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Elliptic Curve $y^2 = x^3 - pqx$ with Rank

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Abstract

We take E_{-pq} as an elliptic curve $y^2 = x^3 - pqx$ where distinct odd primes p and q are the forms $p = (\ )c^4 + (\ )d^2 + (\ )f^2 + (\ )g^2 + (\ )h^2 + (\ )i^2 + (\ )j^2 + (\ )k^2 + (\ )l^2 + (\ )s^2 + (\ )t^2 + (\ )u^2 + (\ )v^2 + (\ )w^2 + (\ )z^2 + (\ ) \cdot c^2d + (\ )c^2f + (\ )c^2g + (\ )c^2h + (\ )c^2i + (\ )c^2j + (\ )c^2k + (\ )c^2l + (\ ) \cdot c^2s + (\ )c^2t + (\ )c^2u + (\ )c^2v + (\ )c^2w + (\ )c^2z + (\ )df + (\ )dg + (\ )dh + (\ )di + (\ )dj + (\ )dk + (\ )dl + (\ )ds + (\ )dt + (\ )du + (\ )dv + (\ )dw + (\ )dz + (\ )fg + (\ )fh + (\ )fi + (\ )fj + (\ )fk + (\ )fl + (\ )fs + (\ )ft + (\ )fu + (\ )fv + (\ )fw + (\ )fz + (\ )gh + (\ )gi + (\ )gj + (\ )gk + (\ )gl + (\ )gs + (\ )gt + (\ )gu + (\ )gv + (\ )gw + (\ )gz + (\ )hi + (\ )hj + (\ )hk + (\ )hl + (\ )hs + (\ )ht + (\ )hu + (\ )hv + (\ )hw + (\ )hz + (\ )ij + (\ )ik + (\ )il + (\ )is + (\ )it + (\ )iu + (\ )iv + (\ )iw + (\ )iz + (\ )jk + (\ )jl + (\ )js + (\ )jt + (\ )ju + (\ )jv + (\ )jw + (\ )jz + (\ )kl + (\ )ks + (\ )kt + (\ )ku + (\ )kv + (\ )kw + (\ )kz + (\ )ls + (\ )lt + (\ )lu + (\ )lv + (\ )lw + (\ )lz + (\ )st + (\ )su + (\ )sv + (\ )sw + (\ )sz + (\ )tu + (\ )tv + (\ )tw + (\ )tz + (\ )uv + (\ )uw + (\ )uz + (\ )vw + (\ )vz + (\ )wz$ and $q = \{\ \}c^4 + \{\ \}d^2 + \{\ \}f^2 + \{\ \}g^2 + \{\ \}h^2 + \{\ \}i^2 + \{\ \}j^2 + \{\ \}k^2 + \{\ \}l^2 + \{\ \}s^2 + \{\ \}t^2 + \{\ \}u^2 + \{\ \}v^2 + \{\ \}w^2 + \{\ \}z^2 + \{\ \}c^2d + \{\ \}c^2f + \{\ \}c^2g + \{\ \}c^2h + \{\ \}c^2i + \{\ \}c^2j + \{\ \}c^2k + \{\ \}c^2l + \{\ \} \cdot c^2s + \{\ \}c^2t + \{\ \}c^2u + \{\ \}c^2v + \{\ \}c^2w + \{\ \}c^2z + \{\ \}df + \{\ \}dg + \{\ \}dh + \{\ \}di + \{\ \}dj + \{\ \}dk + \{\ \}dl + \{\ \}ds + \{\ \}dt + \{\ \}du + \{\ \}dv + \{\ \}dw + \{\ \}dz + \{\ \}fg + \{\ \}fh + \{\ \}fi + \{\ \}fj + \{\ \}fk + \{\ \}fl + \{\ \}fs + \{\ \}ft + \{\ \}fu + \{\ \}fv + \{\ \}fw + \{\ \}fz + \{\ \}gh + \{\ \}gi + \{\ \}gj + \{\ \}gk + \{\ \}gl + \{\ \}gs + \{\ \}gt + \{\ \}gu + \{\ \}gv + \{\ \}gw + \{\ \}gz + \{\ \}hi + \{\ \}hj + \{\ \}hk + \{\ \}hl + \{\ \}hs + \{\ \}ht + \{\ \}hu + \{\ \}hv + \{\ \}hw + \{\ \}hz + \{\ \}ij + \{\ \}ik + \{\ \}il + \{\ \}is + \{\ \}it + \{\ \}iu + \{\ \}iv + \{\ \}iw + \{\ \}iz + \{\ \}jk + \{\ \}jl + \{\ \}js + \{\ \}jt + \{\ \}ju + \{\ \}jv + \{\ \}jw + \{\ \}jz + \{\ \}kl + \{\ \}ks + \{\ \}kt + \{\ \}ku + \{\ \}kv + \{\ \}kw + \{\ \}kz + \{\ \}ls + \{\ \}lt + \{\ \}lu + \{\ \}lv + \{\ \}lw + \{\ \}lz + \{\ \}st + \{\ \}su + \{\ \}sv + \{\ \}sw +$

