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Cyclic Quadruples of Linearly Independent Roots in Root Systems D_l and E_l

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In memory of my friend, Victor Shikovich Aranov

Abstract

We consider the hierarchy of four geometrical objects in the root system: {roots}, {dipoles}, {cyclic quadruples}, and {cyclic configurations}. A pair of orthogonal roots is said to be a *dipole*. A subset consisting of four linearly independent roots forming a cycle is said to be a *cyclic quadruple*. The cyclic quadruple is represented by $D_4(a_1)$, the smallest Carter diagram containing cycles, see [5]. We study the arrangement of two cyclic quadruples of roots in the root system in relation to each other. This mutual arrangement of cyclic quadruples is considered under the action of different groups: *similarities*, *isomorphisms* or *conjugacies*. First, we show that any two cyclic quadruples, up to isomorphisms, have a *common dipole*. After this, we show that all cyclic quadruples in the root system D_l (with $l > 4$) or E_l (resp. D_4) are divided into two (resp. three) non-isomorphic classes which constitute one conjugacy class. At last, we consider *cyclic configurations*, the pairs of cyclic quadruples $\{S_1, S_2\}$ with a common dipole. A priori, up to similarities, there are 81 cyclic configurations. We reduce the number of possible configurations to 12 and list them in Table 1.1.

Keywords: Dynkin diagrams, Carter diagrams, Weyl group, root systems, cycles

1. Introduction

1.1. **Abstract patterns of root subsets.** Let Δ be the root system D_l or E_l. A root subset in Δ consisting of four linearly independent roots and forming a cycle is called a *cyclic quadruple*. The main purpose of the article is to classify the arrangement of two cyclic quadruples in the root system Δ in relation to each other. A pair of orthogonal roots $d = \{\beta_1, \beta_2\}$ in the root system Δ is said to be a *dipole*. Let $\{S_1, S_2\}$ be any pair of cyclic quadruples in Δ. We show that there exist dipoles $d_1 \in S_1$ and $d_2 \in S_2$ which are isomorphic. Then, we transform isomorphically S_2 into S'_2, another cyclic quadruple, so that the image of dipole d_2 coincides with d_1. In this way, we get $\{S_1, S'_2\}$, a new diagram called a *cyclic configuration*. The cyclic configuration consists of two cyclic quadruples with a common dipole. Up to similarities[1], cyclic configurations are classified and listed in Table 1.1.

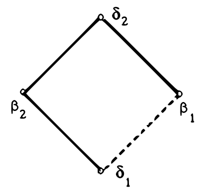

FIG. 1.1. The $D_4(a_1)$-quadruple $S = \{\beta_1, \delta_1, \beta_2, \delta_2\}$

1.1.1. *Another name for a cyclic quadruple.* A cyclic quadruple (resp. cyclic configuration) is also called a $D_4(a_1)$-quadruple (resp. a $D_4(a_1)$-configuration). Why $D_4(a_1)$? The diagram $D_4(a_1)$ is the smallest Carter diagram with cycles, see [5], it is shown in Fig. 1.1. For any Carter diagram, *dotted* (resp. *solid*) edges correspond to the inner product 1 (resp. −1). For $D_4(a_1)$, this means

$$(\beta_1, \delta_1) = 1, \quad (\beta_1, \delta_2) = -1, \quad (\beta_2, \delta_1) = -1, \quad (\beta_2, \delta_2) = -1,$$

where $\beta_1, \beta_2, \delta_1, \delta_2$ are roots in Δ, and $(\cdot, \cdot)$ is the symmetric bilinear form corresponding to the Cartan matrix associated with Δ, see [4, §2.1]. As a corollary of classification of $D_4(a_1)$-configurations, we show that $D_4(a_1)$ determines a single conjugacy class in the Weyl group.

1.1.2. *Some properties of Carter diagrams.* Consider the root subset of linearly independent roots (in the root system associated with a Dynkin diagram) which forms a connected component. Let us call such a subset a Γ-*collection*, where Γ is the diagram representing this subset. In what follows, Γ can be a connection, Dynkin, or Carter diagram. The exact definitions of connection, Carter diagrams and Γ-collections will be provided shortly in §1.3, meanwhile we list the general properties of the Carter diagrams, see [5]:

(a) each cycle of Carter diagram contains an even number of nodes.

(b) any Carter diagram can contain dotted (resp. solid) edges, which correspond to positive (resp. negative) inner products of roots. The Carter diagram without cycles is just a Dynkin diagram.

[1]For a root α in the root system, the reflection $T^\alpha : \alpha \to -\alpha$ is said to be the *similarity*, see §1.3.3.

(c) the roots corresponding to the vertices of Carter diagrams are not necessarily simple, in contrast to the Dynkin diagrams.

(d) each Carter diagram is explicitly or implicitly endowed by the order of vertices Ω. In according with the order Ω, the product of corresponding reflections represents a certain element w from the Weyl group.

(e) each Carter diagram is given up to the following set of transformations: similarity, conjugacy and s-permutation. These transformations determine the equivalence relation on Carter diagrams, [5]. The element w associated with a Carter diagram is considered up to its conjugacy class.

(f) The most interesting case is the Carter diagrams with cycles. The main theorem of [5] is as follows: *Any Carter diagram containing l-cycles, where l > 4, is equivalent to another Carter diagram containing only 4-cycles*, see [5, Theorem 3.1].

The Γ-collection is a certain *abstract pattern* of the root subset which can be embedded into different root systems. For 4-vertex diagrams, instead of "Γ-collection", we use the term Γ-quadruple. A lot of $D_4(a_1)$-quadruples can be found in the root systems B_l, C_l, D_l, E_l and F_4.

1.1.3. *The smallest cycle.* The smallest cycle among the Carter diagrams is the 4-cycle. Let us see how the cycle consisting of 3 linearly independent roots can be easily eliminated. If $\{\alpha,\beta,\gamma\}$ are the linearly independent roots of the 3-cycle then one of the cycle's edges should be dotted, see [5, Lemma A.1]. Assume, the edge $\{\alpha,\beta\}$ is dotted, then by means of one s-permutation[1], we get $w = s_\alpha s_\beta s_\gamma = s_\alpha s_\gamma s_{\beta+\gamma}$, in addition $(\alpha,\beta+\gamma) = (\alpha,\beta) + (\alpha,\gamma) = 1 - 1 = 0$, see Fig. 1.2.

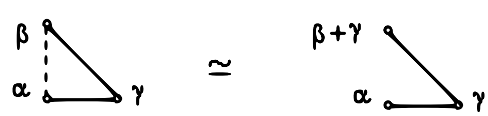

FIG. 1.2. Eliminating the 3-cycle $\{\alpha,\beta,\gamma\}$ by means of one s-permutation

Let us move to 4-cycles. There are two 4-cycles differing by order Ω of vertices: the element $w_1 = s_{\alpha_1}s_{\beta_1}s_{\alpha_2}s_{\beta_2}$ is represented by connection diagram $(\mathcal{G}_4,\Omega_1)$, where Ω_1 is the order $\{\alpha_1,\beta_1,\alpha_2,\beta_2\}$ and $w_2 = s_{\alpha_1}s_{\alpha_2}s_{\beta_1}s_{\beta_2}$ is represented by Carter diagram $D_4(a_1)$ with the order $\{\alpha_1,\alpha_2,\beta_1,\beta_2\}$. The element w_1 can be transformed to the element $w_1' = s_{\alpha_1}s_{\alpha_2}s_{-(\alpha_1+\beta_1+\beta_2)}s_{\beta_2}$ represented by Dynkin diagram D_4, i.e., diagram $(\mathcal{G}_4,\Omega_1)$ is transformed to D_4, see [5, §1.2.4]. The diagram $D_4(a_1)$ representing the element w_2 cannot be isomorphically

[1]Here, the *s-permutation* is the following identity: $s_\beta s_\gamma = s_\gamma s_{\beta+\gamma}$. For the generic case, see §1.3.3.

transformed to the diagram without cycles. The 4-cycle $D_4(a_1)$ is the smallest Carter diagram with a cycle, which cannot be eliminated.

1.1.4. *Linear independence.* Γ-collections have some properties that are invariant with respect to the root system containing these Γ-collections. The first such property is the connection of vertices: The connection diagrams and Carter diagrams are intended for representing this property. Another such property is the *linear independence* of vectors from a Γ-collection.

