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A New Characterization of Several Alternating Simple Groups

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Abstract

In this paper, we studied the influence of centralizers on the structure of groups and demonstrate that the alternating groups A_n (for $n = 13, 14, 15$) can be uniquely determined by two crucial quantitative properties: its even-order components of the group and the set $\pi_{p_m}(G)$. Here, G represents a finite group, and $\pi(G)$ is the set of prime factors of the order of G, and p_m is the largest element in $\pi(G)$,and $\pi_{p_m}(G)$ =$\{|C_G(x)|\big|x \in G$ and $|x| = p_m$ $\}$ denotes the set of orders of centralizers of p_m-order elements in G.

Mathematics Subject Classification: 20D10; 20D20

Keywords: Finite groups; Simple Groups; Order components; Centralizers; Order

1 Introduction

The groups mentioned in this paper are all finite groups.

Let G be a finite group. K. W. Gruenberg and O. Kegel defined the prime graph $\Gamma(G)$ of a finite group G as follows: the vertex set of G is the set of all prime factors of $|G|$, and two vertices p and q are adjacent if and only if G contains an element of order pq (see [9]). The number of connected components in the prime graph of G is denoted by $t(G)$, and the set of connected components in the prime graph of G is denoted by $T(G) = \{\pi_i(G)|i = 1, 2, \ldots, t(G)\}$. When G is an even-order group, it is stipulated that $2 \in \pi_1(G)$. If $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ are all the connected components of G's prime graph, then $|G| = m_1 m_2 \cdots m_{t(G)}$, where the prime factor set of m_i is π_i for $i = 1, 2, \ldots, t(G)$. The numbers $m_1, m_2, \ldots, m_{t(G)}$ are called the order components of G, and $OC(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$ is the set of order components of G (see [2]). For convenience, we denote the even-order component of G as $m_1(G)$. Professor Guiyun Chen gave the order components of all simple groups whose prime graph is disconnected (see [2] Table1-Table4).

Using the concept of order components, Professor G. Y. Chen and other group theorists such as Professor H. G. Shi have studied the following problem: Let G be a finite group and S be a non-abelian simple group. If $OC(G) = OC(S)$, are G and S isomorphic?

Many group theorists have conducted in-depth studies on this problem, and some of their achievements can be found in [2–5, 8]. From these results, it can be seen that the order components are effective quantitative properties for characterizing simple groups.

This paper investigated the impact of the even order component $m_1(G)$ of a group and $\pi_{p_m}(G)$ =$\{|C_G(x)|\big||x \in G$ and $|x| = p_m\}$, which is the set of orders of centralizers of p_m-order elements in G, and utilizes them to characterize the alternating groups A_n (for $n = 13, 14, 15$).

MAIN THEOREM. *Let G be a finite group, and let M be one of the alternating groups A_{13}, A_{14}, or A_{15}. Then G is isomorphic to M if and only if:*

(1) $m_1(G) = m_1(M)$;

(2) $\pi_{p_m}(G) = \pi_{p_m}(M)$.

2 Preliminaries

In this paper, we adopt the following conventions: $\pi(G)$ denotes the set of prime factors of the order of G; p_m stands for the largest element of $\pi(G)$, and $\pi_{p_m}(G)$ represents the set of orders of centralizers of p_m-order elements in G. The symbol $|\pi(G)|$ refers to the number of prime factors of the order of G. All other symbols not explicitly defined are standard and can be found in [6].

The following theorem provides a characterization of the structure of finite groups when $t(G) \geq 2$.

Lemma 2.1 *[9, Corollary] Let G be a finite group with disconnecting prime graph. Then the structure of G is as follows:*

(1) G is a Frobenius group or a 2-Frobenius group;

(2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where H is a nilpotent $\pi_1(G)$-group, G/K is soluble $\pi_1(G)$-group, K/H is a non-abelian simple group, and $|G/K|$ divides $|\mathrm{Out}(K/H)|$.

Definition 2.2 *Let G be a finite group. G is called a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that G/H and K are Frobenius groups with kernels K/H and H, respectively (see [1]).*

The following two lemmas respectively provide characterizations of the structure of even-order Frobenius groups and even-order 2-Frobenius groups.

