cdot D \cdot K(\pi,a_0+\sigma,\tau)] < K^2(a_0,\pi,a_0+\sigma,\tau). \tag{26}$$

Analogously relations (15) mean formation of *abc*-equation, so we must prove:

$$R[K(a_0,\pi,a_0+\sigma,\tau)] \cdot R[N \cdot D \cdot K(\pi,a_0+\sigma,\tau)] < K^2(a_0,\pi,a_0+\sigma,\tau). \tag{27}$$

From (15) and (25) and omitting factor 2:

$$\begin{aligned} R[N \cdot D \cdot K(\pi,a_0+\sigma,\tau)] &= R[N \cdot D \cdot 2K(\rho',\omega') \cdot K(\omega')] \\ &= R[N \cdot D \cdot K(\rho',\omega') \cdot K(\omega')] < K^2(\rho',\omega') < K^2(\rho',\omega') + D \cdot K^2(\omega') \\ &= K(a_0,\pi,a_0+\sigma,\tau). \end{aligned}$$

Multiplying obtained inequality with $R[K(a_0,\pi,a_0+\sigma,\tau)] \leq K(a_0,\pi,a_0+\sigma,\tau)$ gives necessary (27).

3. If equation $K^2(a_0,n\pi,a_0+\rho,\omega) - D \cdot K^2(n\pi,a_0+\rho,\omega) = N$ is an *abc*-triple, then

$$R[N \cdot K(a_0,n\pi,a_0+\rho,\omega) \cdot D \cdot K(n\pi,a_0+\rho,\omega)] < K^2(a_0,n\pi,a_0+\rho,\omega)$$

– this is our initial condition. We must prove that equation $K^2(a_0,2n\pi,a_0+\sigma',\tau') - D \cdot K^2(2n\pi,a_0+\sigma',\tau') = N^2$ also is an *abc*-triple, so

$$\begin{aligned} &R[K(a_0,2n\pi,a_0+\sigma',\tau')] \cdot R[N \cdot D \cdot K(2n\pi,a_0+\sigma',\tau')] \\ &\quad < K^2(a_0,2n\pi,a_0+\sigma',\tau') \end{aligned} \tag{28}$$

Relations (14) confirm coprimality of the terms for the new equation and:

$$\begin{aligned} &R[N \cdot D \cdot K(2n\pi,a_0+\sigma',\tau')] \\ &\quad = R[N \cdot D \cdot 2K(a_0,n\pi,a_0+\rho,\omega) \cdot K(n\pi,a_0+\rho,\omega)] \\ &\quad = R[N \cdot D \cdot K(a_0,n\pi,a_0+\rho,\omega) \cdot K(n\pi,a_0+\rho,\omega)] \\ &\quad < K^2(a_0,n\pi,a_0+\rho,\omega) < K^2(a_0,n\pi,a_0+\rho,\omega) \\ &\qquad + D \cdot K^2(n\pi,a_0+\rho,\omega) = K(a_0,2n\pi,a_0+\sigma',\tau'). \end{aligned} \tag{29}$$

As previously, we omit factor 2. Now $R[K(a_0, 2n\pi, a_0+\sigma', \tau')] \leq K(a_0, 2n\pi, a_0+\sigma', \tau')$ and it's multiplication with (29) gives necessary (28).

If $K^2(a_0, n\pi, a_0 + \rho, \omega) - D \cdot K^2(n\pi, a_0 + \rho, \omega) = -N$ is an *abc*-triple, we can use

$$\begin{aligned} &R[N \cdot D \cdot K(a_0, n\pi, a_0 + \rho, \omega) \cdot K(n\pi, a_0 + \rho, \omega)] \\ &\quad < D \cdot K^2(n\pi, a_0 + \rho, \omega) < K^2(a_0, n\pi, a_0 + \rho, \omega) + D \cdot K^2(n\pi, a_0 + \rho, \omega) \\ &\quad = K(a_0, 2n\pi, a_0 + \sigma', \tau'), \end{aligned}$$

with analogous multiplication, leading to necessary (28).