Lemma 1.1. *Let $S = \{\alpha_1, \dots, \alpha_n\}$ be a n-cycle.*

(i) *If each edge in the cycle S is solid then the roots of S are linearly dependent and*

$$\sum_{i=1}^{n} \alpha_i = 0. \tag{1.1}$$

(ii) *If $\{\alpha_1, \dots, \alpha_n\}$ are linearly independent then the number of dotted edges is odd.*

Proof. The statements (i), (ii) follow from [5, Lemma A.1]. □

In fact, Lemma 1.1 is a special case of a more general statement related to the maximal roots in the root system, see Lemma 2.2. We often use the particular cases of Lemma 1.1 related to triangles (Lemma 2.3) and squares (Lemma 2.5).

1.2. **The main results.**

1.2.1. *Two non-isomorphic classes of* Γ*-quadruples.* Two Γ-collections S_1 and S_2 are said to be *isomorphic* if $TS_1 = S_2$ for some element T in the Weyl group W, where the image of T is determined up to similarities $T^{\alpha_i} : \alpha_i \to -\alpha_i$, $\alpha_i \in S_2$. For isomorphic Γ-collections S_1 and S_2 , we write $S_1 \simeq S_2$. Two Γ-collections S_1 and S_2 are said to be *conjugate* if $Tw_1T^{-1} = w_2$ for some element $T \in W$, where w_1, w_2 are given by Γ-collections S_1 and S_2. For details, see §1.3.3. In **Theorem 3.4**, we show that

- *In D_l (with $l > 4$) or E_l, there are* **two** *non-isomorphic classes of $D_4(a_1)$-quadruples.*
- *In D_4, there are* **three** *non-isomorphic classes of $D_4(a_1)$-quadruples.*
- *In D_l and E_l, there is* **one** *conjugacy class of $D_4(a_1)$-quadruples, see §1.2.3.*

An example of three pairwise non-isomorphic classes in D_4 is given in **Remark 4.6**.

1.2.2. *Common dipoles and $D_4(a_1)$-configurations.* A pair of orthogonal roots in a $D_4(a_1)$-quadruple is said to be a *dipole*. The dipole is constituted by any diagonal of $D_4(a_1)$. Let Δ be a root system D_l or E_l. In **Theorem 3.8**, we show that

- *for any pair of $D_4(a_1)$-quadruples $\{S_1, S_2\}$ in Δ, there exist dipoles $d_1 \in S_1$ and $d_2 \in S_2$, which are isomorphic: $d_1 \simeq d_2$.*

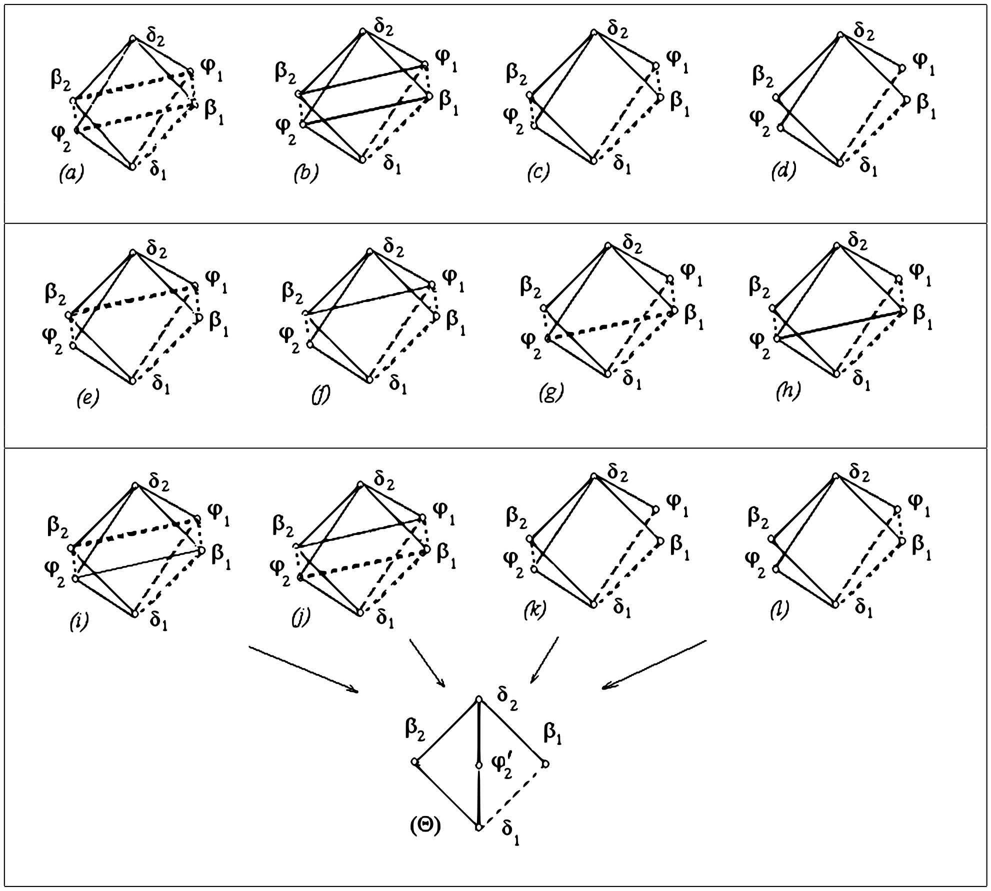

TABLE 1.1. Up to similarities, there are 12 $D_4(a_1)$-configurations $\{S_1, S_2\}$, see Theorem 4.7

We apply Theorem 3.8 for identifying the isomorphic dipoles in S_1 and S_2. In this way, we get the diagram which is said to be a $D_4(a_1)$-*configuration.* The common dipole is denoted by $\{\delta_1, \delta_2\}$. The rectangle $R = \{\beta_1, \varphi_1 \beta_2, \varphi_2\}$ completely determines the $D_4(a_1)$-configuration, see Table 1.1. By **Lemma 4.1** we essentially reduce the number of possible $D_4(a_1)$-configurations:

- *Each of two pairs* $\{\beta_1, \varphi_1\}$ *and* $\{\beta_2, \varphi_2\}$ *is a dotted edge or does not form any edge, see Table 1.1.*
- *If one of the pairs* $\{\beta_1, \varphi_2\}$ *or* $\{\beta_2, \varphi_1\}$ *is connected, then the pairs* $\{\beta_1, \varphi_1\}$ *and* $\{\beta_2, \varphi_2\}$ *constitute dotted edges, see Table 1.1*(*a*),(*b*),(*e*)-(*j*).

Up to similarities, there are 81 possible rectangles R, i.e., 81 possible $D_4(a_1)$-configurations. By Lemma 4.1 the number of possible $D_4(a_1)$-configurations is reduced to 12, they are listed in Table 1.1. In **Theorem 4.7**, we consider $D_4(a_1)$-quadruples $\{S_1, S_2\}$ of Table 1.1:

- *In* $D_4(a_1)$*-configurations* (*a*)-(*h*), S_1 *and* S_2 *are isomorphic.*

- *In $D_4(a_1)$-configurations (i)-(l), S_1 and S_2 are not necessarily isomorphic. These cases are reduced to case (Θ) of Table 1.1.*
- *In all cases (a)-(l), elements $w_1 = s_{\beta_1}s_{\beta_2}s_{\delta_1}s_{\delta_2}$ and $w_2 = s_{\varphi_1}s_{\varphi_2}s_{\delta_1}s_{\delta_2}$ (where w_i is the S_i-associated element, $i = 1, 2$) are conjugate.*

1.2.3. *Uniqueness of the conjugacy class.* From **Theorem 4.7** and **Proposition 4.8** we get that the Carter diagram $D_4(a_1)$ represents a **unique conjugacy class**:

- *For any pair of $D_4(a_1)$-quadruples $\{S_1, S_2\}$, the S_i-associated element w_i, where $i = 1, 2$, are conjugate.*

Dipole type	Representative	Coefficients p_i
2–index dipoles	$\{e_i + e_j,\ e_i - e_j\}$	
4–index dipoles	$\{e_i + e_j,\ e_k + e_n\}$	
8–index dipoles	$\{\sum\limits_{i=1}^{8} p_i e_i,\ \sum\limits_{i=1}^{4} p_i e_i - \sum\limits_{i=5}^{8} p_i e_i\}$	$p_i = (-1)^{\nu(i)}$
2–8–index dipoles	$\{\sum\limits_{i=1}^{8} p_i e_i,\ p_k e_k - p_n e_n\}$	$p_i = (-1)^{\nu(i)}$; $k, n \in \{1, \dots, 8\}$

