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Numeration of Rank in Elliptic Curve

$\boldsymbol{y^2 = x^3 - pqx}$

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Abstract

Suppose that E_{-pq} is an elliptic curve $y^2 = x^3 - pqx$ where p and q are distinct odd primes then, we will investigate the rank of it and compare the consequence with that of curves $y^2 = x^3 - 2px$ and $y^2 = x^3 - 4px$.

Mathematics Subject Classification: 11A41, 11G05

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1 Introduction

In [5], the author verified that rank of $y^2 = x^3 - 2px$ is 1 where prime is $p = 146s^4 + t^4 + 4u^4 + 24s^2t^2 + 48s^2u^2 + 4t^2u^2\ w.i.s.t.u.1$ and $p \equiv 3(mod\ 16)$ and the case $p = 486s^4 + 16t^4 + u^4 - 176s^2t^2 - 44s^2u^2 + 8t^2u^2\ w.i.s.t.u.1$ and $p \equiv 3(mod\ 16)$ and the case $p = 402s^4 + 16t^4 + u^4 - 160s^2t^2 - 40s^2u^2 + 8t^2u^2\ w.i.s.t.u.1$ and $p \equiv 3(mod\ 16)$ and that of curve $y^2 = x^3 - px$ is 1 where prime is such that $p = 1002002u^4 + 18018u^2v^2 + 81v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$. In [7], the author regarded ranks of curves $y^2 = x^3 - pqsx$ where different odd primes p and q and s are given as $p = u^2 + 4v^2$ and $q = u^4 - 10u^2v^2 + 88v^4$ and $s = u^2 + 22v^2\ \ w.i.u.v.1$ and $p \equiv 13(mod\ 16)$, $q \equiv 15(mod\ 16)$, $s \equiv 15(mod\ 16)$ and the case distinct odd primes p and q and s are $p = u^2 + 4v^2$ and $q = 3u^4 + 2u^2v^2 + 104v^4$ and $s = 3u^2 + 26v^2\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$, $q \equiv 13(mod\ 16)$, $s \equiv$

$13(mod\ 16)$ and that of $y^2 = x^3 - pqstx$ where different odd primes p and q and s and t are $p = u^2 + 4v^2$ and $q = 9u^2 + 10v^2$ and $s = 3u^2 + 4v^2$ and $t = 3u^2 + 10v^2$ $w.i.u.v.1$ and $p \equiv 13(mod\ 16)$, $q \equiv 11(mod\ 16)$, $s \equiv 15(mod\ 16)$ and $t \equiv 5(mod\ 16)$. In this paper, we shall consider the rank of $y^2 = x^3 - pqx$ where distinct odd primes p an q have more than 10 variables and 100 terms.

Take E and $\bar{E}$ as elliptic curves $y^2 = x^3 + ax^2 + bx$ and $y^2 = x(x^2 - 2ax + a^2 - 4b)$. Denote $N^2 = b_1M^4 + aM^2e^2 + b_2e^4$ and $N^2 = b_1M^4 - 2aM^2e^2 + b_2e^4$ as relating equations for Γ and $\bar{\Gamma}$ respectively in section 6 of chapter III in [8] and [3]. Let (M, e, N) be a solution of it in [8] and [3] respectively.

Then, we have that $2^r = \frac{\#\alpha(\Gamma)\#\bar{\alpha}(\Gamma)}{4}$ with rank r of E.

We define several notations as follows:

$w.i.c.d.f.g.h.i.j.k.l.s.t.u.v.w.z.1$: with integers c and d and f and g and h and i and j and k and l and s and t and u and v and w and z and $(c, d, f, g, h, i, j, k, l, s, t, u, v, w, z) = 1$([6]).

$w.i.t.u.1$: with integers t and u and $(t, u) = 1$.

$w.i.c.d.f.g.h.i.j.k.l.1$: with integers c and d and f and g and h and i and j and k and l and $(c, d, f, g, h, i, j, k, l) = 1$.

2 Ranks in E_{-pq} and other curves

On account of [3] we only should search the solution of $3) N^2 = pM^4 - qe^4$ for Γ and $5) N^2 = 2pM^4 + 2qe^4$ for $\bar{\Gamma}$ in E_{-pq} and from [2] we needed to find the solution of $4) N^2 = -2M^4 + pe^4$ for Γ in E_{-2p} and from [4] it is necessary to find the solution of $6) N^2 = -4M^4 + pe^4$ for Γ in E_{-4p}. $r4.4$ and $r4.2$ are in [3].