$\{\ \}sz + \{\ \}tu + \{\ \}tv + \{\ \}tw + \{\ \}tz + \{\ \}uv + \{\ \}uw + \{\ \}uz + \{\ \}vw + \{\ \} \cdot vz + \{\ \}wz$ then, we shall compute the rank of curve.

Mathematics Subject Classification: 11G05, 11A41

Keywords: Elliptic curve, distinct odd primes

1 Introduction

In [7], the author treated rank of curve $E_{-pq}: y^2 = x^3 - pqx$ where primes are $p = c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + 3 \cdot z^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2c^2l + 2c^2s + 2c^2 \cdot t + 2c^2u + 2c^2v + 2c^2w + 2c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2 \cdot dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2g \cdot l + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2 \cdot iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2k \cdot t + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2 \cdot su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz$ and $q = c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 - z^2 + 2c^2d + 2c^2f + 2c^2g + 2c^2h + 2c^2i + 2c^2j + 2c^2k + 2 \cdot c^2l + 2c^2s + 2c^2t + 2c^2u + 2c^2v + 2c^2w + 2c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2 \cdot fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2h \cdot k + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2i \cdot t + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz\ w.i.c.d.f.g.h.i.j.k.l.s.t.u.v.w.z.1$ and $p \equiv 3(mod\ 16)$ and $q \equiv 15(mod\ 16)$. The rank was 2 and there were more cases in it. Here, we also consider the rank of E_{-pq} where distinct primes are the same forms.

Denote E and $\bar{E}$ as elliptic curves $y^2 = x^3 + ax^2 + bx$ and $y^2 = x(x^2 - 2ax + a^2 - 4b)$ and $N^2 = b_1M^4 + aM^2e^2 + b_2e^4$, $N^2 = b_1M^4 - 2aM^2e^2 + b_2e^4$ as relating equations for Γ and for $\overline{\Gamma}$ respectivley in section 6 of chapter III in [9] and [5]. Assign the triple (M, e, N) as a solution of these equations that satisfies the conditions in [9] and [5] respectively. Then, we gain $2^r = \frac{\#\alpha(\Gamma)\#\bar{\alpha}(\Gamma)}{4}$ with rank r of E.

We define several notations as follows:

$w.i.c.d.f.g.h.i.j.k.l.s.t.u.v.w.z.1$: with integers c and d and f and g and h and i and j and k and l and s and t and u and v and w and z and $(c, d, f, g, h, i, j, k, l, s, t, u, v, w, z) = 1$([7]).

$w.i.t.u.1$: with integers t and u and $(t, u) = 1$.

$w.i.u.v.1$: with integers u and v and $(u, v) = 1$([6]).

2 Rank in Curve E_{-pq}

Due to [6] and [3] we only needed to find the solution of two equations 3)$N^2 = pM^4 - qe^4$ for Γ and 5)$N^2 = 2pM^4 + 2qe^4$ for $\bar{\Gamma}$. $r4.4$ and $r4.2$ are in [6].