TABLE 1.2. Four types of dipoles

1.2.4. *Four types of dipoles.* In the root system $\Delta = D_l$ or E_l, there exist 2-index roots $\pm e_i \pm e_j$, see §3.1. Besides, in E_l there exist 8-index roots $\sum_{i=1}^{8}(-1)^{\nu(i)}e_i$, see §3.2. Consequently, there exist four types of dipoles: 2-index, 4-index, 8-index, 2-8-index, see Table 1.2. A number of basic properties of dipoles in D_l (resp. E_l) are shown in §3.1 (resp. in §3.2). In our considerations, 4-*index dipoles are crucial.* In particular, in **Lemma 3.1** and **Lemma 3.3**, it is shown that

- *In the root system D_l, any $D_4(a_1)$-quadruple contains a 4-index dipole. It can happen that both dipoles are 4-index.*
- *There exist 2 non-isomorphic classes of 4-index dipoles in D_4.*
- *Any two 4-index dipoles in D_l (with $l > 4$) and E_l are isomorphic.*

In **Lemma 3.6**, for a $D_4(a_1)$-quadruple with a 4-index dipole d_1 and a dipole d_2, we show that

- *If d_2 is an 8-index dipole, there exists an isomorphism preserving d_1 and transforming d_2 into some 2-8-index dipole.*
- *If d_2 is an 2-8-index dipole, there exists an isomorphism preserving d_1 and transforming d_2 into some 4-index dipole.*

In **Corollary 3.7**, it is shown that

- *Any $D_4(a_1)$-quadruple containing 8-index roots can be isomorphically mapped onto a certain $D_4(a_1)$-quadruple without 8-index roots.*

1.3. **Diagrams and transformations.**

1.3.1. *Carter diagrams and* Γ-*collections.* Let Φ be the root system associated with a Weyl group W; let s_{α_i} be the reflection in W corresponding to not necessarily simple root $\alpha_i \in \Phi$. Each element $w \in W$ can be expressed in the form

$$w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}, \text{ where } \alpha_i \in \Phi, \tag{1.2}$$

We denote by $l_C(w)$ the smallest value k in any expression like (1.2), see [2, p. 3]. We always have $l_C(w) \le l(w)$. Recall that $l(w)$ is the smallest value k in any expression like (1.2) such that all roots α_i are simple. The decomposition (1.2) is called *reduced* if $l_C(s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}) = k$.

Lemma 1.2. [2, Lemma 3] *Let $\alpha_1, \alpha_2, \dots, \alpha_k \in \Phi$. Then $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ is reduced if and only if $\alpha_1, \alpha_2, \dots, \alpha_k$ are linearly independent.* ☐

A diagram Γ is said to be *admissible*, see [2, p. 7], if

(a) The nodes of Γ correspond to a set of linearly independent roots in Φ.

(b) If a subdiagram of Γ is a cycle, then it contains an even number of nodes. (1.3)

Any admissible diagram Γ is said to be a *Carter diagram* if any edge connecting a pair of roots $\{\alpha, \beta\}$ with inner product $(\alpha, \beta) > 0$ (resp. $(\alpha, \beta) < 0$) is drawn as dotted (resp. solid) edge. Let

$$S = \{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_h\} \tag{1.4}$$

be any set of linearly independent, not necessarily simple, roots associated with Γ, where roots of the set $S_\alpha := \{\alpha_i \mid i = 1, \dots, k\}$ are mutually orthogonal, roots of the set $S_\beta := \{\beta_j \mid j = 1, \dots, h\}$ are also mutually orthogonal. By (1.3) the partitioning into such a sum of sets S_α and S_β is possible. The set of roots S is said to be a Γ-*collection.* Let

$$w = w_1 w_2, \quad \text{where} \quad w_1 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}, \quad w_2 = s_{\beta_1} s_{\beta_2} \dots s_{\beta_h}. \tag{1.5}$$

Since S is linearly independent, the decomposition (1.5) is reduced, see Lemma 1.2, and $k + h = l_C(w)$. The element w is said to be Γ-*associated*, and also S-*associated*. The decomposition (1.5) is said to be a *bicolored decomposition.* The set of roots S_α (resp. S_β) is said to be the α-*set* (resp. β-*set*) of roots corresponding to the bicolored decomposition (1.5).

1.3.2. *Connection diagrams.* Let Γ be the diagram characterizing connections between roots of a certain set S of linearly independent and not necessarily simple roots, Ω be the order of reflections in the decomposition (1.2). The pair (Γ, Ω) is said to be a *connection diagram*. We omit indicating order Ω in the description of the connection diagram if the order of reflections in the decomposition (1.2) is clear[1]. The connection diagram determines the element w (and its inverse w^{-1}) obtained as the product of all reflections associated with the diagram, while the order Ω (resp. Ω^{-1}) describes the order of reflections in the decomposition of w (resp. w^{-1}). Connection diagrams describe connected sets with any cycles in the diagram, not necessarily even cycles.

1.3.3. *Similarity, isomorphism, congugacy, s-permutation.* We define a number of relationships and transformations on Carter and connection diagrams, Γ-collections and Γ-associated elements.

Similarity. The transformation *similarity* T^α acts only on the root α and the edges attached in the vertex α. All dotted edges with vertex α are transformed into the solid edges and vice versa. The similarity T^α maps the root α to the opposite one:

$$T^\alpha : \alpha \longmapsto -\alpha. \tag{1.6}$$

Two Γ-collections S_1 and S_2 are said to be *similar* if they can be obtained from each other by the sequence of *similarities*.

Isomorphism. Two Γ-collections $S_1 = \{\alpha_1, \dots, \alpha_n\}$ and $S_2 = \{\alpha'_1, \dots, \alpha'_n\}$ are said to be *isomorphic* if there exists $T \in W$ such that

$$T\alpha_i = \alpha'_i \text{ or } T\alpha_i = -\alpha'_i \text{ for } i = 1, \dots, n, \tag{1.7}$$

where α_i and α'_i correspond to the same vertex of Γ.

Conjugacy. Let $S_1 = \{\alpha_1, \dots, \alpha_n\}$ and $S_2 = \{\alpha'_1, \dots, \alpha'_n\}$ be two Γ-collections, w_i be S_i-associated elements:

$$w_1 = \prod_i s_{\alpha_i}, \qquad w_2 = \prod_i s_{\alpha'_i}.$$

The Γ-collections S_1 and S_2 are said to be *conjugate* if there exists $T \in W$ such that $Tw_1T^{-1} = w_2$. If $T : S_1 \to S_2$ is the isomorphism then, since $s_{\alpha_i} = s_{-\alpha_i}$, we have

$$s_{\alpha'_i} = Ts_{\alpha_i}T^{-1} \quad \text{for} \quad i = 1, \dots, n, \quad \text{and} \quad Tw_1T^{-1} = w_2. \tag{1.8}$$

For the Carter diagrams, conjugacy (1.8) follows from the isomorphism (1.7).

[1]Usually, this is the case of a bicolored decomposition, where the order Ω corresponds to (1.5).

<u>*s-Permutation*</u>. The "evenness" of cycles of Carter diagrams can be violated by the transformations of the third type, called *s-permutations*:

$$s_\alpha s_\beta = \begin{cases} s_\beta s_{\alpha+\beta} = s_{\alpha+\beta} s_\alpha & \text{for } (\alpha,\beta) < 0, \\ s_\beta s_{\alpha-\beta} = s_{\alpha-\beta} s_\alpha & \text{for } (\alpha,\beta) > 0, \\ s_\beta s_\alpha & \text{for } (\alpha,\beta) = 0. \end{cases} \tag{1.9}$$

The s-permutation in (1.9) transforms the *solid* (resp. *dotted*) edge $\{\alpha,\beta\}$ into edge $\{\beta,\alpha+\beta\}$ (resp. $\{\beta,\alpha-\beta\}$) or $\{\alpha+\beta,\alpha\}$ (resp. $\{\alpha-\beta,\alpha\}$). We get a new diagram, which is not necessarily a Carter diagram, but only a certain connection diagram.

1.4. **Similarity of $D_4(a_1)$-configurations.**

1.4.1. *Rotations.* Consider a certain $D_4(a_1)$-configuration $\{S_1,S_2\}$, where S_1, S_2 are two $D_4(a_1)$-quadruples with a common dipole $d=\{\delta_1,\delta_2\}$. Let ρ be the rotation of one of $D_4(a_1)$-quadruples about the axis d by 180 degrees. Denote these rotations by $\rho^{\beta_1\beta_2}$ and $\rho^{\varphi_1\varphi_2}$. Rotation $\rho^{\beta_1\beta_2}$ (resp. $\rho^{\varphi_1\varphi_2}$) transposes vertices β_1 and β_2 (resp. φ_1 and φ_2) and preserves corresponding edges, see Fig. 1.3. The purpose of rotations is to move dotted edges to the lower right position.

