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On Algebraic Real Algebras Satisfying Some Conditions

Alassane Diouf and Mbayang Amar

Département de Mathématiques et Informatique
Faculté des Sciences et Techniques
Université Cheikh Anta Diop, 5005 Dakar, Sénégal

Abstract

We study the locally complex partially alternative algebras. We also consider an algebraic real algebra A with no nonzero divisor of zero. If A is a right or left Moufang algebra, we give some conditions for A to be isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. Finally, we show also that if A is unitary and $dim(A) > 1$, then A contains $\mathbb{C}$.

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1 Introduction

Throughout this paper we will deal with nonzero real algebras. The algebras are not assumed to be associative, finite-dimensional, or unital.

The study of real algebras with no nonzero divisor of zero (including division algebras and absolute-valued algebras) began with the discovery of $\mathbb{H}$ (Hamilton's quaternions) in 1843 and $\mathbb{O}$ (Cayley's octonions) in 1843 and, independently, 1845. Frobenius-Zorn showed that every alternative, quadratic real algebra without divisors of zero is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ [7, 8, 14]. If A is a weakly alternative algebraic real algebra with no nonzero joint divisor of zero, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ [6, Corollary 3]. The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only algebraic algebras with no nonzero divisor of zero satisfying the middle Moufang identity [6, Theorem 3]. These four algebras are also the only locally complex alternative algebras [5, Theorem 4.3] and [1, Theorem 4.7]. However, there are several algebras with no nonzero divisor of zero that satisfy the right or left Moufang identity and not isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ [4, Theorems 2.1 and 2.2]. This implies that the additional assumptions of Theorems 3.5 and 3.8 cannot be removed.

2 Notations and Preliminary

Let A be an algebra with product $(x, y) \mapsto xy$. We recall that $(.,.,.)$, $[.,.]$, $A(x)$ denote respectively the associator, the commutator, and the subalgebra of A generated by $x \in A$. An element x of A is said to be a *divisor of zero* in A if there exists $y \in A \setminus \{0\}$ such that $xy = 0$ or $yx = 0$. The algebra A is called division algebra, if $L_x : A \to A \;\; a \mapsto xa$ and $R_x : A \to A \;\; a \mapsto ax$ are bijective for all $x \in A \backslash \{0\}$. A is called power-associative (resp. algebraic) if $A(x)$ is associative (resp. if $A(x)$ is finite-dimensional) for every $x \in A$. An element e in A is called flexible if $(e, x, e) = 0$ for all $x \in A$.

A is called quadratic if A contains a unit element e and e, x, x^2 are linearly dependent for every $x \in A$. If A is quadratic, the set $Im(A) = \{x \in A : x^2 \in \mathbb{R}e \textit{ and } x \notin \mathbb{R}e \setminus \{0\}\}$ of purely imaginary elements of A constitutes a linear subspace of A which is suppementary to $\mathbb{R}e$ [7].

A is called left alternative (resp. right alternative) if $(x, x, y) = 0$ (resp. $(y, x, x) = 0$) for all $x, y \in A$. We recall that A is alternative if A is left and right alternative.

A is called power-commutative (resp. flexible) if $A(x)$ is commutative for all $x \in A$ (resp. $(x, y, x) = 0$ for all $x, y \in A$). Every flexible algebra is power-commutative. The algebra A is called third-power associative if $(x, x, x) = 0$ for every $x \in A$. A linearization of $(x, x, x) = 0$ gives that

$$(x, x, y) + (x, y, x) + (y, x, x) = 0. \tag{2.1}$$

A is said to be locally complex algebra if $A(a)$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ for every $a \in A \setminus \{0\}$. We also remember that A satisfies the right Moufang identity (resp. the left Moufang identity) if $y[(xz)x] = [(yx)z]x$ (resp. if $[x(yx)]z = x[y(xz)]$) for all $x, y, z \in A$. Any alternative algebra satisfies Moufang's identities.

A nonzero element e of A is called a generalized left-unit (resp. generalized right-unit) if $e(xy) = x(ey)$ $\big($resp. $(xy)e = (xe)y\big)$ for all $x, y \in A$, and e

is called a generalized unit if e is both generalized left-unit and generalized right-unit [3]. It is easy to verify that a left-unit is a generalized left-unit and a right-unit is a generalized right-unit. The converses are false, see [3].