Theorem. 2.1. (1). Set p and q are distinct odd primes as $p = 27c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d + 10c^2f$ $+10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2$ $\cdot u + 10c^2v + 10c^2w + 10c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl$ $+2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2$ $\cdot fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl$ $+2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2h$ $\cdot t + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2i$

$\cdot w + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2 \cdot vz + 2wz$ and $q = 23c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d + 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10 \cdot c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2f \cdot z + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2j \cdot v + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2 \cdot lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz\ w.i.c.d.f.g.h.i.j.k.l.s.t.u.v.w.z.1$ and $p \equiv 11 (mod\ 16)$ and $q \equiv 7 (mod\ 16)$ in $E_{-pq}: y^2 = x^3 - pqx$ then, there educed that

$$rank(E_{-(27c^4+d^2+\cdots\cdots+2vz+2wz)(23c^4+d^2+\cdots\cdots+2vz+2wz)}(Q))$$

$$> rank(E_{-2(18t^4-8t^3u-8tu^3+52t^2u^2+27u^4)}(Q))$$

where E_{-2p} is an elliptic curve $y^2 = x^3 - 2px$ with prime as $p = 18t^4 - 8t^3u - 8tu^3 + 52t^2u^2 + 27u^4\ w.i.t.u.1, p \equiv 11 (mod\ 16)$.

(2). We denote p and q as $p = 11c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 6c^2d - 6c^2f + 6c^2g + 6c^2h + 6c^2i + 6 \cdot c^2j + 6c^2k + 6c^2l + 6c^2s + 6c^2t + 6c^2u + 6c^2v + 6c^2w + 6c^2z - 2df + 2 \cdot dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2fv - 2fw - 2f \cdot z + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2j \cdot v + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2 \cdot lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz$ and $q = 7c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 6c^2d - 6c^2f + 6c^2g + 6c^2h + 6c^2i + 6c^2j + 6c^2k + 6c^2l + 6c^2s + 6c^2t + 6c^2u + 6c^2v + 6c^2w + 6c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2d \cdot v + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2fv - 2fw - 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2 \cdot gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2j \cdot s + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2 \cdot kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2$

$\cdot\, tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz\; w.i.c.d.f.$ $g.h.i.j.k.l.s.t.u.v.w.z.1$ and $p \equiv 11(mod\ 16)$ and $q \equiv 7(mod\ 16)$ in elliptic curve $E_{-pq} : y^2 = x^3 - pqx$ then, there is given that

$$rank(E_{-(11c^4+d^2+\cdots\cdots+2vz+2wz)(7c^4+d^2+\cdots\cdots+2vz+2wz)}(Q))$$

$$> rank(E_{-2(38t^4-8t^3u-8tu^3+24t^2u^2+3u^4)}(Q))$$

where elliptic curve $E_{-2p}: y^2 = x^3 - 2px$ is given as $p = 38t^4 - 8t^3u - 8tu^3 + 24t^2u^2 + 3u^4\; w.i.t.u.1, p \equiv 3(mod\ 16)$.

(3). Assume that p and q are $p = c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + 3l^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2c^2l + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2gh + 2gi + 2gj + 2gk + 2gl + 2hi + 2hj + 2hk + 2hl + 2ij + 2ik + 2il + 2jk + 2jl + 2kl$ and $q = c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 - l^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2c^2l + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2gh + 2gi + 2gj + 2gk + 2gl + 2hi + 2hj + 2hk + 2hl + 2ij + 2ik + 2il + 2jk + 2jl + 2kl\; w.i .c.d.f.g.h.i.j.k.l.1$ and $p \equiv 11(mod\ 16)$ and $q \equiv 7(mod\ 16)$ in $E_{-pq} : y^2 = x^3 - pqx$ then, we take that

$$rank(E_{-(c^4+\cdots\cdots+3l^2+\cdots\cdots+2jl+2kl)(c^4+\cdots\cdots-l^2+\cdots\cdots+2jl+2kl)}(Q))$$

$$> rank(E_{-4(68t^4+16t^3u+16tu^3+72t^2u^2+13u^4)}(Q))$$

where $E_{-4p}: y^2 = x^3 - 4px$ is assumed as the form $p = 68t^4 + 16t^3u + 16tu^3 + 72t^2u^2 + 13u^4\; w.i.t.u.1, p \equiv 13(mod\ 16)$.