4. For initial equations $K^2(a_0, n\pi, a_0+\rho', \omega') - D\cdot K^2(n\pi, a_0+\rho', \omega') = \pm N$ as *abc*-triples we can act analogously with previous item, only use relations (18) instead of (14). This completes the proof of Theorem 4.1. □

The following experimental Tables 2 (initial equation $x^2 - 2y^2 = \pm 7^5$) and 3 (equation $x^2 - 2y^2 = \pm 7^{10}$) illustrate induction of *abc*-triples, described in Theorem 4.1. In Table 3 sequences are limited by my laptop's performance, therefore in Table 2 they are truncated at 45 π units. T means "True" – we get an *abc*-triple; F means "False".

Table 2. Equation $x^2 - 2y^2 = \pm 7^5$ and *abc*-triples.

		Number of π units												
Extension	N	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ', ω'	$+7^5$	T		F		F		F		F		F		T
ρ, ω	$+7^5$	T	F		F		F		F		F		F	
ρ, ω	-7^5	T		F		T		F		F		F		F
ρ', ω'	-7^5	T	F		F		F		T		F		T	
		13	14	15	16	17	18	19	20	21	22	23	24	25
ρ', ω'	$+7^5$		F		F		F		F		T		F	
ρ, ω	$+7^5$	F		F		F		F		F		F		F
ρ, ω	-7^5		F		F		F		T		T		F	
ρ', ω'	-7^5	F		F		F		F		F		F		F
		26	27	28	29	30	31	32	33	34	35	36	37	38
ρ', ω'	$+7^5$	F		F		F		F		F		F		F
ρ, ω	$+7^5$		F		F		F		F		F		F	
ρ, ω	-7^5	F		F		F		F		F		T		F
ρ', ω'	-7^5		T		T		F		F		F		T	
		39	40	41	42	43	44	45						
ρ', ω'	$+7^5$		F		F		F							
ρ, ω	$+7^5$	F		F		F		F						
ρ, ω	-7^5		F		T		F							
ρ', ω'	-7^5	F		F		T		F						

Table 3. Equation $x^2 - 2y^2 = \pm 7^{10}$ and *abc*-triples.

Extension	N	Number of π units												
		0	1	2	3	4	5	6	7	8	9	10	11	12
ρ', ω'	$+7^{10}$	T		F		T		T		T		F		F
ρ, ω	$+7^{10}$	T	T		F		F		T		F		F	
ρ, ω	-7^{10}	F		F		F		F		F		F		F
ρ', ω'	-7^{10}	F	T		F		F		F		F		F	
		13	14	15	16	17	18	19	20	21	22	23	24	25
ρ', ω'	$+7^{10}$		T		T		F		F		F		T	
ρ, ω	$+7^{10}$	F		T		F		T		F		T		T
ρ, ω	-7^{10}		F		F		F		F		F		F	
ρ', ω'	-7^{10}	F		F		F		F		F		F		F
		26	27	28	29	30	31	32	33	34	35	36	37	38
ρ', ω'	$+7^{10}$	F		T		F		F		T		F		F
ρ, ω	$+7^{10}$		F		F		T		F		F		T	
ρ, ω	-7^{10}	F		F		F		F		F		F		F
ρ', ω'	-7^{10}		F		T		F		F		F		F	
		39	40	41	42	43	44	45	46	47	48	49	50	51
ρ', ω'	$+7^{10}$		T		F		T		F		F		F	
ρ, ω	$+7^{10}$	T		F		T		T		F		F		F
ρ, ω	-7^{10}		F		F		F		F		F		F	
ρ', ω'	-7^{10}	F		F		F		F		F		F		F
		52	53	54	55	56	57	58	59	60	61	62	63	64
ρ', ω'	$+7^{10}$	T		F		T		F		F		F		T
ρ, ω	$+7^{10}$		F		T		F		T		F		F	
ρ, ω	-7^{10}	F		F		F		F		F		F		F
ρ', ω'	-7^{10}		F		F		F		F		F		F	
		65	66	67	68	69	70	71	72	73	74	75	76	77
ρ', ω'	$+7^{10}$		F		F		T		T		T		T	
ρ, ω	$+7^{10}$	F		T		F		T		T		T		F
ρ, ω	-7^{10}		T		F		F		F		F		F	
ρ', ω'	-7^{10}													
		78	79	80	81	82	83	84	85	86	87	88	89	90
ρ', ω'	$+7^{10}$	F		F		F		T		F		T		F
ρ, ω	$+7^{10}$		T		F		F		F		T		F	
ρ, ω	-7^{10}	F		F		F								
ρ', ω'	-7^{10}													

