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A Note on Real Weak Division Algebras

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Abstract

Let p, q be integers ≥ 2. A finite-dimensional real algebra $\mathcal{A}$ is said to be *weak division of index* (p, q) if the equality $x^p y^q = 0$ implies $x = 0$ or $y = 0$, and there exists non-zero $a, b \in \mathcal{A}$ such that $a^p b^{q-1} = 0$ or $a^{p-1} b^q = 0$. We show that every weak division algebra, whose index is a pair of odd integers ≥ 3, is a real division algebra and that the mapping $x \mapsto x^m$ is onto for all odd integer $m \geq 1$.

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1 Introduction

One of the fundamental and powerful results about real division algebras is the $(1, 2, 4, 8)$-theorem. It states that the dimension of every real division algebra is either $1, 2, 4$ or 8 [2]. It is proved partially by Hopf [3] then finished, independently, by Kervaire [6] and Milnor-Bott [1].

Yang [7, Lemma 1] showed that for every unital real division algebra of dimension ≥ 2, the square mapping $x \mapsto x^2$ is onto. This persists if the unit is replaced by a non-zero central element [4, Remark 1].

Here we introduce a new notion of division, weaker than the ordinary. Let $\mathcal{A}$ be a real algebra of finite dimension. We define the powers to the left of an element $x \in \mathcal{A}$ by: $x^1 = x$ and $x^{n+1} = x x^n$ for all $n \geq 1$. Algebra $\mathcal{A}$ is called *weak division of index* (p, q) (p, q being integers ≥ 2) if the equality $x^p y^q = 0$

implies $x = 0$ or $y = 0$, and there exists $a, b \in \mathcal{A} \setminus \{0\}$ such that $a^p b^{q-1} = 0$ or $a^{p-1} b^q = 0$.

Our objective in the present paper is to give a result analogous to those in ([7, Lemma 1], [4, Remark 1]) for weak division algebras whose index is a pair of odd integers ≥ 3 (Lemma 3). This will allow an extension of the $(1, 2, 4, 8)$-theorem (Corollary 2). It also leads to a surprising surjectivity of the mapping $x \mapsto x^m$ (m being an odd fixed integer ≥ 1) for any real division algebra (Corollary 1).

2 Definitions and notations

Definitions 1 Let $\mathcal{A}$ be a non-associative real algebra of finite dimension and let p, q be integers ≥ 2.

1. The powers to the left of an element $x \in \mathcal{A}$ are defined by: $x^1 = x$ and $x^{n+1} = x x^n$ for all $n \geq 1$.
2. $\mathcal{A}$ is said to be a *real division algebra* if it has no divisors of zero.
3. $\mathcal{A}$ is said to be a *real weak division algebra* of index (p, q) if the equality $x^p y^q = 0$ implies $x = 0$ or $y = 0$, and there exists $a, b \in \mathcal{A} \setminus \{0\}$ such that $a^p b^{q-1} = 0$ or $a^{p-1} b^q = 0$.

The following example shows that there are real weak division algebras which are not real division algebras:

Example 1 Let e_1, e_2, e_3 be the canonical basis of the euclidian space $(\mathbb{R}^3, ||.||)$. Then $\mathbb{R}^3$ equipped with the multiplication defined by the following table:

$\diamond$	e_1	e_2	e_3
e_1	e_1	0	0
e_2	0	e_1	0
e_3	0	0	e_1

is a 3-dimensional real weak division algebra with index $(2, 2)$ having, clearly, a non-zero central idempotent e_1.

Indeed, for every $x = \alpha e_1 + \beta e_2 + \gamma e_3 \in \mathbb{R}^3$, we have:

$$\begin{aligned} x \diamond x &= (\alpha e_1 + \beta e_2 + \gamma e_3) \diamond (\alpha e_1 + \beta e_2 + \gamma e_3) \\ &= (\alpha^2 + \beta^2 + \gamma^2) e_1 \\ &= ||x||^2 e_1. \end{aligned}$$

Now, for $x, y \in \mathbb{R}^3 : (x \diamond x) \diamond (y \diamond y) = ||x||^2 ||y||^2 e_1$. So the equality $(x \diamond x) \diamond (y \diamond y) = 0$ gives $x = 0$ or $y = 0$. Moreover, the equality $(x \diamond x) \diamond y = 0$ holds for $x = e_1 \neq 0$ and $y = e_2 \neq 0$. ∎