Lemma 2.3 *[1, Theorem 1] Let G be an even-order Frobenius group with Frobenius kernel H and Frobenius complement K. Then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$ Moreover, the structure of G is one of the following:*

1) If $2 \in \pi(H)$, then the Sylow subgroups of K are cyclic;

2) If $2 \in \pi(K)$, then H is an abelian group. When K is soluble, the odd-order Sylow subgroups of K are cyclic and the Sylow 2-subgroup is either a cyclic group or a generalized quaternion group. When K is insoluble, there exists $K_0 \leq K$ such that $|K : K_0| \leq 2$ and $K_0 \simeq Z \times SL(2,5)$, where $(|Z|, 30) = 1$ and the Sylow subgroups of Z are cyclic.

Lemma 2.4 *[1, Theorem 2] Let G be an even-order 2-Frobenius group. Then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, $|G/K|$ divides $|Aut(K/H)|$, and both $|G/K|$ and $|K/H|$ are cyclic groups. In particular, $|G/K| \leq |K/H|$ and G is soluble.*

3 Proof of Main Theorem

Given a subgroup K of a group G, it is obvious that $m_1(K)$ divides $m_1(G)$. In the following proof, we will frequently use this result without further explanation.

Lemma 3.1 *Let G be a finite group, and let M be one of the alternating groups A_{13}, A_{14}, or A_{15}. If G satisfies the following conditions:*

(1) $m_1(G) = m_1(M)$;

(2) $\pi_{p_m}(G) = \pi_{p_m}(M)$.

Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is nilpotent, G/K is solvable, and $|G/K| \big| |Out(K/H)|$.

Proof: Case 1 $M \cong A_{13}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$.

In this case, $m_1(G) = m_1(M) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, and $\pi_{p_m}(M) = \{13\}$. Since $m_1(G) = m_1(M) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, we have $t(G) \geq 2$, which implies that G has the following structure according to Lemma 2.1:

1) G is a Frobenius group or a 2-Frobenius group;

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, we have $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7$, where $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7$. Therefore, $|K| \big| |Aut(S_7)|$. However, since $13 \big| |K|$ and $|Aut(S_7)| \big| 6$, we have a contradiction.

2) If $2 \in \pi(K)$, then by the given condition, the Sylow 13-subgroup S_{13} of G is normal in G and has order 13. If we let the Sylow 7-subgroup of G act on S_{13}, we obtain an element of order 91 in G, which contradicts $\pi_{p_m}(M) = \{13\}$. Therefore, G is not a Frobenius group.

G is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, we have $13 \in \pi_2(G)$, which means K contains an element of order 13. If we let the 13-order element in K act on the Sylow 2-subgroup of H or the Sylow 3-subgroup of H or the Sylow 5-subgroup of H or the Sylow 7-subgroup of H, we will obtain a contradiction. Therefore, G is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of G is as follows: G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is a nilpotent group, G/K is a solvable $\pi_1(G)$-group, and K/H is a nonabelian simple group with $|G/K| \big| |Out(K/H)|$.

Case 2 $M \cong A_{14}(2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13)$.

In this case, $m_1(G) = m_1(M) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, and $\pi_{p_m}(M) = \{13\}$. Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, we have $t(G) \geq 2$, which implies that G has the following structure according to Lemma 2.1:

1) G is a Frobenius group or a 2-Frobenius group;

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, we have $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7 \times S_{11}$, where $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7, 11$. Therefore, $|K| \big| |Aut(S_{11})|$. However, since $13 \big| |K|$ and $|Aut(S_{11})| \big| 10$, we have a contradiction.

2) If $2 \in \pi(K)$, then by the given condition, the Sylow 13-subgroup S_{13} of G is normal in G and has order 13. If we let the Sylow 11-subgroup of G act on S_{13}, we obtain an element of order 143 in G, which contradicts $\pi_{p_m}(M) = \{13\}$. Therefore, G is not a Frobenius group.

G is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, we have $13 \in \pi_2(G)$, which means K contains an element of order 13. If we let the 13-order element in K act on the Sylow 2-subgroup of H or the Sylow 3-subgroup of H or the Sylow 5-subgroup of H or the Sylow 7-subgroup of H or the Sylow 11-subgroup, we will obtain a contradiction. Therefore, G is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of G is as follows: G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is a nilpotent group, G/K is a solvable $\pi_1(G)$-group, and K/H is a nonabelian simple group with $|G/K| \big| |Out(K/H)|$.