All *abc*-triples from Table 2, considered as primary in view of Theorem 4.1, induce corresponding secondary *abc*-triples, whose finding in Table 3 we leave to concerned reader.

Theorem 4.2. *Under Property A conditions and with criterion < the following relations exist (for odd N values).*
1. *If fundamental roots* $K(\rho', \omega')/K(\omega')$ *of the generalized Pell's equation* $x^2 - D\cdot y^2 = N$ *produce an abc-triple, then abc-triple is also produced by fundamental roots* $K(\sigma', \tau')/K(\tau')$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = N^2$.
2. *If fundamental roots* $K(\rho, \omega)/K(\omega)$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = -N$ *produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation* $x^2 - D \cdot y^2 = N^2$, *having one palindromic unit* π *(roots* $K(a_0, \pi, a_0 + \sigma, \tau)$ *and* $K(\pi, a_0 + \sigma, \tau)$*).*
3. *If roots* $K(a_0, n\pi, a_0 + \rho, \omega)$ *and* $K(n\pi, a_0 + \rho, \omega)$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = \pm N$, *having* $n > 0$ *palindromic units* π, *produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation* $x^2 - D \cdot y^2 = N^2$, *having* $2n + 1$ *palindromic units* π.
4. *If roots* $K(a_0, n\pi, a_0 + \rho', \omega')$ *and* $K(n\pi, a_0 + \rho', \omega')$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = \pm N$, *having* $n > 0$ *palindromic units* π, *produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation* $x^2 - D \cdot y^2 = N^2$, *having* $2n$ *palindromic units* π.

Proof. The proof for Theorem 4.2 is fully analogous to that for Theorem 4.1, using relations (11), (19), (20) and (21). □

The following experimental Tables 4 (initial equation $x^2 - 65y^2 = \pm 7^4$; truncated at 20π units) and 5 (equation $x^2 - 65y^2 = \pm 7^8$;) illustrate induction of *abc*-triples, described in Theorem 4.2. In Table 5 sequences are limited by my laptop's performance. T means "True" – we get an *abc*-triple; F means "False".

Table 4. Equation $x^2 - 65y^2 = \pm 7^4$ and *abc*-triples.

		Number of π units												
Extension	N	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ', ω'	$+7^4$	T		T		F		T		F		F		F
ρ, ω	$+7^4$	T	F		T		F		F		T		F	
ρ, ω	-7^4	T		F		F		F		T		F		F
ρ', ω'	-7^4	T	F		F		F		F		T		T	
		13	14	15	16	17	18	19	20					
ρ', ω'	$+7^4$		F		F		F		F					
ρ, ω	$+7^4$	F		F		F		T						
ρ, ω	-7^4		F		T		T		F					
ρ', ω'	-7^4	F		T		F		F						

Table 5. Equation $x^2 - 65y^2 = \pm 7^8$ and *abc*-triples.