Theorem. 2.1. (1). Take distinct odd primes p and q as $p = 123c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 22c^2d + 22c^2f + 22c^2g + 22c^2h + 22c^2i + 22c^2j + 22c^2k + 22c^2l + 22c^2s + 22c^2t + 22c^2 \cdot u + 22c^2v + 22c^2w + 22c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2 \cdot fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2h \cdot t + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2i \cdot w + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2 \cdot vz + 2wz$ and $q = 119c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 22c^2d + 22c^2f + 22c^2g + 22c^2h + 22c^2i + 22c^2j + 22c^2k + 22c^2l + 22c^2s + 22c^2t + 22c^2u + 22c^2v + 22c^2w + 22c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2 \cdot dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2 \cdot gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz$ $w.i.c.d.f.g.h.i.j.k.l.s.t.u.v.w.z.1$ and $p \equiv 11(mod\ 16)$ and $q \equiv 7(mod\ 16)$ in $E_{-pq}: y^2 = x^3 - pqx$ and we appoint that p is $p = 66t^4 - 8t^3u - 8tu^3 + 28t^2u^2 + 3u^4$ $w.i.t.u.1$, $p \equiv 3(mod\ 16)$ in elliptic curve $E_{-2p}: y^2 = x^3 - 2px$ then, it is induced that

$$rank(E_{-(123c^4+d^2+\cdots\cdots+z^2+\cdots\cdots+2vw+2vz+2wz)(119c^4+d^2+\cdots\cdots+z^2+\cdots\cdots+2vw+2vz+2wz)}(Q))$$

$$> rank(E_{-2(66t^4-8t^3u-8tu^3+28t^2u^2+3u^4)}(Q)).$$

(2). Let p and q be distinct odd primes as $p = 27c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d - 10c^2f + 10c^2g + 10 \cdot c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v +10c^2w + 10c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt +2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2 \cdot ft - 2fu - 2fv - 2fw - 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt +2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2h \cdot v + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk +2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz$ and $q = 23c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 +z^2 + 10c^2d - 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z - 2df + 2dg + 2dh + 2d \cdot i + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh -2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2fv - 2fw - 2fz + 2gh + 2g \cdot i + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2j \cdot z + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2 \cdot lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz\ w.i.c.d.f.g.h.i.j.k.l.s.t.u.v.w.z.1$ and $p \equiv 3(mod\ 16)$ and $q \equiv 15(mod\ 16)$ in elliptic curve $E_{-pq} : y^2 = x^3 - pqx$ and take the curves E_{-2p} and E_{-p} as $y^2 = x^3 - 2px$ with prime of the form $p = 1371u^4 + 296u^2v^2 + 16v^4\ w.i.u.v.1$ and $p \equiv 11(mod\ 16)$ and $y^2 = x^3 - px$ with prime as $p = 50u^4 + 98u^2v^2 + 49v^4\ w.i.u.v.1$ and $p \equiv 5(mod\ 16)$ respectively then, we are faced with

$$rank(E_{-(27c^4+d^2+\cdots\cdots+z^2+\cdots\cdots+2vw+2vz+2wz)(23c^4+d^2+\cdots\cdots+z^2+\cdots\cdots+2vw+2vz+2wz)}(Q))$$

$$> rank(E_{-2(1371u^4+296u^2v^2+16v^4)}(Q))$$

$$= rank(E_{-(50u^4+98u^2v^2+49v^4)}(Q)).$$

Proof. (1). Equation 3) for Γ is given as $N^2 = (123c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 22c^2d + 22c^2f + 22c^2g + 22c^2h + 22c^2i + 22c^2j + 22c^2k + 22c^2l + 22c^2s + 22c^2t + 22c^2u + 22c^2v +22c^2w + 22c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt +2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2f$