Case 3 $M \cong A_{15}(2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)$.

In this case, $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, and $\pi_{p_m}(M) = \{13\}$. Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, we have $t(G) \geq 2$, which implies that G has the following structure according to Lemma 2.1:

1) G is a Frobenius group or a 2-Frobenius group;

2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with H being a nilpotent $\pi_1(G)$-group, G/K being a solvable $\pi_1(G)$-group, and K/H being a nonabelian simple group.

However, G cannot be a Frobenius group. Otherwise, we have $G = HK$, where H is the Frobenius kernel and K is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since H is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7 \times S_{11}$, where $S_i \trianglelefteq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7, 11$. Therefore, $|K| \big| |Aut(S_{11})|$. However, since $13 \big| |K|$ and $|Aut(S_{11})| \big| 10$, we have a contradiction.

2) If $2 \in \pi(K)$, then by the given condition, the Sylow 13-subgroup S_{13} of G is normal in G and has order 13. If we let the Sylow 11-subgroup of G act on S_{13}, we obtain an element of order 143 in G, which contradicts $\pi_{p_m}(M) = \{13\}$. Therefore, G is not a Frobenius group.

G is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, we have $13 \in \pi_2(G)$, which means K contains an element of order 13. If we let the 13-order element in K act on the Sylow 2-subgroup of H or the Sylow 3-subgroup of H or the Sylow 5-subgroup of H or the Sylow 7-subgroup of H or the Sylow 11-subgroup, we will obtain a contradiction. Therefore, G is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of G is as follows: G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is a nilpotent group, G/K is a solvable $\pi_1(G)$-group, and K/H is a nonabelian simple group with $|G/K|\big||Out(K/H)|$.

Here, we will provide a complete characterization of the alternating group A_n for $n = 13, 14, 15$, that is, to prove $G \cong M$.

Theorem 3.2 *Let G be a finite group, and let M be one of the alternating groups A_{13}, A_{14}, or A_{15}. Then G is isomorphic to M if and only if:*
(1) $m_1(G) = m_1(M)$;
(2) $\pi_{p_m}(G) = \pi_{p_m}(M)$.

Proof: The necessity of the theorem is obvious, so we only need to prove the sufficiency. According to Lemma 3.1, we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is nilpotent, G/K is solvable, and $|G/K|\big||Out(K/H)|$. Now we complete the proof based on different cases of M.

Case 1 $M \cong A_{13}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$.

Since $m_1(G) = m_1(M) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, it follows that $t(G) \geq 2$, which implies that $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$. Additionally, $13 \in \pi(K/H)$. If H is nontrivial, let us assume $H = S_2 \times S_3 \times S_5 \times S_7$, where $S_i \in Syl_i(H)$ for $i = 2, 3, 5, 7$. Since H is a nilpotent group, we have $S_i \trianglelefteq K$ for $i = 2, 3, 5, 7$. By letting the 13-order element of K act on S_i, we obtain $\pi_{p_m}(M) \neq \{13\}$, which leads to a contradiction. Therefore, we conclude that $H = 1$.

In this way, G has a normal nonabelian simple subgroup K with $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7\}$ and $13 \in \pi(K)$. If $|\pi(K)| = 3$, then K is a simple K_3-group. According to [7], the order of all simple K_3-groups does not contain the prime factor 13. If $|\pi(K)| = 4$, then because $\pi_{p_m}(M) = 13$, we have $t(K) \geq 2$. When $t(K) = 2$, by examining Table 3 and Table 4 in [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. When $t(K) \geq 3$, by examining Table 2 and Table 4 of [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. If $|\pi(K)| \geq 5$, then because $\pi_{p_m}(M) = 13$ and $t(K) \geq 3$, by examining Table 2 to Table 4 of [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be deduced that K can only be one of the following groups: A_{13}, A_{14}, A_{15}, and Suz.

If $K \cong Suz$, then $|K| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Since in this case $m_1(K) = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$, it clearly contradicts $m_1(G) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, hence $K \ncong Suz$. Similarly, $K \ncong A_{14}$ and A_{15}.