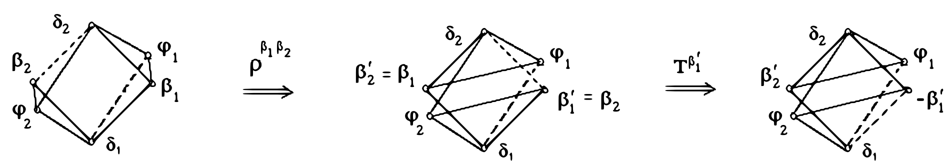

FIG. 1.3. Rotations of $D_4(a_1)$-quadruples

1.4.2. *Examples of similarities.* Two $D_4(a_1)$-configurations are said to be *similar* if they can be obtained from each other by several *similarities* and one of two *rotations* from §1.4.1. We study Γ-configurations *up to similarities*[1] in order to freeze the location of dotted edges and to reduce the number of cases that should be examined, see Fig. 1.4. The $D_4(a_1)$-configurations depicted on the right side in Fig. 1.4(a),(b),(c) we call *basic $D_4(a_1)$-configurations.* Consider the following list:

$$\{\{\beta_1,\varphi_1\},\{\beta_1,\varphi_2\},\{\beta_2,\varphi_1\},\{\beta_1,\varphi_2\}\}. \tag{1.10}$$

Any pair of the list (1.10) can be a solid or dotted edge or does not form any edge at all. Every $D_4(a_1)$-configuration without edges of (1.10) is similar to the basic one on the right side in Fig. 1.4(a),(b). The basic $D_4(a_1)$-configuration contains exactly two dotted edges: one belonging to S_1, and the other belonging to S_2. It is easy to see that any $D_4(a_1)$-configuration with edges from the list (1.10) can be transformed by means of similarities to another one, where the lower right edges are dotted, see Fig. 1.4(c).

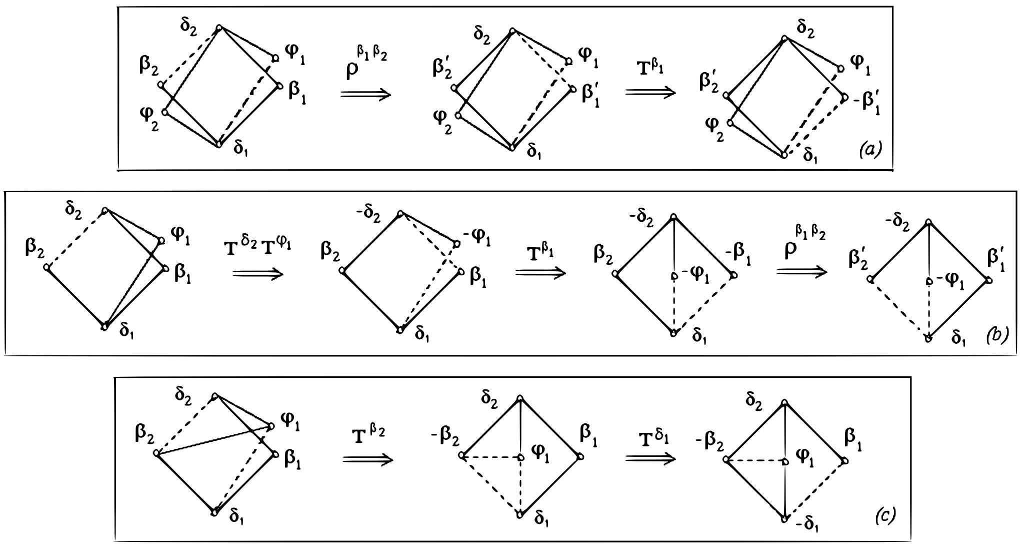

FIG. 1.4. Examples of isomorphic mappings of $D_4(a_1)$-configurations

1.5. **Dipoles, Γ-quadruples and Γ-configurations.** If Γ is a 4-vertex diagram, the Γ-collection is said to be a Γ-*quadruple*. In this article, we study $D_4(a_1)$-quadruples. The pair of roots corresponding to a diagonal of any $D_4(a_1)$-quadruple is said to be a *dipole*. The vertices corresponding to the roots of the dipole are not connected by any edge. For diagram $D_4(a_1)$, not every two dipoles are isomorphic; however, for any two $D_4(a_1)$-quadruples S_1 and S_2, there exist dipoles $d_1 \in S_1$ and $d_2 \in S_2$ such that d_1 and d_2 are isomorphic, see Theorem 3.8.

Let $\{S_1, S_2\}$ be a pair of $D_4(a_1)$-quadruples. We isomorphically map S_2 to S_2' such that S_1 and S_2' have a common dipole. Then, we can assume that from the beginning S_1 and S_2 have a certain common dipole. The pair $\{S_1, S_2\}$ is said to be $D_4(a_1)$-*configuration*.

2. Basic patterns: Triangles and squares

2.1. **The partial Cartan matrix.** We determine the *partial Cartan matrix* corresponding to the Carter diagram Γ as follows:

$$B_\Gamma := \begin{pmatrix} (\tau_1, \tau_1) & \dots & (\tau_1, \tau_n) \\ \dots & \dots & \dots \\ (\tau_n, \tau_1) & \dots & (\tau_n, \tau_n) \end{pmatrix}, \tag{2.1}$$

[1]Hereafter, saying "up to similarities", we mean "up to similarities and rotations".

where $S = \{\tau_1, \dots, \tau_n\}$ is a Γ-collection. The symmetric bilinear form associated with the partial Cartan matrix B_Γ is denoted by $(\cdot,\cdot)_\Gamma$ and the corresponding quadratic form is denoted by $\mathscr{B}_\Gamma$. The subspace $L \subseteq V$ spanned by the Γ-collection S is said to be the *S-associated subspace*. For $S = \{\tau_1, \dots, \tau_l\}$, we write $L = [\tau_1, \dots, \tau_l]$.

Proposition 2.1. *1) The restriction of the bilinear form associated with the Cartan matrix* $\mathbf{B}$ *onto the S-associated subspace L coincides with the bilinear form associated with the partial Cartan matrix B_Γ, i.e., for any pair of vectors $v, u \in L$, we have*

$$(v,u)_\Gamma = (v,u), \text{ and } \mathscr{B}_\Gamma(v) = \mathscr{B}(v). \tag{2.2}$$

2) For every Carter diagram, the matrix B_Γ is positive definite.

Proof. 1) From eq. (2.1) we deduce:

$$(v,u)_\Gamma = (\sum_i t_i\tau_i, \sum_j q_j\tau_j)_\Gamma = \sum_{i,j} t_i q_j (\tau_i,\tau_j)_\Gamma = \sum_{i,j} t_i q_j (\tau_i,\tau_j) = (v,u).$$

2) This follows from 1). □

2.1.1. *The generalized and particular Cartan matrices.* Recall that the $n \times n$ matrix K such that

$$\begin{aligned} &(C1) \quad k_{ii} = 2 \text{ for } i = 1,\dots,n,\\ &(C2) \quad -k_{ij} \in \mathbb{Z} = \{0,1,2,\dots\} \text{ for } i \neq j,\\ &(C3) \quad k_{ij} = 0 \text{ implies } k_{ji} = 0 \text{ for } i,j = 1,\dots,n \end{aligned}$$

is called a *generalized Cartan matrix*, [3], [4, §2.1]. The condition (C2) does not hold for the partial Cartan matrix, because the elements k_{ij} associated with dotted edges are positive.

If the Carter diagram does not contain any cycle, then the Carter diagram is the Dynkin diagram, the corresponding conjugacy class is the conjugacy class of the Coxeter element, and the partial Cartan matrix is the classical Cartan matrix, a particular case of a generalized Cartan matrix.

