A Note on Real Weak Division Algebras

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Abstract

Let $p, q$ be integers $\geq 2$. A finite-dimensional real algebra $A$ is said to be weak division of index $(p, q)$ if the equality $x^p y^q = 0$ implies $x = 0$ or $y = 0$, and there exists non-zero $a, b \in A$ such that $a^pbq^{-1} = 0$ or $a^{p-1}bq = 0$. We show that every weak division algebra, whose index is a pair of odd integers $\geq 3$, is a real division algebra and that the mapping $x \mapsto x^m$ is onto for all odd integer $m \geq 1$.

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1 Introduction

One of the fundamental and powerful results about real division algebras is the $(1, 2, 4, 8)$-theorem. It states that the dimension of every real division algebra is either $1, 2, 4$ or $8$ [2]. It is proved partially by Hopf [3] then finished, independently, by Kervaire [6] and Milnor-Bott [1].

Yang [7, Lemma 1] showed that for every unital real division algebra of dimension $\geq 2$, the square mapping $x \mapsto x^2$ is onto. This persists if the unit is replaced by a non-zero central element [4, Remark 1].

Here we introduce a new notion of division, weaker than the ordinary. Let $A$ be a real algebra of finite dimension. We define the powers to the left of an element $x \in A$ by: $x^1 = x$ and $x^{n+1} = xx^n$ for all $n \geq 1$. Algebra $A$ is called weak division of index $(p, q)$ $(p, q$ being integers $\geq 2$) if the equality $x^py^q = 0$
implies \( x = 0 \) or \( y = 0 \), and there exists \( a, b \in \mathcal{A} \setminus \{0\} \) such that \( a^pb^q-1 = 0 \) or \( a^{p-1}b^q = 0 \).

Our objective in the present paper is to give a result analogous to those in ([7, Lemma 1], [4, Remark 1]) for weak division algebras whose index is a pair of odd integers \( \geq 3 \) (Lemma 3). This will allow an extension of the \((1,2,4,8)\)-theorem (Corollary 2). It also leads to a surprising surjectivity of the mapping \( x \mapsto x^m \) (\( m \) being an odd fixed integer \( \geq 1 \)) for any real division algebra (Corollary 1).

2 Definitions and notations

**Definitions 1** Let \( \mathcal{A} \) be a non-associative real algebra of finite dimension and let \( p, q \) be integers \( \geq 2 \).

1. The powers to the left of an element \( x \in \mathcal{A} \) are defined by: \( x^1 = x \) and \( x^{n+1} = xx^n \) for all \( n \geq 1 \).

2. \( \mathcal{A} \) is said to be a **real division algebra** if it has no divisors of zero.

3. \( \mathcal{A} \) is said to be a **real weak division algebra** of index \( (p,q) \) if the equality \( x^py^q = 0 \) implies \( x = 0 \) or \( y = 0 \), and there exists \( a, b \in \mathcal{A} \setminus \{0\} \) such that \( a^pb^q-1 = 0 \) or \( a^{p-1}b^q = 0 \).

The following example shows that there are real weak division algebras which are not real division algebras:

**Example 1** Let \( e_1, e_2, e_3 \) be the canonical basis of the euclidian space \((\mathbb{R}^3, ||.||)\). Then \( \mathbb{R}^3 \) equipped with the multiplication defined by the following table:

<table>
<thead>
<tr>
<th>( \diamond )</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
<th>( e_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_1 )</td>
<td>( e_1 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( e_2 )</td>
<td>0</td>
<td>( e_1 )</td>
<td>0</td>
</tr>
<tr>
<td>( e_3 )</td>
<td>0</td>
<td>0</td>
<td>( e_1 )</td>
</tr>
</tbody>
</table>

is a 3-dimensional real weak division algebra with index \((2,2)\) having, clearly, a non-zero central idempotent \( e_1 \).

