A New Characterization of Several Alternating Simple Groups

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Abstract

In this paper, we studied the influence of centralizers on the structure of groups and demonstrate that the alternating groups $A_n$ (for $n = 13, 14, 15$) can be uniquely determined by two crucial quantitative properties: its even-order components of the group and the set $\pi_{p_m}(G)$. Here, $G$ represents a finite group, and $\pi(G)$ is the set of prime factors of the order of $G$, and $p_m$ is the largest element in $\pi(G)$, and $\pi_{p_m}(G) = \{|C_G(x)| | x \in G \text{ and } |x| = p_m \}$ denotes the set of orders of centralizers of $p_m$-order elements in $G$.

Mathematics Subject Classification: 20D10; 20D20

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1 Introduction

The groups mentioned in this paper are all finite groups.
Let $G$ be a finite group. K. W. Gruenberg and O. Kegel defined the prime graph $\Gamma(G)$ of a finite group $G$ as follows: the vertex set of $G$ is the set of all prime factors of $|G|$, and two vertices $p$ and $q$ are adjacent if and only if $G$ contains an element of order $pq$ (see [9]). The number of connected components in the prime graph of $G$ is denoted by $t(G)$, and the set of connected components in the prime graph of $G$ is denoted by $\pi(G) = \{\pi_i(G)\mid i = 1, 2, \ldots, t(G)\}$. When $G$ is an even-order group, it is stipulated that $2 \in \pi_1(G)$. If $\pi_1, \pi_2, \ldots, \pi_t(G)$ are all the connected components of $G$'s prime graph, then $|G| = m_1 m_2 \cdots m_t(G)$, where the prime factor set of $m_i$ is $\pi_i$ for $i = 1, 2, \ldots, t(G)$. The numbers $m_1, m_2, \ldots, m_t(G)$ are called the order components of $G$ (see [2]). For convenience, we denote the even-order component of $G$ as $m_1(G)$. Professor Guiyun Chen gave the order components of all simple groups whose prime graph is disconnected (see [2] Table1-Table4).

Using the concept of order components, Professor G. Y. Chen and other group theorists such as Professor H. G. Shi have studied the following problem: Let $G$ be a finite group and $S$ be a non-abelian simple group. If $OC(G) = OC(S)$, are $G$ and $S$ isomorphic?

Many group theorists have conducted in-depth studies on this problem, and some of their achievements can be found in [2–5, 8]. From these results, it can be seen that the order components are effective quantitative properties for characterizing simple groups.

This paper investigated the impact of the even order component $m_1(G)$ of a group and $\pi_{pm}(G) = \{|C_G(x)|\mid x \in G \text{ and } |x| = pm \}$, which is the set of orders of centralizers of $pm$-order elements in $G$, and utilizes them to characterize the alternating groups $A_n$ (for $n = 13, 14, 15$).

**MAIN THEOREM.** Let $G$ be a finite group, and let $M$ be one of the alternating groups $A_{13}, A_{14},$ or $A_{15}$. Then $G$ is isomorphic to $M$ if and only if:

1. $m_1(G) = m_1(M)$;
2. $\pi_{pm}(G) = \pi_{pm}(M)$.

### 2 Preliminaries

In this paper, we adopt the following conventions: $\pi(G)$ denotes the set of prime factors of the order of $G$; $pm$ stands for the largest element of $\pi(G)$, and $\pi_{pm}(G)$ represents the set of orders of centralizers of $pm$-order elements in $G$.

The symbol $|\pi(G)|$ refers to the number of prime factors of the order of $G$. All other symbols not explicitly defined are standard and can be found in [6].

The following theorem provides a characterization of the structure of finite groups when $t(G) \geq 2$. 
Lemma 2.1 [9, Corollary] Let $G$ be a finite group with disconnecting prime graph. Then the structure of $G$ is as follows:

1) $G$ is a Frobenius group or a 2-Frobenius group;
2) $G$ has a normal series $1 \leq H \leq K \leq G$, where $H$ is a nilpotent $\pi_1(G)$-group, $G/K$ is soluble $\pi_1(G)$-group, $K/H$ is a non-abelian simple group, and $|G/K|$ divides $|\text{Out}(K/H)|$.

Definition 2.2 Let $G$ be a finite group. $G$ is called a 2-Frobenius group if there exists a normal series $1 \leq H \leq K \leq G$, such that $G/H$ and $K$ are Frobenius groups with kernels $K/H$ and $H$, respectively (see [1]).