2.2. **Linear dependence and maximal roots.** Let $S = \{\tau_1, \dots, \tau_l\}$ be a Γ-collection, S' be another Γ-collection, and $S' = uS$ for some element $u \in W$. The matrix B_Γ is the same for S and S', since $(u\tau_i, u\tau_j) = (\tau_i, \tau_j)$ for any $\tau_i, \tau_j \in S$. Let γ be a root which is linearly dependent on roots of S as follows:

$$\gamma = t_1\tau_1 + \dots + t_l\tau_l. \tag{2.3}$$

Then, we have

$$\begin{pmatrix} (\gamma,\tau_1) \\ \dots \\ (\gamma,\tau_l) \end{pmatrix} = B_\Gamma \begin{pmatrix} t_1 \\ \dots \\ t_l \end{pmatrix} = B_\Gamma\gamma, \text{ and } \begin{pmatrix} t_1 \\ \dots \\ t_l \end{pmatrix} = B_\Gamma^{-1} \begin{pmatrix} (\gamma,\tau_1) \\ \dots \\ (\gamma,\tau_l) \end{pmatrix}. \tag{2.4}$$

If the root τ_i is replaced by $-\tau_i$ (for any $\tau_i \in S$), then the coefficient t_i is replaced by $-t_i$ in the decomposition (2.3).

Lemma 2.2. *Let $S = \{\tau_1, \dots, \tau_l\}$ be a Γ-collection, where Γ is a simply-laced Dynkin diagram, and let γ be a root linearly dependent on roots of S. Suppose γ is connected exactly to the same point(s) as the maximal (or minimal) root is connected to the simple root(s) in Γ (in the root system A_n, γ is connected to 2 points; in the remaining root systems, γ is connected only to one point). Then γ coincides with the maximal (resp. minimal) root*[1]*.*

Proof. By conditions of lemma, the orthogonality relations (γ, τ_i) in eq. (2.4) coincide with orthogonality relations for the maximal (resp. minimal) root while the edge connecting with γ is dotted (resp. solid). The lemma holds since equation (2.4) has a unique solution. □

In the remaining part of §2, we consider Γ-collections forming the simplest diagrams: dipoles, triangles, squares. For each type of these diagrams, we describe properties helping to understand whether a certain root subset is linearly independent or not, see Lemmas 2.3, 2.5.

2.3. **Triangles and squares.**

Lemma 2.3. *Consider a 3-cycle Γ. Let $S = \{\alpha, \beta, \gamma\}$ be a certain Γ-collection. The triple S is linearly independent if and only if the number of dotted edges of Γ is odd, see Fig. 2.5(c),(d). If all edges of Γ are solid, see Fig. 2.5(a), then*

$$\alpha + \beta + \gamma = 0. \tag{2.5}$$

If only one edge of Γ is solid, for example, $\{\alpha, \gamma\}$ in Fig. 2.5(b), then

$$\alpha - \beta + \gamma = 0. \tag{2.6}$$

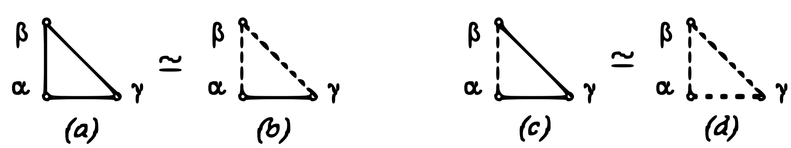

FIG. 2.5. Linearly dependent triples (a), (b); linearly independent triples (c),(d).

Remark 2.4. Note that changing the sign of any root in S does not affect the linear dependence and does not change the element $w \in W$ corresponding to Γ-collection S since $s_\alpha = s_{-\alpha}$. In Fig. 2.5, the case (b) turns into (a) under the similarity T^β, the case (d) turns into (c) under the similarity T^γ. In Fig. 2.6, the case (b) turns into (a) under the similarity T^{α_1}, the case (d) turns into (c) under the similarity T^{β_1}, the case (f) turns into (e) under the similarity $T^{\alpha_2}T^{\alpha_1}$.

[1] Note that for $\Gamma = A_n$, Lemma 1.1(i) is deduced from Lemma 2.2.

Proof of Lemma 2.3. The lemma is the particle case of Lemma 1.1. If S is linearly independent, then there exist 1 or 3 dotted edges, see (c) and (d) in Fig. 2.5, otherwise we have the extended Dynkin diagram $\widetilde{A}_3$ which cannot happen. The root $-\alpha-\beta$ is the minimal root for $A_2 = \{\alpha, \beta\}$ in accordance with Lemma 2.2. Eq. (2.5) follows from (1.1). Eq. (2.6) is obtained from eq. (2.5) by similarity T^β. □

Lemma 2.5. *Consider a 4-cycle* Γ. *Let* $S = \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ *be a* Γ*-quadruple. The* Γ*-quadruple* S *is linearly independent if and only if the number of dotted edges of* Γ *is odd, see Fig.* 2.6(a),(b). *For the* Γ*-quadruple with* 0, 2 *or* 4 *dotted edges, we have*

$$\begin{cases} \alpha_1+\alpha_2+\beta_1+\beta_2=0, & \textit{no dotted edges}, \quad Fig.2.6(c), \\ \alpha_1+\alpha_2-\beta_1+\beta_2=0, & \textit{edges } \{\alpha_1,\beta_1\} \textit{ and } \{\alpha_2,\beta_1\} \textit{ are dotted}, \quad Fig.2.6(d), \\ \alpha_1-\alpha_2-\beta_1+\beta_2=0, & \textit{edges } \{\alpha_1,\beta_1\} \textit{ and } \{\alpha_2,\beta_2\} \textit{ are dotted}, \quad Fig.2.6(e), \\ \alpha_1-\alpha_2+\beta_1-\beta_2=0, & \textit{edges } \{\alpha_1,\beta_2\} \textit{ and } \{\alpha_2,\beta_1\} \textit{ are dotted}, \quad Fig.2.6(f), \\ \alpha_1+\alpha_2-\beta_1-\beta_2=0, & \textit{all edges are dotted}, \quad Fig.2.6(g). \end{cases} \tag{2.7}$$

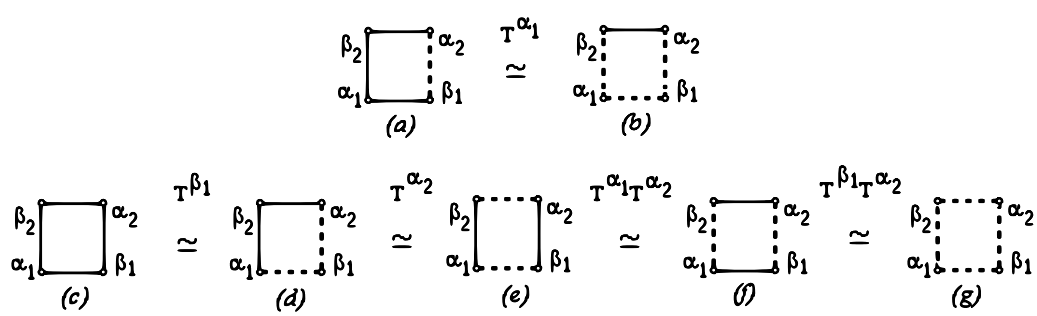

FIG. 2.6. Linearly dependent quadruples (a), (b); linearly independent quadruples (c)-(g).

Proof. This lemma, like Lemma 2.3 above, is a special case of Lemma 1.1. If S is linearly independent, then there exist 1 or 3 dotted edges, see Fig. 2.6(a) and (b), otherwise by similarities we get the diagram $\widetilde{A}_4$, which cannot happen. For case (c) in Fig. 2.6, by (1.1) we have $\beta_2 = -\alpha_1 - \alpha_2 - \beta_1$. This is the minimal root for A_3 in accordance with Lemma 2.2. Case (d) (resp. (e); resp. (f)) is obtained from case (c), see (2.7), by similarity T^{β_1}, (resp., by similarity $T^{\beta_1}T^{\alpha_2}$; resp. by similarity $T^{\beta_2}T^{\alpha_2}$). Case (g) is obtained from case (f) by similarity $T^{\beta_1}T^{\alpha_2}$. □

2.4. **Inconsistent triangles, squares, edges and Γ-configurations.** A diagram is said to be *incomplete* if there exist two disconnected non-orthogonal roots on this diagram. Disconnected non-orthogonal roots must be connected by the dotted or solid edge. A diagram is said to be *incorrect* if there exists an edge which should be removed or changed to the edge of the *opposite type* (i.e., solid to dotted and vice versa). The corresponding edge is also said to be *incorrect*. An incomplete or incorrect diagram or any other diagram (resp. Γ-configuration) with conflicting relationships of vertices and edges are said to be *inconsistent*. The diagram (resp. Γ-configuration) obtained from the inconsistent one by needed actions (connecting points, removing an edge, changing the type) is said to be *fixed*. The fixed diagram or any other diagram (resp. Γ-configuration) without conflicting relationships is said to be *valid*.