The *opposite algebra* $A^{(0)}$ of A is defined as the algebra consisting of the vector space of A and the product $(x, y) \mapsto yx$. In [10, Definition 2.2], the authors introduce the notion of partially alternative algebras which is a generalization of alternative algebras.

Definition 2.1. *Let A be a real algebra with unit element $\mathbf{1}$. An element $q \in A$ is called an imaginary unit if $q^2 = -\mathbf{1}$. Denote by $\mathcal{I}_A$ the set of all imaginary units in A.*

Definition 2.2. *Let A be a real algebra with unit element $\mathbf{1}$ and $\mathcal{I}_A \neq \emptyset$.*

(1) A is called partially left alternative if for all $x \in \mathcal{I}_A$ and $y \in A$, $(x, x, y) = 0$.

(2) A is called partially flexible if for all $x \in \mathcal{I}_A$ and $y \in A$, $(x, y, x) = 0$.

(3) A is called partially right alternative if for all $x \in \mathcal{I}_A$ and $y \in A$, $(y, x, x) = 0$.

The algebra A is called *partially alternative* if it is partially left alternative, flexible, and right alternative.

We have the following preliminary result.

Lemma 2.3. *Let A be a right moufang real unital algebra with no nonzero divisor of zero. Then A is alternative.*

Proof. Taking $z = 1$ in the right Moufang identity $y[(xz)x] = [(yx)z]x$, we have $yx^2 = (yx)x$, that is, A is a right alternative algebra. Therefore, by [6, Corollary 1], A is alternative. □

3 Main results

Theorem 3.1. *Let A be a locally complex partially left and right alternative algebra. Then A is isomorphic to $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. (i) Suppose that A is partially left alternative. So $(u, u, z) = 0$ for every $u \in \mathcal{I}_A$ and $z \in A$ $\big($Definition 2.2(1)$\big)$. By [5, Theorem 3.3], A is quadratic, so $A = \mathbb{R}\mathbf{1} \oplus Im(A)$. By definition, we have $\mathcal{I}_A \neq \emptyset$, $\mathcal{I}_A \subset Im(A)$ and for every

$w \in Im(A)$ there exists $\beta \in \mathbb{R}$ and $v \in \mathcal{I}_A$ such that $w = \beta v$. Let $x, y \in A$. We have $x = \alpha\mathbf{1} + \lambda u$, with $u \in \mathcal{I}_A$ and $(\alpha, \lambda) \in \mathbb{R}^2$. We obtain

$$\begin{aligned}
(x, x, y) &= (\alpha\mathbf{1} + \lambda u, \alpha\mathbf{1} + \lambda u, y) \\
&= (\alpha\mathbf{1}, \alpha\mathbf{1} + \lambda u, y) + (\lambda u, \alpha\mathbf{1} + \lambda u, y) \\
&= (\lambda u, \alpha\mathbf{1} + \lambda u, y) \\
&= (\lambda u, \alpha\mathbf{1}, y) + (\lambda u, \lambda u, y) \\
&= (\lambda u, \lambda u, y) \\
&= \lambda^2(u, u, y) \\
&= 0,
\end{aligned}$$

so A is a left alternative algebra.

(ii) Suppose that A is partially right alternative. It follows from (i) by passing to the opposite algebra that A is a right alternative algebra.

This implies that A is alternative. Therefore, by [5, Theorem 4.3], A is isomorphic to $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. □

Theorem 3.2. *Let A be a locally complex algebra. The following assertions are equivalent:*

(i) A is partially left alternative and partially flexible,

(ii) A is partially right alternative and partially flexible,

(iii) A is isomorphic to $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

Proof. $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are obvious.

$(i) \Rightarrow (iii)$. Suppose that A is partially left alternative and partially flexible. The first part of the proof of Theorem 3.1 proves that A is a left alternative algebra. This implies that A is third-power associative, and the equaliy (2.1) gives that $(x, y, x) + (y, x, x) = 0$ for every $x, y \in A$.