Hence, we have $K \cong A_{13}$, which implies $1 \trianglelefteq A_{13} \trianglelefteq G$. In this case, it is clear that $C_G(A_{13}) = 1$ and $|Out(A_{13})| = 2$. Therefore, we either have $G \cong A_{13}$ or $G \cong Aut(A_{13})$. If $G \cong Aut(A_{13})$, then we have evidently that $m_1(G) > m_1(A_{13}) = m_1(M)$, which contradicts our assumption. Hence, $G \cong A_{13}$.

Case 2 $M \cong A_{14}(2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13)$.

Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, we have $t(G) \geq 2$. Therefore, $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K/H)$. Similar to Case 1, we can prove that $H = 1$, implying that G has a normal nonabelian simple subgroup K with $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K)$. If $|\pi(K)| = 3$, then K is a simple K_3-group. According to [7], the order of all simple K_3-groups does not contain the prime factor 13. If $|\pi(K)| = 4$, then because $\pi_{p_m}(M) = 13$, we have $t(K) \geq 2$. When $t(K) = 2$, by examining Table 3 and Table 4 in [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. When $t(K) \geq 3$, by examining Table 2 and Table 4 of [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. If $|\pi(K)| \geq 5$, then because $\pi_{p_m}(M) = 13$ and $t(K) \geq 3$, by examining Table 2 to Table 4 of [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be deduced that K can only be one of the following groups: A_{13}, A_{14}, A_{15}, and Suz.

If $K \cong A_{13}$, then $|K| = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Since $m_1(G) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$ and $m_1(K) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, we have $2 \cdot 7 \big| |G/K| \big| |Out(A_{13})| = 2$, which leads to a contradiction. Therefore, K is not isomorphic to A_{13}.

If $K \cong Suz$, then $|K| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Since in this case $m_1(K) = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$, it clearly contradicts $m_1(G) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, hence $K \ncong Suz$. Similarly, $K \ncong A_{15}$.

Hence, we have $K \cong A_{14}$, which implies $1 \trianglelefteq A_{14} \trianglelefteq G$. In this case, it is clear that $C_G(A_{14}) = 1$ and $|Out(A_{14})| = 2$. Therefore, we either have $G \cong A_{14}$ or $G \cong Aut(A_{14})$. If $G \cong Aut(A_{14})$, then we have evidently that $m_1(G) > m_1(A_{14}) = m_1(M)$, which contradicts our assumption. Hence, $G \cong A_{14}$.

Case 3 $M \cong A_{15}(2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)$.

Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, we have $t(G) \geq 2$. Therefore, $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K/H)$. Similar to Case 1, we can prove that $H = 1$, implying that G has a normal nonabelian simple subgroup K with $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K)$. If $|\pi(K)| = 3$, then K is a simple K_3-group. According to [7], the order of all simple K_3-groups does not contain the prime factor 13. If $|\pi(K)| = 4$, then because $\pi_{p_m}(M) = 13$, we have $t(K) \geq 2$. When $t(K) = 2$, by examining Table 3 and Table 4 in [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. When $t(K) \geq 3$, by examining Table 2 and Table 4 of [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. If $|\pi(K)| \geq 5$, then because $\pi_{p_m}(M) = 13$ and $t(K) \geq 3$, by examining Table 2 to Table 4 of [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be deduced that K can only be one of the following groups: A_{13}, A_{14}, A_{15}, and Suz.

If $K \cong A_{13}$, then $|K| = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Since $m_1(G) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$ and $m_1(K) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, we have $2 \cdot 3 \cdot 5 \cdot 7 \big| |G/K| \big| |Out(A_{13})| = 2$, which leads to a contradiction. Therefore, K is not isomorphic to A_{13}. Similarly, $K \ncong A_{14}$.

If $K \cong Suz$, then $|K| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Since in this case $m_1(K) = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$, it clearly contradicts $m_1(G) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, hence $K \ncong Suz$.

Hence, we have $K \cong A_{15}$, which implies $1 \trianglelefteq A_{15} \trianglelefteq G$. In this case, it is clear that $C_G(A_{15}) = 1$ and $|Out(A_{15})| = 2$. Therefore, we either have $G \cong A_{15}$ or $G \cong Aut(A_{15})$. If $G \cong Aut(A_{15})$, then we have evidently that $m_1(G) > m_1(A_{15}) = m_1(M)$, which contradicts our assumption. Hence, $G \cong A_{15}$.

The proof has been completed.

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