3 Preliminary results

We have the following preliminary results:

Lemma 1 *Every real division algebra $\mathcal{A}$ is a real weak division algebra whose index is an arbitrary pair of integers ≥ 2.*

Proof. Let p, q be arbitrary integers ≥ 2 and let $x, y \in \mathcal{A}$. We have

$$\begin{aligned} x^p y^q = 0 \quad &\Rightarrow \quad x^p = 0 \text{ or } y^q = 0 \text{ because } \mathcal{A} \text{ has no divisors of zero} \\ &\Rightarrow \quad x = 0 \text{ or } y = 0 \text{ for the same reason.} \end{aligned}$$

□

Lemma 2 *Let $\mathcal{A}$ be a real weak division algebra whose index is a pair (p, q) of integers ≥ 2. Then $\mathcal{A}$ has no nilpotent element of index $\leq \min(p, q)$.*

Proof. Assume that $p \leq q$ and let x be in $\mathcal{A}$ such that $x^r = 0 \neq x^{r-1}$ with $2 \leq r \leq p$. Let L_x be the left-multiplication operator by x. Now, for arbitrary nonzero $y \in \mathcal{A}$:

$$\begin{aligned} 0 &= L_x^{p-(r-1)}(x^{r-1}) \quad \text{because } p-(r-1) \geq 1 \text{ and } x^r = 0 \\ &= x^p \\ &= x^p y^q. \end{aligned}$$

This gives $x = 0$ since $\mathcal{A}$ has index (p, q) and $y \neq 0$, which contradicts the fact that $x^{r-1} \neq 0$. The result is the same if $q < p$. □

4 Main result

Let $\mathbb{S}^{n-1}$ be the unit sphere of Euclidian space $(\mathbb{R}^n, ||.||)$. A continuous mapping $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ induces a homomorphism $f_* : H_{n-1}(\mathbb{S}^{n-1}) \to H_{n-1}(\mathbb{S}^{n-1})$, where $H_{n-1}(\mathbb{S}^{n-1})$ is the $(n-1)^{th}$ homology group of the sphere $\mathbb{S}^{n-1}$. It is well known that $H_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$ [5, Theorem 9.1.9], so f_* has the form $f_* : x \mapsto \alpha x$ for some fixed $\alpha \in \mathbb{Z}$. The integer α is called the *degree of* f denoted by $\deg(f)$ [5, p. 199].

We have the following key result:

Lemma 3 *Let $\mathcal{A}$ be a real algebra, with underlying space $\mathbb{R}^n$, and assume that it is a weak division algebra of index a pair (p, q) of odd integers ≥ 3. Then both mappings $x \mapsto x^p$, $x \mapsto x^q$ are onto.*

Proof. The mapping $g : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ $x \mapsto ||x^p||^{-1} x^p$ is well defined by Lemma 2. It is continuous and odd. So $\deg(g) \neq 0$ [5, Proposition 10.2.5] and then g is onto. Therefore the mapping $\mathcal{A} \to \mathcal{A}$ $x \mapsto x^p$ is onto. □

Theorem 1 *Let $\mathcal{A}$ be a finite-dimensional real algebra. Then the following assertions are equivalent:*

1. *$\mathcal{A}$ is a division algebra,*
2. *$\mathcal{A}$ is weak division algebra of index a pair (p, q) of odd integers ≥ 3.*

Proof. $(2) \Rightarrow (1)$. Let $x, y \in \mathcal{A}$ such that $xy = 0$. There exists $z, t \in \mathcal{A}$ such that $(x, y) = (z^p, t^q)$ by Lemma 3. Now $z^p t^q = xy = 0$. So $z = 0$ or $t = 0$, that is $x = 0$ or $y = 0$. □

As consequences

Corollary 1 *Let $\mathcal{A}$ be a real division algebra and m an odd integer ≥ 1. Then the mapping $\mathcal{A} \to \mathcal{A}$ $x \mapsto x^m$ is onto.*

Corollary 2 *Let $\mathcal{A}$ be a weak division algebra of index a pair (p, q) of odd integers ≥ 3. Then the dimension of $\mathcal{A}$ is either $1, 2, 4$ or 8.*

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