$\cdot t + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2h \cdot v + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)M^4 - (119c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 22c^2d + 22c^2f + 22c^2g + 22c^2h + 22c^2i + 22c^2j + 22c^2k + 22c^2l + 22c^2s + 22c^2t + 22c^2u + 22c^2v + 22c^2w + 22c^2z + 2df + 2dg + 2dh + 2d \cdot i + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2g \cdot i + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2j \cdot z + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2 \cdot lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)e^4$. The triple $(1, 1, 2c^2)$ satisfies the solution of this equation. Wherefore, we have that $\#\alpha(\Gamma) = 4$. Next, equation 5) for $\bar{\Gamma}$ is written as $N^2 = 2(123c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 22c^2d + 22c^2f + 22c^2g + 22c^2h + 22c^2i + 22c^2j + 22 \cdot c^2k + 22c^2l + 22c^2s + 22c^2t + 22c^2u + 22c^2v + 22c^2w + 22c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2f \cdot z + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2j \cdot v + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2 \cdot lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)M^4 + 2(119c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 22c^2d + 22c^2f + 22c^2g + 22c^2h + 22c^2i + 22c^2j + 22c^2k + 22c^2l + 22c^2s + 22c^2t + 22c^2u + 22c^2v + 22c^2w + 22c^2z + 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2 \cdot ds + 2dt + 2du + 2dv + 2dw + 2dz + 2fg + 2fh + 2fi + 2fj + 2fk + 2fl + 2fs + 2ft + 2fu + 2fv + 2fw + 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2g \cdot s + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2 \cdot iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2k \cdot u + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2 \cdot sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)e^4$. From the terms $246c^4$ and $238c^4$ in coefficients of M^4 and e^4 we can anticipate an appearance of $484c^4$. And there are common terms from $2d^2$ to $2z^2$ in each coefficient. Thus, we select the components of N as $22c^2$ and $2d$ and $2f$ and $2g$ and $2h$ and $2i$ and $2j$ and $2k$ and $2l$ and $2s$ and $2t$ and $2u$ and $2v$ and $2w$ and $2z$. Now taking a pair in each case (multiplication) derives from the terms

$88c^2d$ to $88c^2z$. And other terms are also derived similarly. Consequently, the solution of equation is deduced as $(1,\, 1,\, 22c^2 + 2d + 2f + 2g + 2h + 2i + 2j + 2k + 2l + 2s + 2t + 2u + 2v + 2w + 2z)$. Accordingly, we gain $\#\bar{\alpha}(\bar{\Gamma}) = 4$. Thus, we take that $rank(E_{-(123c^4+d^2+\cdots\cdots+2vz+2wz)(119c^4+d^2+\cdots\cdots+2vz+2wz)}(Q)) = 2$ from $r4.4$. In addition, it is sufficient that we find the solution of equation 4)$N^2 = -2M^4 + (66t^4 - 8t^3u - 8tu^3 + 28t^2u^2 + 3u^4)e^4$ for Γ from [2]. Take $M = t - u$ and $e = 1$ then, we are faced with $-2(t-u)^4 + (66t^4 - 8t^3u - 8tu^3 + 28t^2u^2 + 3u^4) = 64t^4 + 16t^2u^2 + u^4$. It yields that $N = 8t^2 + u^2$. Thus, the triple $(t - u,\, 1,\, 8t^2 + u^2)$ is deduced as the solution of equation 4). It gives that $\#\alpha(\Gamma) = 4$ and so $rank(E_{-2(66t^4-8t^3u-8tu^3+28t^2u^2+3u^4)}(Q)) = 1$ is given from $r4.2$.