2.4.1. *Correction of inconsistent diagrams.* We use Lemmas 2.6, 2.7, 2.8 for construction of potential common dipoles and common triples for pairs of Γ-quadruples S_1 and S_2 in §4.

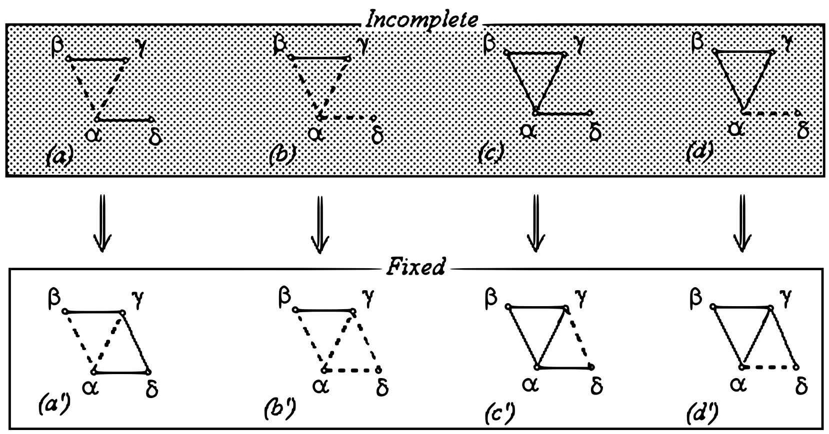

FIG. 2.7. Triangles $\{\alpha,\beta,\gamma\}$ with $\delta\perp\beta$: Incomplete (a)-(d) and fixed (a')-(d')

Lemma 2.6 (Incomplete diagrams with 3-cycles)**.** *Each of diagrams* (a)-(d) *in Fig. 2.7 contains the* 3*-cycle* $\{\alpha,\beta,\gamma\}$ *and the edge* $\{\delta,\alpha\}$ *attached in the vertex* α. *For cases* $(a),(c)$ *(resp.* (b), (d)*), the edge* $\{\delta,\alpha\}$ *is solid (resp. dotted). Assume* $\delta\perp\beta$. *Then diagrams* (a)-(d) *are incomplete.*

To fix diagrams $(a),(d)$ *it is necessary to add the solid edge* $\{\delta,\gamma\}$.

To fix diagrams $(b),(c)$ *it is necessary to add the dotted edge* $\{\delta,\gamma\}$.

The fixed diagrams are shown in (a')-(d').

Proof. For all cases, by Lemma 2.3 the roots $\{\alpha,\beta,\gamma\}$ are linearly dependent.

Cases (a), (b). By (2.6) we have $\alpha=\beta+\gamma$. Then $(\delta,\alpha)=(\delta,\beta+\gamma)=(\delta,\gamma)$, since $\delta\perp\beta$, i.e., $\{\delta,\gamma\}$ is solid (resp. dotted) since $\{\delta,\alpha\}$ is solid (resp. dotted).

Cases (c), (d). By (2.5) we have $\alpha=-(\beta+\gamma)$. Then $(\delta,\alpha)=(\delta,-(\beta+\gamma))=-(\delta,\gamma)$, i.e., $\{\delta,\gamma\}$ is solid (resp. dotted) since $\{\delta,\alpha\}$ is dotted (resp. solid). $\square$

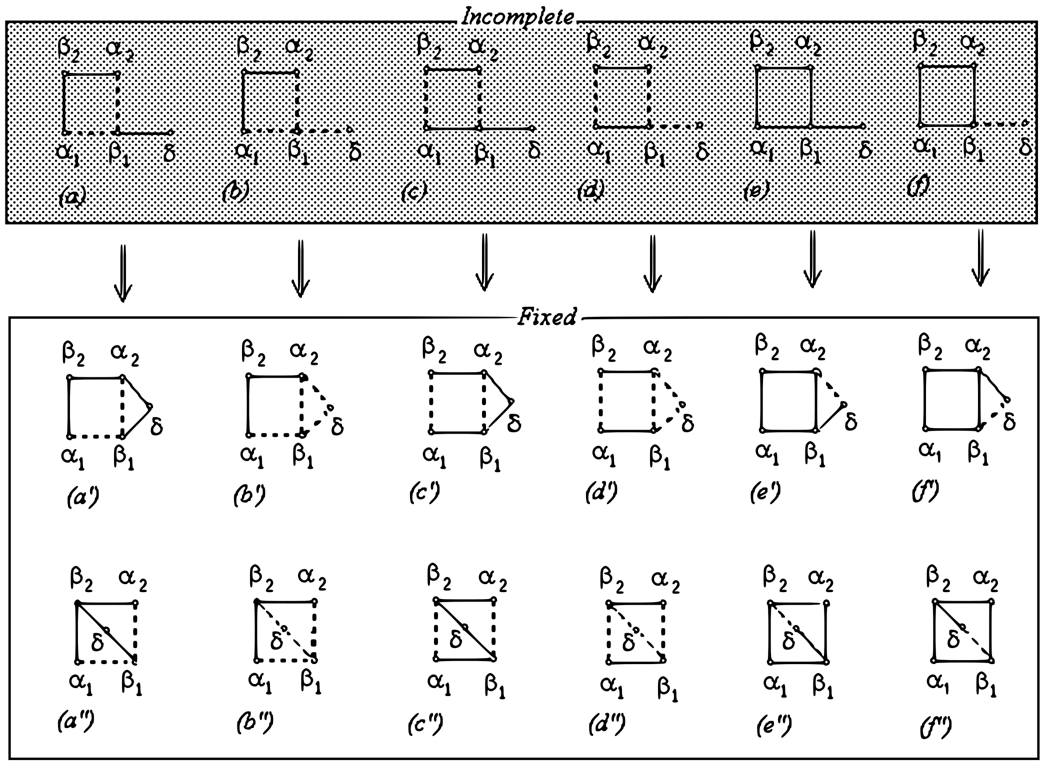

FIG. 2.8. Squares $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ with $\delta \perp \alpha_1$: Incomplete (a)-(f) and fixed (a')-(f'), (a'')-(f'')

Lemma 2.7 (Incomplete diagrams with 4-cycles). *Each of diagrams (a)-(f) in Fig. 2.8 contains the 4-cycle $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ and the edge $\{\delta, \beta_1\}$ attached in the vertex β_1. For cases (a), (c), (e) (resp. (b), (d), (f)) the edge $\{\delta, \beta_1\}$ is solid (resp. dotted). Assume $\delta \perp \alpha_1$. Then diagrams (a)-(f) are incomplete.*

To fix diagrams (a), (c), (f) it is necessary to add the solid edge $\{\delta, \alpha_2\}$ or $\{\delta, \beta_2\}$.

To fix diagrams (b), (d), (e) it is necessary to add the dotted edge $\{\delta, \alpha_2\}$ or $\{\delta, \beta_2\}$.

The fixed diagrams are shown in (a')-(f') and (a'')-(f'').

Proof. For all cases, by Lemma 2.5 quadruples of roots are linearly dependent.

Cases (a), (b). By (2.7) we have

$$\beta_1 = \alpha_1 + \beta_2 + \alpha_2 \quad \Rightarrow \quad (\delta, \beta_1) = (\delta, \alpha_1 + \beta_2 + \alpha_2) = (\delta, \beta_2) + (\delta, \alpha_2) \quad \text{since} \quad \delta \perp \alpha_1.$$

Thus (δ, β_2) or (δ, α_2) coincides with (δ, β_1), the second summand is zero. In other words, $\{\delta, \beta_2\}$ (or $\{\delta, \alpha_2\}$) is solid (resp. dotted) since $\{\delta, \beta_1\}$ is solid (resp. dotted).

Cases (c), (d). By (2.7) we have

$$\beta_1 = -\alpha_1 + \beta_2 + \alpha_2 \quad \Rightarrow \quad (\delta, \beta_1) = (\delta, \beta_2) + (\delta, \alpha_2) \quad \text{since} \quad \delta \perp \alpha_1.$$

As above, $\{\delta, \beta_2\}$ (or $\{\delta, \alpha_2\}$) is solid (resp. dotted) since $\{\delta, \beta_1\}$ is solid (resp. dotted).

