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MI-Injective and MI-Flat Modules[1]

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Abstract

We introduce MI-injective modules and MI-flat modules. The properties and characterizations of these two classes of modules are provided. We also characterize rings such that all min-injective modules are min-flat in term of MI-flat modules.

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1 Introduction

Min-injective modules were first introduced by Harada [3] and then studied by Nicholson and Yousif in [5]. In [4], Mao introduced min-flat modules and studied min-coherent rings in term of min-injective modules and min-flat modules. In this paper, we would like introduce two new classes of modules based from min-injective modules, namely MI-injective modules and MI-flat modules. The properties of these two classes of modules are provided. We also characterize MI-injective modules as kernels of some precovers and MI-flat modules as cokernels of some preenvelopes. As a main result, we prove that

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every MI-injective module is a direct sum of a reduced MI-injective module and an injective module. We also characterize rings such that all min-injective modules are min-flat in term of MI-flat modules.

Throughout the paper, R is an associative ring with identity and all modules are unitary. For an R-module M, the character module M^+ is defined by $M^+ := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})^+$.

For two left R-modules M and N, we use $\mathrm{Hom}(M,N)$ to stand for $\mathrm{Hom}_R(M,N)$. Similarly, we have notations $M \otimes N$, $\mathrm{Ext}^k(M,N)$ and $\mathrm{Tor}_k(M,N)$ for $k \geqslant 1$.

Let $\mathcal{L}$ be a class of R-modules. $\mathcal{L}$ is said to be closed under *extensions*, if for any exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Z \in \mathcal{L}$, it holds that $Y \in \mathcal{F}$ too.

Let $\mathcal{C}$ be a class of R-modules and M is a R-module. Following [2], we say that a homomorphism $\phi : M \to C$ is a $\mathcal{C}$-preenvelope if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a $\mathcal{C}$-envelope if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. Dually we have the definitions of a $\mathcal{C}$-precover and a $\mathcal{C}$-cover. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Let R be a ring. A left R-module M is said to be finitely generated (finitely presented, resp.) if there is an exact sequence $P_0 \to M \to 0$ ($P_1 \to P_0 \to M \to 0$, resp.), where each P_i is finitely generated projective left R-modules. In particular, a simple module is always finitely generated.

Recall that a ring R is called *left coherent* if every finitely generated left ideal is finitely presented.A ring R is called *left min-coherent* if every simple left ideal of R is finitely presented.

2 MI-injective and MI-flat modules

We firstly recall the following definition, see for instance [5].

Definition 2.1 *Let R be a ring. A left R-module M is called min-injective if* $\mathrm{Ext}^1(R/I, M) = 0$ *or equivalently,the sequence* $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M) \to 0$ *is exact for any simple left ideal I of R.*

In the following, we introduce two new classes of modules.

Definition 2.2 *(1) A left R-module M is called MI-injective if* $\mathrm{Ext}^1(G, M) = 0$ *for any min-injective left R-module G.*

(2) A right R-module N is said to be MI-flat if $\mathrm{Tor}_1(N, G) = 0$ *for any min-injective left R-module G.*

Immediately from the standard isomorphism: $\mathrm{Ext}^1(N, M^+) \cong \mathrm{Tor}_1(M, N)^+$ for any left R-module N and right R-module M, we have the following results.

Proposition 2.3 *A right R-module M is MI-flat if and only if M^+ is MI-injective.*

The following result gives some properties of MI-injective modules and MI-flat modules. The proof is also easy, so we omit it.

Proposition 2.4 *(1) The class of MI-injective modules is closed under direct products, direct summands and extensions.*

(2) The class of MI-flat modules is closed under direct sums, direct summands and extensions.

Recall that, the min-injective dimension of M, denoted by $mid(M)$, is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}^{n+1}(R/I, M) = 0$ for every simple left ideal I (if no such n exists, set $mid(M) = \infty$). For instance, an R-module M has the min-injective dimension 0 if and only if M itself is min-injective.

It is easy to see that injective modules are MI-injective and that flat modules are MI-flat. The following result consider the inverse part in the case of coherent rings.

Proposition 2.5 *We have the following result for a left coherent ring R:*

(1) A left R-module M is injective if and only if M is MI-injective and $mid(M) \leqslant 1$.

(2) A right R-module N is flat if and only if N is MI-flat and $fd(M) \leqslant 1$.