Proo$f$. (1). Relating equation 3) for Γ is $N^2 = (27c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d + 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2f \cdot t + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2h \cdot v + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)M^4 - (23c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d + 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z + 2df + 2dg + 2dh + 2d \cdot i + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh$

$+2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2g \cdot i + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2j \cdot z + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2 \cdot lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)e^4$. The solution of this equation is gotten as (1, 1, $2c^2$). Thus, it implies that $\#\alpha(\Gamma) = 4$. In the next step, equation 5) for $\bar{\Gamma}$ is $N^2 = 2(27c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d + 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10 \cdot c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z + 2df + 2dg + 2d \cdot h + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg +2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2g \cdot h + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi +2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il +2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2 \cdot jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)M^4 + 2(23c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d + 10c^2f + 10c^2g +10c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2 \cdot v + 10c^2w + 10c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs +2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2 \cdot gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu +2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2 \cdot kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2w \cdot z)e^4$. It is possible to guess that the square $100c^4$ will be appeared from two terms $54c^4$ and $46c^4$ in coefficients of M^4 and e^4. And there are terms from $2d^2$ to $2z^2$ in both coefficients of M^4 and e^4. Next, there exist the terms from $20c^2d$ to $20c^2z$ in both coefficients. Thus, we select first component of N as $10c^2$. Next, there exist common terms from $4df$ to $4wz$ in both coefficients of M^4 and e^4. Thereby, we choose other components as $2d$ and $2f$ and $2g$ and $2h$ and $2i$ and $2j$ and $2k$ and $2l$ and $2s$ and $2t$ and $2u$ and $2v$ and $2w$ and $2z$. And this selection can be done under the hypothesis that $M = e = 1$. Henceforth, the triple (1, 1, $10c^2 + 2d + 2f + 2g + 2h + 2i + 2j + 2k + 2l + 2s + 2t + 2u + 2v + 2w + 2z$) is given as the solution of equation. Whence, it shows that $\#\bar{\alpha}(\bar{\Gamma}) = 4$. And it gives that $rank(E_{-(27c^4+d^2+\cdots\cdots+2vz+2wz)(23c^4+d^2+\cdots\cdots+2vz+2wz)}(Q)) = 2$ from $r4.4$. Now we consider the rank of curve E_{-2p}. In arithmetical values $-2M^4 + 18t^4e^4$ and $-2M^4 + 27u^4e^4$ we can expect an appearance of squares $16t^4$ and $25u^4$. For this in value M there must exist t and u. We choose M as $t - u$. In computation $-2(t-u)^4$ we can gain the terms $8t^3u$ and $8tu^3$. From the calculation $8t^3u + 8tu^3 - 8t^3u - 8tu^3$ the terms $-8t^3u$ and $-8tu^3$ are erased. Now we needed to consider remaining terms $-12t^2u^2 + 52t^2u^2$. Eventually, we get the numerical value $16t^4{+}40t^2u^2 + 25u^4$. Wherefore, the triple $(t-u,\, 1,\, 4t^2 + 5u^2)$ satisfies the solution. Henceforth, we take that $\#\alpha(\Gamma) = 4$ and $r4.2$. Accordingly, we acquire the result $rank(E_{-2(18t^4-8t^3u-8tu^3+52t^2u^2+27u^4)}(Q)) = 1$. Thus, we achieved the proof of (1).

(2). There is gotten relating equation 3) for Γ as $N^2 = (11c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 6c^2d - 6c^2f + 6c^2g + 6c^2h + 6c^2i + 6c^2j + 6c^2k + 6c^2l + 6c^2s + 6c^2t + 6c^2u + 6c^2v + 6c^2w + 6c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2d \cdot v + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - -2fv - 2fw - 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2 \cdot gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2j \cdot s + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2 \cdot kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2 \cdot tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)M^4 - (7c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 6c^2 \cdot d - 6c^2f + 6c^2g + 6c^2h + 6c^2i + 6c^2j + 6c^2k + 6c^2l + 6c^2s + 6c^2t + 6c^2 \cdot u + 6c^2v + 6c^2w + 6c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2 \cdot ds + 2dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2fv - 2fw - 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2g \cdot s + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2 \cdot iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2k \cdot u + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2 \cdot sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)e^4$. We have the triple $(1, 1, 2c^2)$ as the solution of 3). Accordingly, we gain $\#\alpha(\Gamma) = 4$. Next, equation 5) for $\bar{\Gamma}$ is $N^2 = 2(11c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 6c^2d - 6c^2f + 6c^2g + 6 \cdot c^2h + 6c^2i + 6c^2j + 6c^2k + 6c^2l + 6c^2s + 6c^2t + 6c^2u + 6c^2v + 6c^2w + 6 \cdot c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2f \cdot v - 2fw - 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2h \cdot z + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2 \cdot tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)M^4 + 2(7c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 6c^2d - 6 \cdot c^2f + 6c^2g + 6c^2h + 6c^2i + 6c^2j + 6c^2k + 6c^2l + 6c^2s + 6c^2t + 6c^2u + 6 \cdot c^2v + 6c^2w + 6c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2 \cdot dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2fv - 2fw - 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2$