		Number of π units												
Extension	N	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ', ω'	$+7^8$	T		T		T		T		F		F		T
ρ, ω	$+7^8$	T	T		F		F		T		F		F	
ρ, ω	-7^8	F		F		F		F		F		F		T
ρ', ω'	-7^8	F	F		F		T		F		F		F	
		13	14	15	16	17	18	19	20	21	22	23	24	25
ρ', ω'	$+7^8$		T		F		T		F		T		F	
ρ, ω	$+7^8$	F		F		T		T		F		F		F
ρ, ω	-7^8		F		F		F		F		F		F	
ρ', ω'	-7^8	F		F		F		F		F				
		26	27	28	29	30	31	32	33	34	35	36	37	38
ρ', ω'	$+7^8$	F		F		T		F						
ρ, ω	$+7^8$		F		F		F		T		T			
ρ, ω	-7^8	F												
ρ', ω'	-7^8													

Finding of corresponding primary-secondary relations between *abc*-triples in Tables 4 and 5 is left to concerned reader.

In Tables 3 and 5 we can notice relative concentration of *abc*-triples in roots of the generalized Pell's equations $x^2 - D \cdot y^2 = N^{2k}$, with $k = 1, 2, 3...$.

In view of Theorems 4.1 and 4.2 we always can find for every primary *abc*-triple in the set of roots for equation $x^2 - D \cdot y^2 = \pm N$ it's corresponding secondary *abc*-triple in the set of roots for equation $x^2 - D \cdot y^2 = N^2$; here N is odd and Property A conditions are satisfied. Similarly we can treat equation $x^2 - D \cdot y^2 = N^2$ as primary and find corresponding secondary *abc*-triples in the set of roots for equation $x^2 - D \cdot y^2 = N^4$; etc. As the result – squaring gives rise to infinite sequences of primary-secondary interrelated *abc*-triples, which can be derived for every *abc*-triple from the set of roots of initial equation $x^2 - D \cdot y^2 = \pm N$.

4.3 Even N values

Now N is even (see subsection 4.1), so both fundamental roots $K(\sigma', \tau')$ and $K(\tau')$ will be even numbers (from relations (10) or (11)) and we do not get an *abc*-equation. Suppose that criterion is $>$, so $K^2(\rho, \omega) + D \cdot K^2(\omega) \equiv 2 \pmod 4$ and $2K(\rho, \omega) \cdot K(\omega) \equiv 2 \pmod 4$. Then new relations

$$\begin{cases} K(\sigma', \tau') & = & \frac{1}{2}[K^2(\rho, \omega) + D \cdot K^2(\omega)] \\ K(\tau') & = & K(\rho, \omega) \cdot K(\omega). \end{cases} \tag{30}$$

with halved previous roots will give correct *abc*-equation

$$K^2(\sigma',\tau') - D\cdot K^2(\tau') = \frac{N^2}{4}. \tag{31}$$

How about induction of *abc*-triples in this case?

Substitution from relations (10) to (30) is critical. Instead of (14) we have

$$\begin{cases} K(a_0, 2n\pi, a_0+\sigma', \tau') &= \frac{1}{2}[K^2(a_0, n\pi, a_0+\rho, \omega) \\ & \qquad +D\cdot K^2(n\pi, a_0+\rho,\omega)], \\ K(2n\pi, a_0+\sigma',\tau') &= K(a_0, n\pi, a_0+\rho,\omega)\cdot K(n\pi, a_0+\rho,\omega). \end{cases} \tag{32}$$

Here $n = 1, 2, 3, ...,$ Property A conditions and criterion > are obligatory.

Relations (15) become

$$\begin{cases} K(a_0, \pi, a_0+\sigma,\tau) &= \frac{1}{2}[K^2(\rho',\omega') + D\cdot K^2(\omega')], \\ K(\pi, a_0+\sigma,\tau) &= K(\rho',\omega')\cdot K(\omega'), \end{cases} \tag{33}$$

but instead of (18) we have

$$\begin{cases} K(a_0, (2n+1)\pi, a_0+\sigma,\tau) &= \frac{1}{2}[K^2(a_0, n\pi, a_0+\rho',\omega') \\ & \qquad +D\cdot K^2(n\pi, a_0+\rho',\omega')], \\ K((2n+1)\pi, a_0+\sigma,\tau) &= K(a_0, n\pi, a_0+\rho',\omega') \\ & \qquad \cdot K(n\pi, a_0+\rho',\omega'). \end{cases} \tag{34}$$

Here $n = 1, 2, 3, ...,$ Property A conditions and criterion > are obligatory. All these modifications can be easily tracked in subsection 3.4 calculations.