(2). Relating equation 3) for Γ is $N^2 = (27c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d - 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2fv - 2fw - 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2 \cdot gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2j \cdot s + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2 \cdot kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2 \cdot tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)M^4 - (23c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10 \cdot c^2d - 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2fv - 2fw - 2fz + 2gh + 2gi + 2gj + 2g \cdot k + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2 \cdot iu + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2k \cdot s + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2 \cdot st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)e^4$. The triple $(1,\, 1,\, 2c^2)$ is induced as the solution of equation. Thus, it is derived that $\#\alpha(\Gamma) = 4$. In the next step, equation 5) for $\bar{\Gamma}$ is $N^2 = 2(27c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d - 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10 \cdot c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z - 2df + 2dg + 2d \cdot h + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2ft - 2fu - 2fv - 2fw - 2fz + 2g \cdot h + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2hv + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2j \cdot w + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2 \cdot lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz +$

$2uv + 2uw + 2uz + 2vw + 2vz + 2wz)M^4 + 2(23c^4 + d^2 + f^2 + g^2 + h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + z^2 + 10c^2d - 10c^2f + 10c^2g + 10c^2h + 10c^2i + 10c^2j + 10c^2k + 10c^2l + 10c^2s + 10c^2t + 10c^2u + 10c^2v + 10c^2w + 10c^2z - 2df + 2dg + 2dh + 2di + 2dj + 2dk + 2dl + 2ds + 2dt + 2du + 2dv + 2dw + 2dz - 2fg - 2fh - 2fi - 2fj - 2fk - 2fl - 2fs - 2f \cdot t - 2fu - 2fv - 2fw - 2fz + 2gh + 2gi + 2gj + 2gk + 2gl + 2gs + 2gt + 2gu + 2gv + 2gw + 2gz + 2hi + 2hj + 2hk + 2hl + 2hs + 2ht + 2hu + 2h \cdot v + 2hw + 2hz + 2ij + 2ik + 2il + 2is + 2it + 2iu + 2iv + 2iw + 2iz + 2jk + 2jl + 2js + 2jt + 2ju + 2jv + 2jw + 2jz + 2kl + 2ks + 2kt + 2ku + 2kv + 2kw + 2kz + 2ls + 2lt + 2lu + 2lv + 2lw + 2lz + 2st + 2su + 2sv + 2sw + 2sz + 2tu + 2tv + 2tw + 2tz + 2uv + 2uw + 2uz + 2vw + 2vz + 2wz)e^4$.

From two terms $54c^4$ and $46c^4$ in coefficients of M^4 and e^4 there can be deduced the square $100c^4$. Next, there are common terms from $2d^2$ to $2z^2$ in each coefficient. Furthermore, there are common terms from $20c^2d$ to $20c^2z$ in each coefficient. Hence in selecting the component of N we take $10c^2$. And there exist common negative terms $-20c^2f$ and $-4df$ and $-4fg$, $-4fh$, $-4fi$, $-4fj$, $-4fk$, $-4fl$, $-4fs$, $-4ft$, $-4fu$, $-4fv$, $-4fw$, $-4f$z. In all these terms, the variable f exists, thus we choose $-2f$ as a component of N. Next, we select other components as $2d$ and $2g$ and $2h$ and $2i$ and $2j$ and $2k$ and $2l$ and $2s$ and $2t$ and $2u$ and $2v$ and $2w$ and $2z$. Henceforth, the triple (1, 1, $10c^2 + 2d - 2f + 2g + 2h + 2i + 2j + 2k + 2l + 2s + 2t + 2u + 2v + 2w + 2z$) is gotten as the solution of equation. Thus, it reveals that both $\#\bar{\alpha}(\bar{\Gamma}) = 4$ and $r4.4$. Thereby, the consequence $rank(E_{-(27c^4+d^2+\cdots\cdots+2wz)(239c^4+d^2+\cdots\cdots+2wz)}(Q)) = 2$ is induced. Moreover, from [2] there is given only equation 4)$N^2 = -2M^4 + (1371u^4 + 296u^2v^2 + 16v^4)e^4$ for Γ which requires to check the solvability. We take that $M = u$ and $e = 1$ then, we say that $-2u^4 + (1371u^4 + 296u^2v^2 + 16v^4) = 1369u^4 + 296u^2v^2 + 16v^4$. Thus, we acquire the integer N as $37u^2 + 4v^2$. Thereby, it induces that #α(Γ) = 4. On that account, we are faced with $r4.2$. To conclude, we reach the result $rank(E_{-2(1371u^4+296u^2v^2+16v^4)}(Q)) = 1$. Finally, owing to [4], we must search the solution of equation 2)$N^2 = -M^4 + (50u^4 + 98u^2v^2 + 49v^4)e^4$ for Γ. The existence of square $49v^4$ in coefficient of e^4 makes possible to induce a square in resultant. If we suppose that both $98u^2v^2$ and $49v^4$ are consisted of resultant then, there ought to be appeared $49u^4$. In arithmetical value $-M^4 + 50u^4e^4$ we assume that $e = 1$. Hence, if M is gotten as u then, we get that $-u^4 + 50u^4 = 49u^4$. Therefore, N is educed as $7u^2 + 7v^2$. Thus, we reach that #α(Γ) = 4. Consequentially, we acquire the conclusion $r4.2$. And it gives that $rank(E_{-(50u^4+98u^2v^2+49v^4)}(Q)) = 1$. □