Cases (e), (f). By (2.7) we have

$$\beta_1 = -(\alpha_1 + \beta_2 + \alpha_2) \quad \Rightarrow \quad (\delta, \beta_1) = -(\delta, \beta_2) - (\delta, \alpha_2) \quad \text{since} \quad \delta \perp \alpha_1, \tag{2.8}$$

i.e., $\{\delta, \beta_2\}$ (or $\{\delta, \alpha_2\}$) is solid (resp. dotted) since $\{\delta, \beta_1\}$ is dotted (resp. solid). □

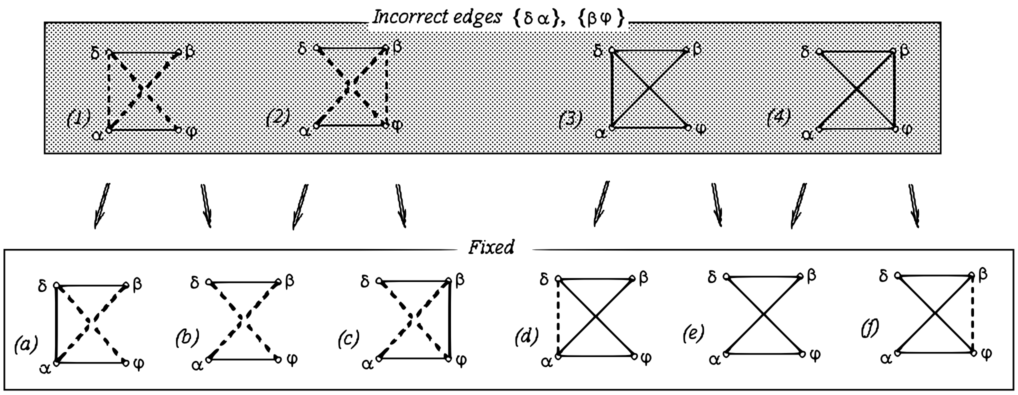

FIG. 2.9. Edges $\{\delta, \alpha\}$ and $\{\beta, \varphi\}$: Incorrect diagrams (1)-(4) and fixed (a)-(f)

Lemma 2.8. *Let* $\{\alpha, \beta, \delta, \varphi\}$ *be linearly independent roots in diagrams* (1)-(4) *of Fig. 2.9. The edge* $\{\delta, \alpha\}$ *(resp.* $\{\beta, \varphi\}$*) is incorrect in cases* (1), (3) *(resp.* (2), (4)*). We can fix the incorrect edge as follows:*

(1) *Remove the dotted edge* $\{\delta, \alpha\}$ *or change it to solid, see Fig. 2.9*(a),(b).
(2) *Remove the dotted edge* $\{\beta, \varphi\}$ *or change it to solid, see Fig. 2.9*(b),(c).
(3) *Remove the solid edge* $\{\delta, \alpha\}$ *or change it to dotted, see Fig. 2.9*(d),(e).
(4) *Remove the solid edge* $\{\beta, \varphi\}$ *or to change it to dotted, see Fig. 2.9*(e),(f).

Proof. Consider case (1). By Lemma 2.3 from triangles $\{\delta, \alpha, \varphi\}$ and $\{\delta, \alpha, \beta\}$ we have $\delta = \alpha + \varphi$ and $\alpha = \delta + \beta$. Then $\varphi = -\beta$, which is contradiction. We fix the incorrect edge $\{\delta, \alpha\}$ by removing it, see Fig. 2.9(b), or changing it to solid, see Fig. 2.9(a). Cases (2),(3),(4) are similarly treated. □

2.5. **Corrective mappings.** There are 8 possible triangles with solid or dotted edges, see Fig. 2.10. By Lemma 2.3 diagrams (a), (c), (f) and (h) in Fig. 2.10 constitute linearly independent triples. Four remaining triangles constitute linearly dependent triples.

Lemma 2.9. *Let us consider triangles* (a)-(h) *in Fig. 2.10. Triangles* (a), (c), (f), (h) *(resp.* (b), (d), (e), (g)*) represent the linearly independent triples (resp. linearly dependent) of roots* $\{\alpha_1, \alpha_2, \beta\}$*. For all triangles, we assume that* α_1 *and* α_2 *are linearly independent.*

(i) *For triangles* (a)-(d)*, we have* $(\alpha_1, \alpha_2) = -1$ *and* $\alpha_1 + \alpha_2$ *is a root. Then*

$$\begin{cases} s_{\alpha_1+\alpha_2}(\alpha_1) = -\alpha_2, \\ s_{\alpha_1+\alpha_2}(\alpha_2) = -\alpha_1, \end{cases} \quad \begin{cases} s_{\alpha_1+\alpha_2}(\beta) = \beta, & \textit{for } (a),(c), \\ s_{\alpha_1+\alpha_2}(\beta) = -\beta, & \textit{for } (b),(d), \end{cases} \quad \begin{cases} s_{\alpha_1+\alpha_2}s_\beta = s_\beta s_{\alpha_1+\alpha_2}, \\ s_{\alpha_1+\alpha_2}s_{\alpha_1} = s_{\alpha_2}s_{\alpha_1+\alpha_2}. \end{cases} \tag{2.9}$$

(ii) *For triangles* (e)-(h)*, we have* $(\alpha_1, \alpha_2) = 1$ *and* $\alpha_1 - \alpha_2$ *is a root. Then*

$$\begin{cases} s_{\alpha_1-\alpha_2}(\alpha_1) = \alpha_2, \\ s_{\alpha_1-\alpha_2}(\alpha_2) = \alpha_1, \end{cases} \quad \begin{cases} s_{\alpha_1-\alpha_2}(\beta) = \beta, & \textit{for } (f),(h), \\ s_{\alpha_1-\alpha_2}(\beta) = -\beta, & \textit{for } (e),(g), \end{cases} \quad \begin{cases} s_{\alpha_1-\alpha_2}s_\beta = s_\beta s_{\alpha_1-\alpha_2}, \\ s_{\alpha_1-\alpha_2}s_\alpha = s_\alpha s_{\alpha_1-\alpha_2}. \end{cases} \tag{2.10}$$

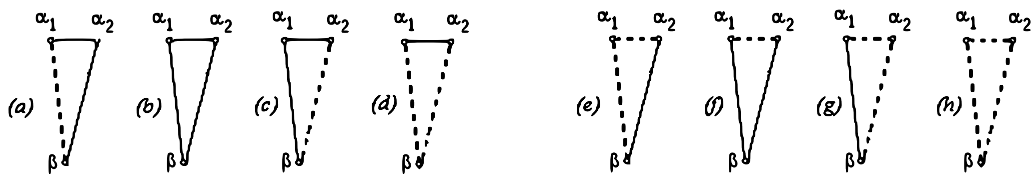

FIG. 2.10. The eight possible triangles each edge of which is either solid or dotted.

Proof. (i) Since $(\alpha, \alpha) = 2$ for any root α, we have

$$s_{\alpha_1+\alpha_2}(\alpha_1) = \alpha_1 - \frac{2(\alpha_1+\alpha_2, \alpha_1)(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2, \alpha_1+\alpha_2)} = \alpha_1 - (2-1)(\alpha_1+\alpha_2) = -\alpha_2. \tag{2.11}$$

Similarly, $s_{\alpha_1+\alpha_2}(\alpha_2) = -\alpha_1$. For linearly independent triples (a) and (c), we get $(\alpha_1, \beta) = -(\alpha_2, \beta)$, i.e., $\alpha_1 + \alpha_2 \perp \beta$, and $s_{\alpha_1+\alpha_2}(\beta) = \beta$. Then $s_{\alpha_1+\alpha_2}$ and s_β commute. For linearly dependent triples (b) and (d), by Lemma 2.3, $\alpha_1 + \alpha_2 = \pm\beta$, i.e., $s_{\alpha_1+\alpha_2} = s_\beta$ and $s_{\alpha_1+\alpha_2}(\beta) = -\beta$. The last relation $s_{\alpha_1+\alpha_2}s_{\alpha_1} = s_{\alpha_2}s_{\alpha_1+\alpha_2}$ follows from (1.9).