Now the impact of mentioned modifications to all four Theorem 4.1 statement analogues.

1. If initial equation $K^2(\rho,\omega) - D\cdot K^2(\omega) = -N$ is an *abc*-triple, then

$$R[K(\rho,\omega)]\cdot R[N]\cdot R[D\cdot K(\omega)] < D\cdot K^2(\omega). \tag{35}$$

As $4|N$ and $16|N^2$, $R[N]$ is not significant, because the new equation is (31). For equation (31) to be an *abc*-triple, we need

$$R[K(\sigma',\tau')]\cdot R[N]\cdot R[D\cdot K(\tau')] < K^2(\sigma',\tau').$$

As $R[K(\sigma',\tau')] \le K(\sigma',\tau')$, we need

$$R[N]\cdot R[D\cdot K(\tau')] < K(\sigma',\tau') = \frac{K^2(\rho,\omega) + D\cdot K^2(\omega)}{2}.$$

As $K^2(\rho,\omega) < D\cdot K^2(\omega)$, we need stronger inequality than (35), which sometimes is not reached. So the first statement analogue is not justified.

Some examples, where mentioned in this statement analogue *abc*-triple induction is not confirmed, include D/N combinations 17/2048, 13/2116 and 17/4864, but for D/N combinations 265/64, 65/10976 and 193/8 new *abc*-triples of type (31) are induced.

2. Now $K^2(\rho', \omega') - D \cdot K^2(\omega') = N$ is an *abc*-triple, so

$$R[K(\rho', \omega')] \cdot R[N] \cdot R[D \cdot K(\omega')] < K^2(\rho', \omega').$$

Again $R[N]$ is not significant, but the new equation is

$$K^2(a_0, \pi, a_0 + \sigma, \tau) - D \cdot K^2(\pi, a_0 + \sigma, \tau) = \frac{N^2}{4}. \tag{36}$$

From (33) we can see that it is an *abc*-equation, but for (36) to be an *abc*-triple, the inequality

$$R[K(a_0, \pi, a_0 + \sigma, \tau)] \cdot R[N] \cdot R[D \cdot K(\pi, a_0 + \sigma, \tau)] < K^2(a_0, \pi, a_0 + \sigma, \tau)$$

must be confirmed. Once more from (33):

$$R[N] \cdot R[D \cdot K(\pi, a_0 + \sigma, \tau)] = R[N] \cdot R[D \cdot K(\rho', \omega') \cdot K(\omega')] < K^2(\rho', \omega').$$

As $K^2(\rho', \omega') > \dfrac{K^2(\rho', \omega') + D \cdot K^2(\omega')}{2} = K(a_0, \pi, a_0 + \sigma, \tau)$, we have

$$R[N] \cdot R[D \cdot K(\pi, a_0 + \sigma, \tau)] < K(a_0, \pi, a_0 + \sigma, \tau).$$

Together with $R[K(a_0, \pi, a_0 + \sigma, \tau)] \leq K(a_0, \pi, a_0 + \sigma, \tau)$ this confirms (36) as *abc*-triple and justifies the second statement analogue.

3. If equation $K^2(a_0, n\pi, a_0 + \rho, \omega) - D \cdot K^2(n\pi, a_0 + \rho, \omega) = N$ is an *abc*-triple, then we have for initial condition

$$R[K(a_0, n\pi, a_0 + \rho, \omega)] \cdot R[N] \cdot R[D \cdot K(n\pi, a_0 + \rho, \omega)] < K^2(a_0, n\pi, a_0 + \rho, \omega).$$