The following two lemmas respectively provide characterizations of the structure of even-order Frobenius groups and even-order 2-Frobenius groups.

Lemma 2.3 [1, Theorem 1] Let $G$ be an even-order Frobenius group with Frobenius kernel $H$ and Frobenius complement $K$. Then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$. Moreover, the structure of $G$ is one of the following:

1) If $2 \in \pi(H)$, then the Sylow subgroups of $K$ are cyclic;
2) If $2 \in \pi(K)$, then $H$ is an abelian group. When $K$ is soluble, the odd-order Sylow subgroups of $K$ are cyclic and the Sylow 2-subgroup is either a cyclic group or a generalized quaternion group. When $K$ is insoluble, there exists $K_0 \leq K$ such that $|K : K_0| \leq 2$ and $K_0 \simeq Z \times SL(2,5)$, where $(|Z|, 30) = 1$ and the Sylow subgroups of $Z$ are cyclic.

Lemma 2.4 [1, Theorem 2] Let $G$ be an even-order 2-Frobenius group. Then $t(G) = 2$ and $G$ has a normal series $1 \leq H \leq K \leq G$, such that $\pi(K/H) = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, $|G/K|$ divides $|\text{Aut}(K/H)|$, and both $|G/K|$ and $|K/H|$ are cyclic groups. In particular, $|G/K| \leq |K/H|$ and $G$ is soluble.

3 Proof of Main Theorem

Given a subgroup $K$ of a group $G$, it is obvious that $m_1(K)$ divides $m_1(G)$. In the following proof, we will frequently use this result without further explanation.

Lemma 3.1 Let $G$ be a finite group, and let $M$ be one of the alternating groups $A_{13}$, $A_{14}$, or $A_{15}$. If $G$ satisfies the following conditions:

1) $m_1(G) = m_1(M)$;
2) $\pi_{p_m}(G) = \pi_{p_m}(M)$.

Then $G$ has a normal series $1 \leq H \leq K \leq G$ such that $H$ and $G/K$ are $\pi_1$-groups, $K/H$ is a non-abelian simple group, $H$ is nilpotent, $G/K$ is soluble, and $|G/K||\text{Out}(K/H)|$. 
Proof: Case 1 $M \cong A_{13}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$.

In this case, $m_1(G) = m_1(M) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, and $\pi_{pm}(M) = \{13\}$. Since $m_1(G) = m_1(M) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, we have $t(G) \geq 2$, which implies that $G$ has the following structure according to Lemma 2.1:

1) $G$ is a Frobenius group or a 2-Frobenius group;
2) $G$ has a normal series $1 \subseteq H \subseteq K \subseteq G$, with $H$ being a nilpotent $\pi_1(G)$-group, $G/K$ being a solvable $\pi_1(G)$-group, and $K/H$ being a nonabelian simple group.

However, $G$ cannot be a Frobenius group. Otherwise, we have $G = HK$, where $H$ is the Frobenius kernel and $K$ is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since $H$ is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7$, where $S_i \leq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7$. Therefore, $|K'||Aut(S_7)|$. However, since $13||K|$ and $|Aut(S_7)||6$, we avoid a contradiction.

2) If $2 \in \pi(K)$, then by the given condition, the Sylow 13-subgroup $S_{13}$ of $G$ is normal in $G$ and has order 13. If we let the Sylow 7-subgroup of $G$ act on $S_{13}$, we obtain an element of order 91 in $G$, which contradicts $\pi_{pm}(M) = \{13\}$. Therefore, $G$ is not a Frobenius group.

$G$ is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and $G$ has a normal series $1 \subseteq H \subseteq K \subseteq G$, such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, we have $13 \in \pi_2(G)$, which means $K$ contains an element of order 13. If we let the 13-order element in $K$ act on the Sylow 2-subgroup of $H$ or the Sylow 3-subgroup of $H$ or the Sylow 5-subgroup of $H$ or the Sylow 7-subgroup of $H$, we will obtain a contradiction. Therefore, $G$ is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of $G$ is as follows: $G$ has a normal series $1 \subseteq H \subseteq K \subseteq G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, $H$ is a nilpotent group, $G/K$ is a solvable $\pi_1(G)$-group, and $K/H$ is a nonabelian simple group with $|G/K||Out(K/H)|$.