Proof. (1) Suppose that M is an MI-injective left R-module. Consider an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Then there exists an exact sequence

$$0 = \mathrm{Ext}^1(R/I, E) \to \mathrm{Ext}^1(R/I, L) \to \mathrm{Ext}^2(R/I, M)$$

for any simple left ideal I. Since $mid(M) \leqslant 1$, we have that $\mathrm{Ext}^2(R/I, M) = 0$. Hence $\mathrm{Ext}^1(R/I, L) = 0$, that is, L is min-injective. It follows that $\mathrm{Ext}^1(L, M) = 0$ since M is MI-*injective*. So the exact sequence $0 \to M \to E \to L \to 0$ is split by the definition. Thus M is isomorphic to a direct summand of E, and hence injective.

(2) Suppose N is any MI-flat right R-module. Then N^+ is MI-injective by Proposition 2.3. Note that $\mathrm{Ext}^{n+1}(R/I, N^+) \cong \mathrm{Tor}_{n+1}(N, R/I)^+$ for any simple left ideal I, since R is coherent. So we have that $mid(N^+) \leqslant 1$ since $fd(N) \leqslant 1$ by the hypothesis. Then N^+ is injective by (1), which implies that N is flat, as desired. □

Proposition 2.6 *The following are equivalent for a left R-module M.*

(1) M is MI-injective.

(2) For every exact sequence $0 \to M \to E \to L \to 0$ with E min-injective, $E \to L$ is a min-injective precover of L.

(3) M is a kernel of a min-injective precover $f:A \to B$ with A injective.

(4) M is injective with respect to every exact sequence $0 \to A \to B \to C \to 0$, where C is min-injective.

Proof. (1) $\Rightarrow$ (2) Note that there exists an exact sequence $\mathrm{Hom}(G, E) \to \mathrm{Hom}(G, L) \to \mathrm{Ext}^1(G, M)$ for any min-injective left R-module G and that $\mathrm{Ext}^1(G, M) = 0$ as M is MI-injective, so $\mathrm{Hom}(G, E) \to \mathrm{Hom}(G, L)$ is an epimorphism. It follows that $E \to L$ is a min-injective precover of L by the definition.

(2) $\Rightarrow$ (3) It is trivial since for any left R-module M, there exists a short exact sequence $0 \to M \to E \to E/M \to 0$, where E is the injective envelope of M.

(3) $\Rightarrow$ (1). Suppose M is the kernel of a min-injective precover $f : A \to B$ with A injective. Then there exists an exact sequence $0 \to M \to A \xrightarrow{\pi} im(f) \to 0$. For any any min-injective left R-module N, we get the induced exact sequence

$$\mathrm{Hom}(N, A) \xrightarrow{\mathrm{Hom}(N,\pi)} \mathrm{Hom}(N, im(f)) \longrightarrow \mathrm{Ext}^1(N, M) \longrightarrow \mathrm{Ext}^1(N, A) = 0 .$$

It is easy to verify that $\pi : A \to im(f)$ is also a min-injective precover. So $\mathrm{Hom}(N, \pi)$ is suejective by the definition of precovers. Then $\mathrm{Ext}^1(N, M) = 0$ and M is MI-injective by the definition.

(1) $\Rightarrow$ (4) Clearly, there exists an exact sequence

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to \mathrm{Ext}^1(C, M).$$

Note that C is min-injective and that M is MI-injective by the assumption, so we have that $\mathrm{Ext}^1(C, M) = 0$. Hence (4) follows.

(4) $\Rightarrow$ (1). Take any a min-injective left R-module N and consider a short exact sequence $0 \to K \to P \to N \to 0$ with P projective. Then there is an induced exact sequence

$$\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to \mathrm{Ext}^1(N, M) \to \mathrm{Ext}^1(P, M) = 0.$$

But $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M)$ is surjective by the assumption of (4). Therefore $\mathrm{Ext}^1(N, M) = 0$ and M is MI-injective, as desired. □

The following result characterizes kernels of min-injective covers. Recall that a left R-module M is called *reduced* if M has no nonzero injective submodules.

Proposition 2.7 *Suppose R is a left coherent ring. Then the following are equivalent for a left R-module M:*

(1) M is a reduced MI-injective left R-module.

(2) M is a kernel of a min-injective cover $f : A \to B$ with A injective.

Proof. (1) $\Rightarrow$ (2) Note that the natural map $\pi : E \to E/M$ is a min-injective precover by Proposition 2.6(3), where E is the injective envelope of M. Since R is left coherent, the module E/M has a min-injective cover [4]. Moreover,

E has no nonzero direct summand K contained in M since M is reduced. So we can get that $\pi : E \to E/M$ is indeed a min-injective cover by [6, Corollary 1.2.8], and then (2) follows.