We must confirm that for an equation

$$K^2(a_0, 2n\pi, a_0 + \sigma', \tau') - D \cdot K^2(2n\pi, a_0 + \sigma', \tau') = \frac{N^2}{4} \tag{37}$$

we have

$$R[K(a_0, 2n\pi, a_0 + \sigma', \tau')] \cdot R[N] \cdot R[D \cdot K(2n\pi, a_0 + \sigma', \tau')] < K^2(a_0, 2n\pi, a_0 + \sigma', \tau').$$

Again relations (32) confirm formation of an *abc*-equation and

$$\begin{aligned} &R[N] \cdot R[D \cdot K(2n\pi, a_0 + \sigma', \tau')] \\ &\quad = R[N] \cdot R[D \cdot K(a_0, n\pi, a_0 + \rho, \omega) \cdot K(n\pi, a_0 + \rho, \omega)] \\ &\quad\quad < K^2(a_0, n\pi, a_0 + \rho, \omega) > K(a_0, 2n\pi, a_0 + \sigma', \tau'). \end{aligned}$$

Together with $R[K(a_0, 2n\pi, a_0+\sigma', \tau')] \le K(a_0, 2n\pi, a_0+\sigma', \tau')$ this confirms (37) as *abc*-triple and justifies the third statement analogue.

For initial *abc*-triple being $K^2(a_0, n\pi, a_0+\rho, \omega) - D \cdot K^2(n\pi, a_0+\rho, \omega) = -N$ we analogously can get $D \cdot K^2(n\pi, a_0+\rho, \omega) > K(a_0, 2n\pi, a_0+\sigma', \tau')$ with further justification of the statement analogue.

4. For initial equations $K^2(a_0, n\pi, a_0+\rho', \omega') - D \cdot K^2(n\pi, a_0+\rho', \omega') = \pm N$ as *abc*-triples we can act analogously with previous item, only use relations (34) instead of (32). This completes confirmation of

Theorem 4.3. *Under Property A conditions and with criterion > the following relations exist (for even N values).*

1. *If fundamental roots $K(\rho', \omega')/K(\omega')$ of the generalized Pell's equation $x^2 - D \cdot y^2 = N$ produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation $x^2 - D \cdot y^2 = N^2/4$, having one palindromic unit π (roots $K(a_0, \pi, a_0+\sigma, \tau)$ and $K(\pi, a_0+\sigma, \tau)$).*

2. *If roots $K(a_0, n\pi, a_0+\rho, \omega)$ and $K(n\pi, a_0+\rho, \omega)$ of the generalized Pell's equation $x^2 - D \cdot y^2 = \pm N$, having $n > 0$ palindromic units π, produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation $x^2 - D \cdot y^2 = N^2/4$, having $2n$ palindromic units π.*

3. *If roots $K(a_0, n\pi, a_0+\rho', \omega')$ and $K(n\pi, a_0+\rho', \omega')$ of the generalized Pell's equation $x^2 - D \cdot y^2 = \pm N$, having $n > 0$ palindromic units π, produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation $x^2 - D \cdot y^2 = N^2/4$, having $2n+1$ palindromic units π.*

The following experimental Tables 6 (initial equation $x^2 - 17y^2 = \pm 2^{11}$) and 7 (equation $x^2 - 17y^2 = \pm 2^{20}$) illustrate induction of *abc*-triples, described in Theorem 4.3. In Table 7 sequences are limited by my laptop's performance, therefore in Table 6 they are truncated at 30 π units. T means "True" – we get an *abc*-triple; F means "False".

Table 6. Equation $x^2 - 17y^2 = \pm 2^{11}$ and *abc*-triples.

		Number of π units												
Extension	N	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ', ω'	$+2^{11}$	F		F		F		F		T		F		F
ρ, ω	$+2^{11}$	F	F		F		F		F		F		F	
ρ, ω	-2^{11}	T		F		T		F		T		F		F
ρ', ω'	-2^{11}	T	F		T		F		F		F		F	
		13	14	15	16	17	18	19	20	21	22	23	24	25
ρ', ω'	$+2^{11}$		F		F		F		F		F		F	
ρ, ω	$+2^{11}$	F		F		F		F		F		F		T
ρ, ω	-2^{11}		T		F		F		F		F		F	
ρ', ω'	-2^{11}	F		T		T		F		T		F		T

(continued)

Table 6 *(continuation)*.