Case 2 $M \cong A_{14}(2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13)$.

In this case, $m_1(G) = m_1(M) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, and $\pi_{pm}(M) = \{13\}$. Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, we have $t(G) \geq 2$, which implies that $G$ has the following structure according to Lemma 2.1:

1) $G$ is a Frobenius group or a 2-Frobenius group;
2) $G$ has a normal series $1 \subseteq H \subseteq K \subseteq G$, with $H$ being a nilpotent $\pi_1(G)$-group, $G/K$ being a solvable $\pi_1(G)$-group, and $K/H$ being a nonabelian simple group.

However, $G$ cannot be a Frobenius group. Otherwise, we have $G = HK$, where $H$ is the Frobenius kernel and $K$ is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$. 

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1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since $H$ is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7 \times S_{11}$, where $S_i \leq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7, 11$. Therefore, $|K| = |\text{Aut}(S_{11})|$. However, since $13||K|$ and $|\text{Aut}(S_{11})||10$, we have a contradiction.

2) If $2 \in \pi(K)$, then by the given condition, the Sylow 13-subgroup $S_{13}$ of $G$ is normal in $G$ and has order 13. If we let the Sylow 11-subgroup of $G$ act on $S_{13}$, we obtain an element of order 143 in $G$, which contradicts $\pi_{pm}(M) = \{13\}$. Therefore, $G$ is not a Frobenius group.

$G$ is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and $G$ has a normal series $1 \leq H \leq K \leq G$, such that $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 11$, we have $13 \in \pi_2(G)$, which means $K$ contains an element of order 13. If we let the 13-order element in $K$ act on the Sylow 2-subgroup of $H$ or the Sylow 3-subgroup of $H$ or the Sylow 5-subgroup of $H$ or the Sylow 7-subgroup of $H$ or the Sylow 11-subgroup, we will obtain a contradiction. Therefore, $G$ is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of $G$ is as follows: $G$ has a normal series $1 \leq H \leq K \leq G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, $H$ is a nilpotent group, $G/K$ is a solvable $\pi_1(G)$-group, and $K/H$ is a nonabelian simple group with $|G/K||\text{Out}(K/H)|$.

**Case 3** $M \cong A_{15}(2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)$.

In this case, $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, and $\pi_{pm}(M) = \{13\}$. Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, we have $t(G) \geq 2$, which implies that $G$ has the following structure according to Lemma 2.1:

1) $G$ is a Frobenius group or a 2-Frobenius group;

2) $G$ has a normal series $1 \leq H \leq K \leq G$, with $H$ being a nilpotent $\pi_1(G)$-group, $G/K$ being a solvable $\pi_1(G)$-group, and $K/H$ being a nonabelian simple group.

However, $G$ cannot be a Frobenius group. Otherwise, we have $G = HK$, where $H$ is the Frobenius kernel and $K$ is the Frobenius complement, and $T(G) = \{\pi(H), \pi(K)\}$.

1) If $2 \in \pi(H)$, then $\pi(H) = \pi_1(G)$. Since $H$ is a nilpotent group, we have $H = S_2 \times S_3 \times S_5 \times S_7 \times S_{11}$, where $S_i \leq G$ and $S_i \in Syl_i(G)$ for $i = 2, 3, 5, 7, 11$. Therefore, $|K| = |\text{Aut}(S_{11})|$. However, since $13||K|$ and $|\text{Aut}(S_{11})||10$, we have a contradiction.

2) If $2 \in \pi(K)$, then by the given condition, the Sylow 13-subgroup $S_{13}$ of $G$ is normal in $G$ and has order 13. If we let the Sylow 11-subgroup of $G$ act on $S_{13}$, we obtain an element of order 143 in $G$, which contradicts $\pi_{pm}(M) = \{13\}$. Therefore, $G$ is not a Frobenius group.

$G$ is also not a 2-Frobenius group. Otherwise, we have $t(G) = 2$ and $G$ has a normal series $1 \leq H \leq K \leq G$, such that $\pi(K/H) = \pi_2(G)$ and $\pi(H) \cup \pi(G/K) = \pi_1(G)$. Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, we
have $13 \in \pi_2(G)$, which means $K$ contains an element of order 13. If we let the 13-order element in $K$ act on the Sylow 2-subgroup of $H$ or the Sylow 3-subgroup of $H$ or the Sylow 5-subgroup of $H$ or the Sylow 7-subgroup of $H$ or the Sylow 11-subgroup, we will obtain a contradiction. Therefore, $G$ is not a 2-Frobenius group.