(2) $\Rightarrow$ (1). Suppose M is the kernel of a min-injective cover $\alpha : A \to B$ with A injective. Then M is MI-injective by Proposition 2.6. Now suppose K is an injective submodule of M. Then K is also an injective submodule of A and hence a direct summand of A. Let $A = K \oplus L$ for some L. Assume that $p : A \to L$ is the projection and $i : L \to A$ is the inclusion. It is easy to see that $\alpha(ip) = \alpha$. In fact, let $a = (k + l) \in A$, then $\alpha(ip)(a) = \alpha(ip)(k + l) = \alpha i(l) = \alpha(l) = \alpha(k + l) = \alpha(a)$, since $\alpha(K) = 0$. Hence ip is an isomorphism since α is a cover. It follows that i is epic. Thus $A = L$ and then $K = 0$. So M is reduced. $\square$

Theorem 2.8 *Let R be a left coherent ring. Then a left R-module M is MI-injective if and only if M is a direct sum of a reduced MI-injective left R-module and an injective left R-module.*

Proof. The if part is obvious.

Only if part. Let M be an MI-injective left R-module and consider the exact sequence $0 \to M \to E \to E/M \to 0$, where E is the injective envelope of M. We know that $E \to E/M$ is a min-injective precover of E/M by Proposition 2.6. Since R is left coherent, E/M also has a min-injective cover $L \to E/M$. So we get the following commutative diagram with exact rows for some K:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E/M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \gamma} & & \| & & \\
0 & \longrightarrow & M & \xrightarrow{\alpha} & E & \longrightarrow & E/M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E/M & \longrightarrow & 0.
\end{array}$$

Note that $\beta\gamma$ is an isomorphism since $L \to E/M$ is a cover. Then $E \simeq L \oplus ker(\beta)$. It follows that L and $ker(\beta)$ is injective. Therefore K is a reduced MI-injective module by Proposition 2.7. Since $\sigma\phi$ is also an isomorphism by the Five Lemma, we have that σ is epic and ϕ is monic and that $M = ker(\sigma) \oplus im(\phi)$, where $im(\phi) \cong K$ since ϕ is monic. In addition, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & ker(\sigma) & \longrightarrow & ker(\beta) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\alpha} & E & \longrightarrow & E/M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E/M & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Hence $ker(\sigma) \cong ker(\beta)$ is injective and we have $M = ker(\sigma) \bigoplus im(\phi)$, where $im(\phi) \cong K$ is a reduced MI-injective left R-module and $ker(\sigma)$ is injective. $\square$

Proposition 2.9 *Suppose R is a left coherent ring.*

(1) If L is a cokernel of a min-flat preenvelope $f : K \to F$ of a right R-module K with F being MI-flat. then L is MI-flat.

(2) If M is a finitely presented MI-flat right R-module, then M is a cokernel of a flat preenvelope.

Proof. (1) Consider the exact sequence $0 \to im(f) \xrightarrow{i} F \to L \to 0$ obtained from the assumption. It is easy to check that $i : im(f) \to F$ is also a min-flat preenvelope. Let N be any min-injective left R-module, then N^+ is min-flat by [4, Lemma 3.2]. Thus we get an exact sequence $\mathrm{Hom}(F, N^+) \to \mathrm{Hom}(im(f), N^+) \to 0$ by the definition of preenvelope. Equivalently, we have the exact sequence $(F \bigotimes N)^+ \to (im(f) \bigotimes N)^+ \to 0$. This implies that the sequence $0 \to im(f) \bigotimes N \to F \bigotimes N$ is exact. On the other hand, since F is MI-flat, we have $\mathrm{Tor}_1(F, N) = 0$. Thus, from the induced long exact sequence

$$0 = \mathrm{Tor}_1(F, N) \to \mathrm{Tor}_1(L, N) \to im(f) \bigotimes N \to F \bigotimes N,$$

we get that $\mathrm{Tor}_1(L, N) = 0$. So L is MI-flat, as desired.

(2) Since M is a finitely presented right R-module, we have an exact sequence $0 \to K \to P \to M \to 0$ with P projective and both P and K finitely generated. It is enough to show that $K \to P$ is a flat preenvelope. In fact, suppose F is any flat right R-module, then F^+ is injective. Obviously, it is also min-injective. Hence $\mathrm{Tor}_1(M, F^+) = 0$, and so we have the following commutative diagram with the first row exact:

$$\begin{array}{ccccc}
0 \longrightarrow & K \bigotimes F^+ & \xrightarrow{\alpha} & P \bigotimes F^+ & \\
 & \downarrow{\scriptstyle \tau_{K,F}} & & \downarrow{\scriptstyle \tau_{P,F}} & \\
 & \mathrm{Hom}(K, F)^+ & \xrightarrow{\theta} & \mathrm{Hom}(P, F)^+. &
\end{array}$$