Extension	N	Number of π units 26	27	28	29	30
ρ', ω'	$+2^{11}$	F		F		F
ρ, ω	$+2^{11}$		F		F	
ρ, ω	-2^{11}	T		F		F
ρ', ω'	-2^{11}		F		F	

Table 7. Equation $x^2 - 17y^2 = \pm 2^{20}$ and *abc*-triples.

		Number of π units												
Extension	N	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ', ω'	$+2^{20}$	F		F		F		F		T		T		F
ρ, ω	$+2^{20}$	F	T		F		T		T		F		F	
ρ, ω	-2^{20}	T		F		F		F		F		F		F
ρ', ω'	-2^{20}	T	T		F		F		F		F		F	
		13	14	15	16	17	18	19	20	21	22	23	24	25
ρ', ω'	$+2^{20}$		F		T		F		F		T		F	
ρ, ω	$+2^{20}$	T		F		T		T		F		F		T
ρ, ω	-2^{20}		F		F		F		F		F		F	
ρ', ω'	-2^{20}	F		F		F		T		F		F		F
		26	27	28	29	30	31	32	33	34	35	36	37	38
ρ', ω'	$+2^{20}$	F		T		T		F		F		F		F
ρ, ω	$+2^{20}$		F		F		T		F		T		F	
ρ, ω	-2^{20}	F		F		F		F		T				
ρ', ω'	-2^{20}		F		F		F		T		F		F	
		39	40	41	42	43	44	45	46	47	48	49	50	51
ρ', ω'	$+2^{20}$		F		F									
ρ, ω	$+2^{20}$	F		F		T		F		F		F		T
ρ, ω	-2^{20}													
ρ', ω'	-2^{20}													
		52	53	54	55	56	57	58	59	60	61	62	63	64
ρ', ω'	$+2^{20}$													
ρ, ω	$+2^{20}$		F		F		T							
ρ, ω	-2^{20}													
ρ', ω'	-2^{20}													

All *abc*-triples from Table 6, considered as primary in view of Theorem 4.3, induce corresponding secondary *abc*-triples, whose finding in Table 7 we leave to concerned reader.

In the situation with $N = 4$ initial equation (primary *abc*-triples) coincides with the new equation (secondary *abc*-triples), as for $D/N = 5/4$. Obtained formulae in Theorem 4.3 agrees with our previous results in [2].

Now suppose that criterion is < . In full analogy we can get correct *abc*-equation, changing relations (11) to

$$\begin{cases} K(\sigma',\tau') & = & \frac{1}{2}[K^2(\rho',\omega') + D \cdot K^2(\omega')] \\ K(\tau') & = & K(\rho',\omega') \cdot K(\omega') \end{cases} \tag{38}$$

with halved previous roots, then subsequent changes in relations (19), (20) and (21). As previously, the analogue to first statement of Theorem 4.2 is not confirmed: no *abc*-triple induction for D/N values 17/77824 and 577/82944, but induction for 17/9728 and 13/8748 takes place. Remaining analogues to statements 2–4 of Theorem 4.2 are valid and this proves

Theorem 4.4. *Under Property A conditions and with criterion < the following relations exist (for even N values).*
1. *If fundamental roots* $K(\rho,\omega)/K(\omega)$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = -N$ *produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation* $x^2 - D \cdot y^2 = N^2/4$, *having one palindromic unit* π *(roots* $K(a_0,\pi,a_0+\sigma,\tau)$ *and* $K(\pi,a_0+\sigma,\tau)$*).*
2. *If roots* $K(a_0,n\pi,a_0+\rho,\omega)$ *and* $K(n\pi,a_0+\rho,\omega)$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = \pm N$, *having* $n > 0$ *palindromic units* π, *produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation* $x^2 - D \cdot y^2 = N^2/4$, *having* $2n+1$ *palindromic units* π.
3. *If roots* $K(a_0,n\pi,a_0+\rho',\omega')$ *and* $K(n\pi,a_0+\rho',\omega')$ *of the generalized Pell's equation* $x^2 - D \cdot y^2 = \pm N$, *having* $n > 0$ *palindromic units* π, *produce an abc-triple, then abc-triple is also produced by roots of the generalized Pell's equation* $x^2 - D \cdot y^2 = N^2/4$, *having* $2n$ *palindromic units* π.