The coefficients of c^4 are 123, 119 and 27, 23 in (1) and (2) respectively. In [7] the coefficients of c^4 are all 1. The consequences in [7] are earlier than the above, thus here it is possible to anticipate that the coefficients are gotten as 7, 3 or 11, 7. But the consequence is 27, 23. And for above (1) due to 27, 23 it seems to be derived the coefficients as 51, 47 or 71, 67 but 123, 119 is given. In computation of rank in elliptic curve there is prerequisite that there is no general method. This can be applied to detail treatment when calculating the rank of elliptic curve. In above consequences it also can be applied. The number of variables and terms in primes are 15 and 120 respectively in the above. There is a probability that more variables and terms in primes that derives rank 2 will be appeared. Namely, there exist a possibility that the primes with having more complex forms can be given. Here there is a premise that variables are arranged as part of monomial and multiplication. But we cannot exclude the potentiality that new primes. In detail, in multiplication part there are three variables or more.

Remark 2.2. We can get more primes in curve $y^2 = x^3 - pqx$ as in the above. That is, there can be educed rank 2 in $y^2 = x^3 - pqx$ where primes p and q are consisted of 15 variables and 120 terms.

Remark 2.3. In [Exercise 2.12.4 in Chapter 2, 8], if C and E are curves defined, respectively, by $C: V^2 = U^4 + 1$ and $E: y^2 = x^3 - 4x$ and ψ be the map defined by $\psi(U,V) = (\frac{2(V+1)}{U^2}, \frac{4(V+1)}{U^3})$ then, if $U \neq 0$ and $(U,V) \in C(Q)$, then $\psi(U,V) \in E(Q)$. We show about this here. It is sufficient that we proved that $16(V+1)^2 = 8(V+1)^3 - 8(V+1)U^4$. Now from the notation $C: V^2 = U^4 + 1$ we confront to RHS as $8(V+1)^3 - 8(V+1)U^4 = 8(V+1)^3 - 8(V+1)(V^2-1) = 8(V+1)\left((V+1)^2 - (V^2-1)\right) = 8(V+1)(2V+2)$ and this is same to LHS.

3 Examples

In this section 3, we suggest examples of previous consequences. Primality was checked in [1].

Examples of theorem 2.1(1) are the followings:

$(p, q, c, d, f, g, h, i, j, k, l, s, t, u, v, w, z)$:

(13691, 13687, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 93) and

(56171, 56167, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 213);

(p, t, u): (1091, 2, 1), (10507043, 20, 1).

Examples of theorem 2.1(2) are the next things:

$(p, q, c, d, f, g, h, i, j, k, l, s, t, u, v, w, z)$:

(1091, 1087, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 17) and

(74531, 74527, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 257);

(p, u, v): (121963, 3, 2), (157771, 3, 4);

(p, u, v): (197, 1, 1), (33749, 5, 1), (1444661, 13, 1).

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