Note that, by [1, Lemma 2], $\tau_{K,F}$ is an epimorphism since K is finitely generated and $\tau_{P,F}$ is an isomorphism since P is finitely presented. Since $\theta\tau_{K,F} = \tau_{P,F}\alpha$, we obtain that $\tau_{K,F}$ is also a monomorphism and hence an isomorphism. Thus θ is a monomorphism. It follows that the homomorphism $\mathrm{Hom}(P, F) \to \mathrm{Hom}(K, F)$ is epic. So $K \to P$ is a flat preenvelope and M is a cokernel of a flat preenvelope. □

We call that R is said to be a left *MIF ring* if every min-injective left R-module is min-flat.

An exact sequence $0 \to A \to B \to C \to 0$ is said to be min-pure exact, if for any simple left ideal I, the induced sequence $0 \to \mathrm{Hom}(R/I, A) \to \mathrm{Hom}(R/I, B) \to \mathrm{Hom}(R/I, C) \to 0$ is exact, or equivalently, the induced sequence $0 \to A \bigotimes R/I \to B \bigotimes R/I \to C \bigotimes R/I \to 0$ is exact (by [1, Lemma 2], since R/I is finitely presented).

M is called a min-pure-injective left R-module, if the functor $\mathrm{Hom}(-, M)$ preserves the exactness of all min-pure exact sequences.

Theorem 2.10 *The following are equivalent for a ring R.*

(1) R is a left MIF ring.

(2) Every min-pure-injective left R-module is MI-injective.

(3) Every right R-module is MI-flat.

(4) Every finitely presented right R-module is MI-flat.

Proof. (1) ⇒ (2). Let M be an arbitrary min-pure-injective left R-module. Take any min-injective left R-module N and consider an exact sequence $0 \to N \to E \to L \to 0$ with E injective. The sequence is in fact min-pure from the definition, since there is an induced exact sequence

$$0 \to \mathrm{Hom}(R/I, N) \to \mathrm{Hom}(R/I, E) \to \mathrm{Hom}(R/I, L) \to \mathrm{Ext}^1(R/I, N) = 0.$$

On the other hand, there exists an exact sequence $0 \to K \to P \to N \to 0$ with P projective. Note that the exact sequence is also min-pure exact sequence, since N is also min-flat by the assumption and we have an induced exact sequence $0 = \mathrm{Tor}_1(N, R/I) \to K \bigotimes R/I \to P \bigotimes R/I \to N \bigotimes R/I \to 0$. Applying the functor $\mathrm{Hom}(M, -)$, we obtain an induced long exact sequence

$$0 \to \mathrm{Hom}(P, K) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to \mathrm{Ext}^1(N, M) \to \mathrm{Ext}^1(P, M) = 0.$$

As M is min-pure-injective, the sequence $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to 0$ is exact. It follows that $\mathrm{Ext}^1(N, M) = 0$ and that M is MI-injective.

(2) ⇒ (3). Take any right R-module M, then M^+ is pure injective. It is obviously min-pure injective. So it is also *MI-injective* by the assumption. Then M is MI-flat by Proposition 2.3.

(3) ⇒ (4) is obvious.

(4) ⇒ (1). Suppose E is any min-injective left R-module and let M be any finitely presented right R-module. Then M is MI-flat by the assumption. Hence $\mathrm{Tor}_1(M, E) = 0$. It follows that E is flat for the arbitrarity of M. Obviously it is also min-flat. □

References

[1] E. E. Enochs, A note on absolutely pure modules, *Canad. Math. Bull.*, **19** (1976), 361-362. https://doi.org/10.4153/cmb-1976-054-5

[2] E.E. Enochs and O.M.G. Jenda, *Relative Homological Algebra*, De Gruyter Expositions in Mathematics, Vol. 30, de Gruyter, Berlin, 2000. https://doi.org/10.1515/9783110803662

[3] M. Harada, Self mini-injective rings, *Osaka J. Math.*, **19** (1982), 587-597.

[4] L. Mao, Min-Flat Modules and Min-coherent Rings, *Comm. Algebra*, **35** (2007), no. 2, 635-650. https://doi.org/10.1080/00927870601074889

[5] W. K. Nicholson and M. F. Yousif, Mininjective rings, *J. Algebra*, **187** (1997), 548-578. https://doi.org/10.1006/jabr.1996.6796

[6] J. Xu, *Flat Covers of Modules*, Lecture Notes in Math., vol. 1634, Springer Verlag, Berlin, 1996. https://doi.org/10.1007/bfb0094173

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