The following experimental Tables 8 (initial equation $x^2 - 17y^2 = \pm 2^9$) and 9 (equation $x^2 - 17y^2 = \pm 2^{16}$) illustrate induction of *abc*-triples, described in Theorem 4.4. In Table 9 sequences are limited by my laptop's performance, therefore in Table 8 they are truncated at 30 π units. T means "True" – we get an *abc*-triple; F means "False".

Table 8. Equation $x^2 - 17y^2 = \pm 2^9$ and *abc*-triples.

		Number of π units												
Extension	N	0	1	2	3	4	5	6	7	8	9	10	11	12
ρ',ω'	$+2^9$	F		F		F		F		F		F		F
ρ,ω	$+2^9$	F	F		F		F		F		T		T	
ρ,ω	-2^9	F		F		F		F		F		F		F
ρ',ω'	-2^9	F	T		F		T		F		F		F	

(continued)

Table 8 *(continuation)*.

Extension	N	Number of π units												
		13	14	15	16	17	18	19	20	21	22	23	24	25
ρ', ω'	$+2^9$		T		F		F		F		T		F	
ρ, ω	$+2^9$	F		F		F		F		F		T		F
ρ, ω	-2^9		F		T		F		F		F		F	
ρ', ω'	-2^9	F		F		F		T		F		F		F
		26	27	28	29	30								
ρ', ω'	$+2^9$	F		F		F								
ρ, ω	$+2^9$		F		F									
ρ, ω	-2^9													
ρ', ω'	-2^9		F		F									

Table 9. Equation $x^2 - 17y^2 = \pm 2^{16}$ and *abc*-triples.

Extension	N	Number of π units												
		0	1	2	3	4	5	6	7	8	9	10	11	12
ρ', ω'	$+2^{16}$	F		T		F		F		F		T		F
ρ, ω	$+2^{16}$	F	F		F		F		F		F		F	
ρ, ω	-2^{16}	F		F		F		T		F		F		F
ρ', ω'	-2^{16}	F	F		F		F		F		F		F	
		13	14	15	16	17	18	19	20	21	22	23	24	25
ρ', ω'	$+2^{16}$		F		F		F		T		F		T	
ρ, ω	$+2^{16}$	F		T		F		T		F		T		T
ρ, ω	-2^{16}		T		F		F		F		F		F	
ρ', ω'	-2^{16}	F		F		F		F		F		F		F
		26	27	28	29	30	31	32	33	34	35	36	37	38
ρ', ω'	$+2^{16}$	F		T		F		F		F		F		T
ρ, ω	$+2^{16}$		F		F		F		T		F		F	
ρ, ω	-2^{16}	F		F		F								
ρ', ω'	-2^{16}		T		F									
		39	40	41	42	43	44	45	46	47	48	49	50	51
ρ', ω'	$+2^{16}$		F		F		T		F		T			
ρ, ω	$+2^{16}$	F		F		F		F		T				
ρ, ω	-2^{16}													
ρ', ω'	-2^{16}													

All *abc*-triples from Table 8, considered as primary in view of Theorem 4.4, induce corresponding secondary *abc*-triples, whose finding in Table 9 we leave to concerned reader.

In view of Theorems 4.3 and 4.4 we analogously can construct infinite sequences of primary-secondary interrelated *abc*-triples, derived from every primary *abc*-triple in the set of roots for the initial equation $x^2 - D \cdot y^2 = \pm N$.

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