Since A is locally complex, by [5, Theorem 3.3], A is quadratic, so $A = \mathbb{R}\mathbf{1} \oplus Im(A)$, and hence $x = \alpha\mathbf{1} + \lambda u$ with $(\alpha, \lambda) \in \mathbb{R}^2$ and $u \in \mathcal{I}_A$. Keeping in mind that $(u, y, u) = 0$, we obtain

$$\begin{aligned}
0 &= (x, y, x) + (y, x, x) \\
&= (\alpha\mathbf{1} + \lambda u, y, \alpha\mathbf{1} + \lambda u) + (y, x, x) \\
&= (\alpha\mathbf{1}, y, \alpha\mathbf{1} + \lambda u) + (\lambda u, y, \alpha\mathbf{1} + \lambda u) + (y, x, x) \\
&= (\lambda u, y, \alpha\mathbf{1} + \lambda u) + (y, x, x) \\
&= (\lambda u, y, \alpha\mathbf{1}) + (\lambda u, y, \lambda u) + (y, x, x) \\
&= (\lambda u, y, \lambda u) + (y, x, x) \\
&= \lambda^2(u, y, u) + (y, x, x) \\
&= (y, x, x),
\end{aligned}$$

so A is a right alternative algebra. We realize that A is alternative. Therefore, by [5, Theorem 4.3], A is isomorphic to $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$.

(ii) $\Rightarrow$ *(iii)*. Suppose that A is partially right alternative and partially flexible. The result follows from *(i)* $\Rightarrow$ *(iii)* by passing to the opposite algebra. □

Corollary 3.3. *Let A be a locally complex partially alternative algebra. Then A is isomorphic to $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Theorem 3.4. *Let A be a power-associative algebraic partially alternative real algebra with no nonzero divisor of zero. Then A is isomorphic to $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. Let $x \in A \setminus \{0\}$, the subalgebra $A(x)$ is associative. Therefore, by [2, Theorem 2.5.29], $A(x)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Since $deg(\mathbb{H}) = 2$, so $A(x)$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and hence A is locally complex. Therefore, by Corollary 3.3, A is isomorphic to $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. □

Theorem 3.5. *Let A be a finite-dimensional division algebra containing a generalized unit. If A satisfies the right or left Moufang identity, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. (i) Suppose that A is a right Moufang algebra and let e a generalized unit of A. Since all the real algebras with finite dimension are normable, the proof of Theorem 1.1 in [4] shows that A contains a left-unit element e_0. Putting $y = x = e$ in $y[(xz)x] = [(yx)z]x$, we get $e[(ez)e] = (e^2z)e$. We have also $e[(ez)e] = e(e^2z)$ because e is a generalized right-unit, and so $(e^2z)e = e(e^2z)$. As $L_{e^2} : A \to A$ $b \mapsto e^2b$ is bijective, we have $ye = ey$ for all y in A, and hence $ee_0 = e_0e = e$. For all x in A, we have $e(xe_0) = x(ee_0)$ because e is a generalized left-unit, hence $e(xe_0) = xe = ex$, and so $xe_0 = x$ because A has no nonzero divisors of zero. So $e_0x = xe_0 = x$ for every $x \in A$, and hence A is alternative because of Lemma 2.3. Therefore, by [2, Theorem 2.5.29], A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

(ii) Suppose that A is a left Moufang algebra. This case follows from (i) by passing to the opposite algebra. □

Proposition 3.6. *Let A be an algebraic real algebra with no nonzero divisor of zero containing a nonzero idempotent e. Then the following assertions are equivalent:*

(i) *A satisfies the right Moufang identity and e is a generalized right-unit,*

(ii) *A satisfies the left Moufang identity and e is a generalized left-unit,*

(iii) *A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are obvious.

$(i) \Rightarrow (iii)$. Suppose that A is a right Moufang algebra and e is a generalized right-unit. Taking $x = y = e$ in the right Moufang identity $y[(xz)x] = [(yx)z]x$, we obtain $e[(ez)e] = (e^2z)e = (ez)e$. Keeping in mind that e is a generalized right unit, we have also $e[(ez)e] = e(e^2z) = e(ez)$ and $(ez)e = e^2z = ez$, hence $ez = e(ez)$. This implies that $ez = z$ because A has no nonzero divisors of zero. The equality $ez = z$ gives that $(ez)e = ze$, so $ez = ze$, and hence $ez = ze = z$ for every z in A. This implies that A is alternative (Lemma 2.3). Therefore, by [2, Theorem 2.5.29], A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