$\cdot gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2 \cdot kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2w \cdot z)e^4$. Due to two terms $22c^4$ and $14c^4$ in both coefficients of M^4 and e^4 the square $36c^4$ can be derived. In addition, in both coefficients there exist common terms from $12c^2d$ to $12c^2z$ and between these there is a negative term $-12c^2f$. We also can find negative terms $-4df$ and from $-4fg$ to $-4fz$. When there is a variable f then, there is a negative term. Hence, in choosing the components of N we take it as $6c^2$ and $2d$ and $-2f$ and $2g$ and $2h$ and $2i$ and $2j$ and $2k$ and $2l$ and $2s$ and $2t$ and $2u$ and $2v$ and $2w$ and $2z$. Then, this selection is matched to primes. And the selection was done on the supposition that $M = e = 1$. Consequently, the solution of equation 5) is induced as $(1, 1, 6c^2 + 2d - 2f + 2g + 2h + 2i + 2j + 2k + 2l + 2s + 2t + 2u + 2v + 2w + 2z)$. And it induces both $\#\bar{\alpha}(\bar{\Gamma}) = 4$ and $r4.4$. On this account, we acquire the consequence $rank(E_{-(11c^4+d^2+\cdots\cdots+2wz)(7c^4+d^2+\cdots\cdots+2wz)}(Q)) = 2$. In the next step, we treat rank of E_{-2p}. From the numerical values $-2M^4 + 38t^4e^4$ and $-2M^4 + 3u^4e^4$ there exists a potentiality that squares $36t^4$ and u^4 will be deduced. For it the integer M should take the variables t and u. Assign the value e as 1. As in (1) in the above, there are the terms $-8t^3u$ and $-8tu^3$ in coefficient of e^4. Accordingly, take M as $t - u$. We already checked the terms $-8t^3u$ and $-8tu^3$ are crossed out. And there is left $-12t^2u^2 + 24t^2u^2$. And we are faced with the terms $12t^2u^2$ and $36t^4$ and u^4. Henceforth, we gain the square $36t^4 + 12t^2u^2 + u^4$. Consequentially, it is educed the triple $(t - u, 1, 6t^2 + u^2)$ as the solution. Thus, we arrive at that $\#\alpha(\Gamma) = 4$ and $r4.2$. Accordingly, it deduces that $rank(E_{-2(38t^4-8t^3u-8tu^3+24t^2u^2+3u^4)}(Q)) = 1$. Henceforth, we completed the proof of (2).

(3). We only needed to find the solution of equations 3) for Γ as $N^2 = (c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + 3l^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2c^2l + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2gh + 2gi + 2gj + 2gk + 2gl + 2hi + 2hj + 2hk + 2hl + 2ij + 2ik + 2il + 2jk + 2jl + 2kl)M^4 - (c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 - l^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2 \cdot c^2l + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2gh + 2gi + 2gj + 2gk + 2gl + 2hi + 2hj + 2hk + 2hl + 2ij + 2ik + 2il + 2jk + 2jl + 2kl)e^4$ and 5) for $\bar{\Gamma}$ as $N^2 = 2(c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + 3l^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2c^2l + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2fg + 2fh + 2fi + 2 \cdot fj + 2fk + 2fl + 2gh + 2gi + 2gj + 2gk + 2gl + 2hi + 2hj + 2hk + 2hl + 2ij + 2ik + 2il + 2jk + 2jl + 2kl)M^4 + 2(c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 - l^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2c^2l + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2gh + 2gi + 2gj + 2gk + 2gl + 2hi + 2hj + 2hk + 2hl + 2ij + 2ik + 2il +$