Indeed, for every \( x = \alpha e_1 + \beta e_2 + \gamma e_3 \in \mathbb{R}^3 \), we have:

\[
x \diamond x = (\alpha e_1 + \beta e_2 + \gamma e_3) \diamond (\alpha e_1 + \beta e_2 + \gamma e_3) = (\alpha^2 + \beta^2 + \gamma^2)e_1 = ||x||^2e_1.
\]

Now, for \( x, y \in \mathbb{R}^3 : (x \diamond x) \diamond (y \diamond y) = ||x||^2||y||^2e_1 \). So the equality \( (x \diamond x) \diamond (y \diamond y) = 0 \) gives \( x = 0 \) or \( y = 0 \). Moreover, the equality \( (x \diamond x) \diamond y = 0 \) holds for \( x = e_1 \neq 0 \) and \( y = e_2 \neq 0 \).
3 Preliminary results

We have the following preliminary results:

Lemma 1 Every real division algebra $A$ is a real weak division algebra whose index is an arbitrary pair of integers $\geq 2$.

Proof. Let $p, q$ be arbitrary integers $\geq 2$ and let $x, y \in A$. We have

\[ x^p y^q = 0 \Rightarrow x^p = 0 \text{ or } y^q = 0 \] because $A$ has no divisors of zero

\[ \Rightarrow x = 0 \text{ or } y = 0 \] for the same reason. \qed

Lemma 2 Let $A$ be a real weak division algebra whose index is a pair $(p, q)$ of integers $\geq 2$. Then $A$ has no nilpotent element of index $\leq \min(p, q)$.

Proof. Assume that $p \leq q$ and let $x$ be in $A$ such that $x^r = 0 \neq x^{r-1}$ with $2 \leq r \leq p$. Let $L_x$ be the left-multiplication operator by $x$. Now, for arbitrary nonzero $y \in A$:

\[ 0 = L_x^{p-(r-1)}(x^{r-1}) \text{ because } p - (r - 1) \geq 1 \text{ and } x^r = 0 \]

\[ = x^p \]

\[ = x^p y^q. \]

This gives $x = 0$ since $A$ has index $(p, q)$ and $y \neq 0$, which contradicts the fact that $x^{r-1} \neq 0$. The result is the same if $q < p$. \qed

4 Main result

Let $S^{n-1}$ be the unit sphere of Euclidian space $(\mathbb{R}^n, ||.||)$. A continuous mapping $f : S^{n-1} \to S^{n-1}$ induces a homomorphism $f_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$, where $H_{n-1}(S^{n-1})$ is the $(n-1)^{th}$ homology group of the sphere $S^{n-1}$. It is well known that $H_{n-1}(S^{n-1}) = \mathbb{Z}$ [5, Theorem 9.1.9], so $f_*$ has the form $f_* : x \mapsto \alpha x$ for some fixed $\alpha \in \mathbb{Z}$. The integer $\alpha$ is called the degree of $f$ denoted by \text{deg}(f) [5, p. 199].

We have the following key result:

Lemma 3 Let $A$ be a real algebra, with underlying space $\mathbb{R}^n$, and assume that it is a weak division algebra of index a pair $(p, q)$ of odd integers $\geq 3$. Then both mappings $x \mapsto x^p$, $x \mapsto x^q$ are onto.

Proof. The mapping $g : S^{n-1} \to S^{n-1}$ $x \mapsto ||x^p||^{-1} x^p$ is well defined by Lemma 2. It is continuous and odd. So \text{deg}(g) \neq 0 [5, Proposition 10.2.5] and then $g$ is onto. Therefore the mapping $A \to A$ $x \mapsto x^p$ is onto. \qed
Theorem 1 Let $\mathcal{A}$ be a finite-dimensional real algebra. Then the following assertions are equivalent:

1. $\mathcal{A}$ is a division algebra,

2. $\mathcal{A}$ is weak division algebra of index a pair $(p, q)$ of odd integers $\geq 3$.

Proof. $(2) \Rightarrow (1)$. Let $x, y \in \mathcal{A}$ such that $xy = 0$. There exists $z, t \in \mathcal{A}$ such that $(x, y) = (z^p, t^q)$ by Lemma 3. Now $z^pt^q = xy = 0$. So $z = 0$ or $t = 0$, that is $x = 0$ or $y = 0$. □

As consequences

Corollary 1 Let $\mathcal{A}$ be a real division algebra and $m$ an odd integer $\geq 1$. Then the mapping $\mathcal{A} \to \mathcal{A}$ $x \mapsto x^m$ is onto.

Corollary 2 Let $\mathcal{A}$ be a weak division algebra of index a pair $(p, q)$ of odd integers $\geq 3$. Then the dimension of $\mathcal{A}$ is either 1, 2, 4 or 8.

References


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