According to Lemma 2.1(2), the structure of $G$ is as follows: $G$ has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, $H$ is a nilpotent group, $G/K$ is a solvable $\pi_1(G)$-group, and $K/H$ is a nonabelian simple group with $|G/K||\text{Out}(K/H)|$.

Here, we will provide a complete characterization of the alternating group $A_n$ for $n = 13, 14, 15$, that is, to prove $G \cong M$.

**Theorem 3.2** Let $G$ be a finite group, and let $M$ be one of the alternating groups $A_{13}$, $A_{14}$, or $A_{15}$. Then $G$ is isomorphic to $M$ if and only if:

1. $m_1(G) = m_1(M)$;
2. $\pi_{pn}(G) = \pi_{pn}(M)$.

**Proof:** The necessity of the theorem is obvious, so we only need to prove the sufficiency. According to Lemma 3.1, we know that $G$ has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$ such that $H$ and $G/K$ are $\pi_1$-groups, $K/H$ is a non-abelian simple group, $H$ is nilpotent, $G/K$ is solvable, and $|G/K||\text{Out}(K/H)|$. Now we complete the proof based on different cases of $M$.

**Case 1** $M \cong A_{13}(2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13)$.

Since $m_1(G) = m_1(M) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, it follows that $t(G) \geq 2$, which implies that $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7\}$. Additionally, $13 \in \pi(K/H)$. If $H$ is nontrivial, let us assume $H = S_{i} \times S_{i} \times S_{i} \times S_{i}$, where $S_{i} \in Syl_{m}(H)$ for $i = 2, 3, 5, 7$. Since $H$ is a nilpotent group, we have $S_{i} \leq K$ for $i = 2, 3, 5, 7$. By letting the 13-order element of $K$ act on $S_{i}$, we obtain $\pi_{pm}(M) \neq \{13\}$, which leads to a contradiction. Therefore, we conclude that $H = 1$.

In this way, $G$ has a normal nonabelian simple subgroup $K$ with $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7\}$ and $13 \in \pi(K)$. If $|\pi(K)| = 3$, then $K$ is a simple $K_3$-group. According to [7], the order of all simple $K_3$-groups does not contain the prime factor 13. If $|\pi(K)| = 4$, then because $\pi_{pm}(M) = 13$, we have $t(K) \geq 2$. When $t(K) = 2$, by examining Table 3 and Table 4 in [2] and utilizing the condition $\pi_{pm}(M) = 13$, it can be concluded that such a simple group does not exist. When $t(K) \geq 3$, by examining Table 2 and Table 4 of [2] and utilizing the condition $\pi_{pm}(M) = 13$, it can be concluded that such a simple group does not exist. If $|\pi(K)| \geq 5$, then because $\pi_{pm}(M) = 13$ and $t(K) \geq 3$, by examining Table 2 to Table 4 of [2] and utilizing the condition $\pi_{pm}(M) = 13$, it can be deduced that $K$ can only be one of the following groups: $A_{13}$, $A_{14}$, $A_{15}$, and $Suz$. 
If $K \cong Suz$, then $|K| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Since in this case $m_1(K) = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$, it clearly contradicts $m_1(G) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7$, hence $K \not\cong Suz$. Similarly, $K \not\cong A_{14}$ and $A_{15}$.

Hence, we have $K \cong A_{13}$, which implies $1 \leq A_{13} \leq G$. In this case, it is clear that $C_G(A_{13}) = 1$ and $|Out(A_{13})| = 2$. Therefore, we either have $G \cong A_{13}$ or $G \cong Aut(A_{13})$. If $G \cong Aut(A_{13})$, then we have evidently that $m_1(G) > m_1(A_{13}) = m_1(M)$, which contradicts our assumption. Hence, $G \cong A_{13}$.