$2jk + 2jl + 2kl)e^4$. Equation 3) has a solution $(1, 1, 2l)$, hence we are faced with $\#\alpha(\Gamma) = 4$. In the next step, the triple $(1, 1, 2c^2 + 2d + 2f + 2g + 2h + 2i + 2j + 2k + 2l)$ satisfies the solution of relating equation 5). Wherefore, it implies that $\#\bar{\alpha}(\bar{\Gamma}) = 4$. Thereby, it gives that $r4.4$. Accordingly, there derived that $rank(E_{-(c^4+\cdots\cdots+3l^2+\cdots\cdots+2jl+2kl)(c^4+\cdots\cdots-l^2+\cdots\cdots+2jl+2kl)}(Q)) = 2$. Next, we compute rank of curve E_{-4p}. The noticeable thing in coefficient of e^4 is symbols of terms for t^3u and tu^3 are positive. Therefore, select the value M as $t + u$. Owing to numeration $-4(t + u)^4$ the numerical value $-16t^3u - 16tu^3$ is given. Whence, terms for t^3u and tu^3 are crossed out. Under the hypothesis that e and M as 1 and $t + u$ there are remained $64t^4$ and $9u^4$ and $48t^2u^2$. Thus, the value N is given as $8t^2 + 3u^2$. For this reason, because of $\#\alpha(\Gamma) = 4$ and $r4.2$ we conclude that $rank(E_{-4(68t^4+16t^3u+16tu^3+72t^2u^2+13u^4)}(Q)) = 1$. On this account, we accomplished the proof. □

The consequences of (1) and (2) were succeeded of it in [6]. In result of (3) the numbers of variables are 9 and that of terms are 45. There is left the probability that the numbers of variables and terms in primes p and q between (1), (2) and (3) can be deduced. As a conjecture that there exists an elliptic curve which has large rank there is a possibility that there can be induced an elliptic curve $E_{-pq} : y^2 = x^3 - pqx$ with rank 2 where distinct odd primes p and q have many variables and terms. We have no idea that ‘many’ denotes how large value is.

Remark 2.2. In E_{pq} there deduced the consequences of rank at least 2 and rank 4. But in relative the rank of 3 is found less. We can attain rank at least 2 when primes are $p \equiv 1 (mod\ 8)$ and $q \equiv 3 (mod\ 16)$, $p \equiv 1 (mod\ 8)$ and $q \equiv 7 (mod\ 16)$, $p \equiv 1 (mod\ 8)$ and $q \equiv 15 (mod\ 16)$ and rank 4 can be given in $p \equiv 1 (mod\ 8)$ and $q \equiv 1 (mod\ 8)$.

Remark 2.3. In curve $E_{-4p} : y^2 = x^3 - 4px$ prime is gotten as $p \equiv 13 (mod\ 16)$ in the above. In $E_{-p} : y^2 = x^3 - px$ theoretically when prime is $p \equiv 5$, $13 (mod\ 16)$ then, rank can be derived 1 but in practically we confront to rank 1 when prime is $p \equiv 5 (mod\ 16)$ mainly. In $p \equiv 13 (mod\ 16)$ finding rank 1 is not simple in curve $E_{-p} : y^2 = x^3 - px$. Meanwhile in $E_{-4p} : y^2 = x^3 - 4px$ we can obtain rank 1 often when primes are $p \equiv 5, 13 (mod\ 16)$.

Remark 2.4. In curve $E_{2p} : y^2 = x^3 + 2px$ there derived rank 0 and 2 but finding rank 1 is not simple in relative. In $p \equiv 3, 5, 11, 13 (mod\ 16)$ the rank 0 is induced.

3 Examples

In this section 3, examples will be given. Primality was done in [1].

There appeared examples of theorem 2.1(1):

$(p, q, c, d, f, g, h, i, j, k, l, s, t, u, v, w, z)$:

(443, 439, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3) and

(13691, 13687, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 99);

(p, t, u):

(443, 2, 1) and (72923, 8, 1).

There deduced examples of theorem 2.1(2):

$(p, q, c, d, f, g, h, i, j, k, l, s, t, u, v, w, z)$:

(443, 439, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 7) and

(13691, 13687, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 103);

(p, t, u):

(1091, 2, 3) and (11027, 4, 3).

Examples of theorem 2.1(3) are followings:

$(p, q, c, d, f, g, h, i, j, k, l)$:

(139, 103, 1, 1, 1, 1, 1, 1, 1, 1, 3) and (1451, 7, 1, 1, 1, 1, 1, 1, 1, 1, 19);

(p, t, u):

(1549, 2, 1) and (5981, 2, 3) and (19661, 4, 1) and (33629, 4, 3).

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