$(ii) \Rightarrow (iii)$. Suppose that A is a left Moufang algebra and e is a generalized left-unit. This case follows $(i) \Rightarrow (iii)$ by passing to the opposite algebra. □

Proposition 3.7. *Let A be a real algebra with no nonzero divisor of zero containing a nonzero flexible idempotent e.*

(i) *If A is a right Moufang algebra, then A is left-unit.*

(ii) *If A is a left Moufang algebra, then A is right-unit.*

(iii) *If A is a middle Moufang algebra, then A is a unital alternative algebra.*

Proof. (i) Suppose that A is a right Moufang algebra. Putting $x = y = e$ in the identity $y[(xz)x] = [(yx)z]x$, we get $e[(ez)e] = (e^2z)e = (ez)e$ for every z in A. Keeping in mind that A has no nonzero divisors of zero and e is flexible, we have $e[(ez)e] = (ez)e$, so $e[(ez)e] = e(ze)$, and hence $ez = z$.

(ii) Suppose that A is a left Moufang algebra. This case follows from (i) by passing to the opposite algebra.

(iii) Suppose that A is a middle Moufang algebra. Taking $x = y = e$ in the middle Moufang identity $(xy)(zx) = [x(yz)]x$, we get $e(ze) = [e(ez)]e$, hence $(ez)e = [e(ez)]e$ because e is flexible. This implies that $ez = e(ez)$, and so $z = ez$ because A has no nonzero divisors of zero. Therefore, by [6, Proposition 2(ii)], A is unital. Taking $y = e$ in the middle Moufang identity $(xy)(zx) = [x(yz)]x$, we have also $(x, z, x) = 0$, that is, A is a flexible algebra. Keeping in mind that A is a middle Moufang algebra and A is a unital flexible algebra, we obtain the successive identities

$$\begin{aligned}
\big((x+e)y\big)\big(x(x+e)\big) &= (x+e)(yx)(x+e)\\
(xy+y)(x^2+x) &= (xyx+yx)(x+e)\\
(xy)x^2+xyx+yx^2+yx &= (xyx)x+xyx+(yx)x+yx\\
yx^2 &= (yx)x,
\end{aligned}$$

so A is a right alternative algebra, and hence A is alternative because of Lemma 2.3. □

Theorem 3.8. *Let A be a nonzero power-commutative algebraic real algebra with no nonzero divisor of zero. If A satisfies the right or left Moufang identity, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof. (i) Suppose that A is a right Moufang algebra. Let $x \in A \setminus \{0\}$. Then the subalgebra $A(x)$ is a finite-dimensional division commutative real algebra, and hence $dim(A(x)) \leq 2$ [9]. Since every finite-dimensional real algebra can be provided with an algebra norm [2, Proposition 1.1.7], it follows from [4, Theorem 2.1] that $A(x)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or the algebra $\mathcal{B}$ of basis $\{e, i\}$ and multiplication table $e^2 = e$, $ei = i = -ie$. Since $\mathcal{B}$ is not commutative, so $A(x)$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and hence $A(x)$ is associative. Now, it follows from the arbitrariness of x in $A \setminus \{0\}$ that A is power-associative. Therefore, by [2, Proposition 2.5.10(ii)], A is quadratic, and so A is alternative because of Lemma 2.3. Then, by [2, Theorem 2.5.29], A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

(ii) Suppose that A is a left Moufang algebra. This case follows from Case 1 by passing to the opposite algebra. □

Corollary 3.9. *Let A be a nonzero flexible algebraic real algebra with no nonzero divisor of zero. If A satisfies the right or left Moufang identity, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Theorem 3.10. *Let A be a nonzero algebraic real unital algebra with no nonzero divisor of zero and $dim(A) > 1$. Then A contains $\mathbb{C}$.*

Proof. Let e the unit of A and x a nonzero element of A not colinear to e. Then the subalgebra $A(x)$ is a finite-dimensional division real algebra. This implies that $A(x)$ contains a nonzero idempotent [12]. Since e is the unique nonzero idempotent of A, hence $e \in A(x)$, and so $dim(A(x)) \geq 2$ because x and e are not colinear. The Yang-Petro's theorem proves that $A(x)$ contains $\mathbb{C}$, and hence A contains $\mathbb{C}$. □

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