**Case 2** $M \cong A_{14}(2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13)$.

Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, we have $t(G) \geq 2$. Therefore, $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K/H)$. Similar to Case 1, we can prove that $H = 1$, implying that $G$ has a normal nonabelian simple subgroup $K$ with $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K)$. If $|\pi(K)| = 3$, then $K$ is a simple $K_3$-group. According to [7], the order of all simple $K_3$-groups does not contain the prime factor 13. If $|\pi(K)| = 4$, then because $\pi_{p_m}(M) = 13$, we have $t(K) \geq 2$. When $t(K) = 2$, by examining Table 3 and Table 4 in [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. When $t(K) \geq 3$, by examining Table 2 and Table 4 of [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. If $|\pi(K)| = 5$, then because $\pi_{p_m}(M) = 13$ and $t(K) \geq 3$, by examining Table 2 to Table 4 of [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be deduced that $K$ can only be one of the following groups: $A_{13}, A_{14}, A_{15}$, and $Suz$.

If $K \cong A_{13}$, then $|K| = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Since $m_1(G) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$ and $m_1(K) = 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, we have $2 \cdot 7 \cdot |G/K||Out(A_{13})| = 2$, which leads to a contradiction. Therefore, $K$ is not isomorphic to $A_{13}$.

If $K \cong Suz$, then $|K| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. Since in this case $m_1(K) = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$, it clearly contradicts $m_1(G) = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11$, hence $K \not\cong Suz$. Similarly, $K \not\cong A_{15}$.

Hence, we have $K \cong A_{14}$, which implies $1 \leq A_{14} \leq G$. In this case, it is clear that $C_G(A_{14}) = 1$ and $|Out(A_{14})| = 2$. Therefore, we either have $G \cong A_{14}$ or $G \cong Aut(A_{14})$. If $G \cong Aut(A_{14})$, then we have evidently that $m_1(G) > m_1(A_{14}) = m_1(M)$, which contradicts our assumption. Hence, $G \cong A_{14}$.

**Case 3** $M \cong A_{15}(2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13)$.

Since $m_1(G) = m_1(M) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11$, we have $t(G) \geq 2$. Therefore, $\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K/H)$. Similar to Case 1, we can prove that $H = 1$, implying that $G$ has a normal nonabelian simple subgroup $K$ with $\pi(G/K) \subseteq \pi_1(G) = \{2, 3, 5, 7, 11\}$ and $13 \in \pi(K)$. If $|\pi(K)| = 3$, then $K$ is a simple $K_3$-group. According to [7], the order of all simple $K_3$-groups does not contain the prime factor 13. If $|\pi(K)| = 4$, then because $\pi_{p_m}(M) = 13$, we have $t(K) \geq 2$. When $t(K) = 2$, by examining Table 3 and Table 4 in [2] and utilizing the condition $\pi_{p_m}(M) = 13$, it can be concluded that such a simple group does not exist. When $t(K) \geq 3$, by examining Table 2 and Table 4
of [2] and utilizing the condition \( \pi_{pm}(M) = 13 \), it can be concluded that such a simple group does not exist. If \(|\pi(K)| \geq 5\), then because \( \pi_{pm}(M) = 13 \) and \( t(K) \geq 3 \), by examining Table 2 to Table 4 of [2] and utilizing the condition \( \pi_{pm}(M) = 13 \), it can be deduced that \( K \) can only be one of the following groups: \( A_{13}, A_{14}, A_{15}, \) and \( Suz \).

If \( K \cong A_{13} \), then \(|K| = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \). Since \( m_1(G) = 2^7 \cdot 3^5 \cdot 7^2 \cdot 11 \), we have \( 2^3 \cdot 5 \cdot 7 | |G/K| |Out(A_{13})| = 2 \), which leads to a contradiction. Therefore, \( K \) is not isomorphic to \( A_{13} \). Similarly, \( K \not\cong A_{14} \).

If \( K \cong Suz \), then \(|K| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \). Since in this case \( m_1(K) = 2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \), it clearly contradicts \( m_1(G) = 2^7 \cdot 3^5 \cdot 7^2 \cdot 11 \), hence \( K \not\cong Suz \).

Hence, we have \( K \cong A_{15} \), which implies \( 1 \leq A_{15} \leq G \). In this case, it is clear that \( C_G(A_{15}) = 1 \) and \(|Out(A_{15})| = 2 \). Therefore, we either have \( G \cong A_{15} \) or \( G \cong Aut(A_{15}) \). If \( G \cong Aut(A_{15}) \), then we have evidently that \( m_1(G) > m_1(A_{15}) = m_1(M) \), which contradicts our assumption. Hence, \( G \cong A_{15} \).

The proof has been completed.

References


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