(ii) Here, we get

$$s_{\alpha_1-\alpha_2}(\alpha_1) = \alpha_1 - \frac{2(\alpha_1-\alpha_2, \alpha_1)(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2, \alpha_1-\alpha_2)} = \alpha_1 - (2-1)(\alpha_1-\alpha_2) = \alpha_2. \tag{2.12}$$

Similarly, $s_{\alpha_1-\alpha_2}(\alpha_2) = \alpha_1$. For linearly independent triples (f) and (h), we get $(\alpha_1, \beta) = (\alpha_2, \beta)$, i.e., $\alpha_1 - \alpha_2 \perp \beta$, and $s_{\alpha_1-\alpha_2}(\beta) = \beta$. Then $s_{\alpha_1-\alpha_2}$ and s_β commute. For linearly dependent triples (e) and (g), by Lemma 2.3, $\alpha_1 - \alpha_2 = \pm\beta$. Thus, $s_{\alpha_1-\alpha_2} = s_\beta$ and $s_{\alpha_1-\alpha_2}(\beta) = -\beta$. The relation $s_{\alpha_1-\alpha_2}s_{\alpha_1} = s_{\alpha_2}s_{\alpha_1-\alpha_2}$ follows from (1.9). □

Two edges are said to be edges of the *opposite type* if one of them is solid and the second is dotted. Two edges are said to be edges of the *same type* if both are solid or both are dotted. Recall that for edges of opposite (resp. the same) types, the corresponding inner products differ (resp. coincide) in sign; for a solid (resp. dotted) edge the inner product is -1 (resp. 1).

Corollary 2.10. *Let $S_1 = \{\alpha_1, \beta_1, \beta_2\}$ and $S_2 = \{\alpha_2, \beta_1, \beta_2\}$ be triples with common vertices $\{\beta_1, \beta_2\}$. Consider two following quadruples $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ containing S_1 and S_2:*

$$\begin{aligned} &\text{(i)} \quad (\alpha_1, \beta_1) = -(\alpha_2, \beta_1), \quad (\alpha_1, \beta_2) = -(\alpha_2, \beta_2), \quad (\alpha_1, \alpha_2) = -1, \\ &\qquad\qquad \textit{(Fig. 2.11(a),(b))}, \\ &\text{(ii)} \quad (\alpha_1, \beta_1) = (\alpha_2, \beta_1), \quad (\alpha_1, \beta_2) = (\alpha_2, \beta_2), \quad (\alpha_1, \alpha_2) = 1, \\ &\qquad\qquad \textit{(Fig. 2.11(c),(d))}. \end{aligned} \tag{2.13}$$

Let w_i be S_i-associated elements:

$$w_1 = s_{\alpha_1} s_{\beta_1} s_{\beta_2}, \quad w_2 = s_{\alpha_2} s_{\beta_1} s_{\beta_2}. \tag{2.14}$$

For quadruple (i) *(resp.* (ii)*), reflection $s_{\alpha_1+\alpha_2}$ (resp. $s_{\alpha_1-\alpha_2}$) realises the isomorphism S_1 onto S_2. The elements w_1 and w_2 are conjugate:*

$$\begin{aligned} s_{\alpha_1+\alpha_2} w_1 s_{\alpha_1+\alpha_2} = w_2 \quad &\textit{for quadruple } \text{(i)}, \\ s_{\alpha_1-\alpha_2} w_1 s_{\alpha_1-\alpha_2} = w_2 \quad &\textit{for quadruple } \text{(ii)}. \end{aligned} \tag{2.15}$$

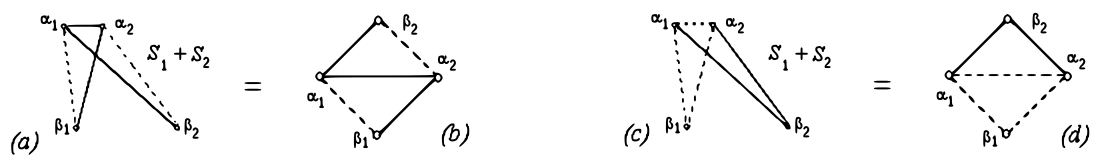

FIG. 2.11. Corrective mapping: $s_{\alpha_1+\alpha_2}$ for cases (a),(b), see eq. (2.13)(i); $s_{\alpha_1-\alpha_2}$ for cases (c),(d), see eq. (2.13)(ii).

Proof. By (2.13) we have

$$\begin{aligned} (\beta_1, \alpha_1 + \alpha_2) = 0, \quad (\beta_2, \alpha_1 + \alpha_2) = 0 \quad &\text{for quadruple (i)}, \\ (\beta_1, \alpha_1 - \alpha_2) = 0, \quad (\beta_2, \alpha_1 - \alpha_2) = 0 \quad &\text{for quadruple (ii)}. \end{aligned}$$

For β_i and α_i where $i = 1, 2$, we get the commutation relations:

$$\begin{aligned} s_{\beta_i} s_{\alpha_1+\alpha_2} = s_{\alpha_1+\alpha_2} s_{\beta_i}, \quad s_{\alpha_1+\alpha_2} s_{\alpha_1} s_{\alpha_1+\alpha_2} = s_{\alpha_2}, \quad &\text{for quadruple (i)}, \\ s_{\beta_i} s_{\alpha_1-\alpha_2} = s_{\alpha_1-\alpha_2} s_{\beta_i}, \quad s_{\alpha_1-\alpha_2} s_{\alpha_1} s_{\alpha_1-\alpha_2} = s_{\alpha_2}, \quad &\text{for quadruple (ii)}. \end{aligned}$$

Then, for $i = 1, 2$, the statement (2.15) holds. □

The reflections $s_{\alpha_1+\alpha_2}$ and $s_{\alpha_1-\alpha_2}$ behave like maps correcting the set $\{\alpha_1, \beta_1, \beta_2\}$ to the set $\{\alpha_2, \beta_1, \beta_2\}$. This is the reason to call the reflection $s_{\alpha_1+\alpha_2}$ (resp. $s_{\alpha_1-\alpha_2}$) a *corrective mapping.*

3. **Combinatorics of roots and dipoles in D_l and E_l**

In what follows, we consider $D_4(a_1)$-quadruples in the root systems D_l and E_l.

3.1. **The roots and dipoles in D_l.** In this section, we consider only $D_4(a_1)$-quadruples in the root systems D_l. The roots in the root system D_l are as follows:

$$\pm e_i \pm e_j \ (1 \le i < j \le l), \text{ where } \{e_i \mid i = 1, \dots l\} \text{ mutually orthogonal unit vectors in } \mathbb{R}^l, \tag{3.1}$$

see [1, Table IV]. Any dipole of each $D_4(a_1)$-quadruple in D_l is either

$$\{e_k - e_n, e_k + e_n\}, \text{ where } k \ne n, \tag{3.2}$$

or

$$\{e_i \pm e_j, e_k \pm e_n\}, \text{ where } i, j, k, n \text{ are different.} \tag{3.3}$$

We call a dipole of type (3.2) (resp. (3.3)) a 2-*index dipole* (resp. a 4-*index dipole*). For properties of 2-index and 4-index dipoles, see Lemma 3.1 and Corollary 3.2.

3.1.1. *The Weyl group $W(D_4)$.* The Weyl group $W(D_4)$ is constructed as semidirect product, see [1, ch.6, §4, n^o 8, (IX) and (X)]. The set of three reflections $\{s_{e_1-e_2}, s_{e_2-e_3}, s_{e_3-e_4}\}$ generates the symmetric group S_4, the subgroup of the order 24 in $W = W(D_4)$. The reflection $s_{e_i-e_j}$ acts as transposition $(e_i e_j)$ and the product of two reflections $s_{ij} = s_{e_i-e_j} s_{e_i+e_j}$ maps $e_i \mapsto -e_i$, $e_j \mapsto -e_j$, and do not change vectors e_k for $k \ne i, j$. Elements $\{s_{12}, s_{23}, s_{34}\}$ generate the abelian group $(\mathbb{Z}/2\mathbb{Z})^3$, the subgroup of the order 8 in $W = W(D_4)$. The Weyl group $W(D_4)$ is the semidirect product of these subgroups:

$$W(D_4) = (\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_4.$$

3.1.2. *The 2-index and 4-index dipoles in $D_4(a_1)$-quadruples.* Let S be a certain set of indices $\{i_1, \dots, i_l\}$ of basis vectors $\{e_{i_1}, \dots e_{i_l}\}$ generating the root system D_l, see (3.1). We denote by $W(S)$ or $W(D_l)$ the group generated by reflections $\{s_{\pm e_k \pm e_n}\}$, where $k, n \in S$. For any pair of 4-index dipoles d_1, d_2 on the same basis vectors $S = \{e_i, e_j, e_k, e_n\}$, we introduce an invariant $\Psi(d_1, d_2)$. Consider the number of different signs in basis vectors entering in d_1 and d_2. This number modulo 2 is denoted by $\Psi(d_1, d_2)$. For example,

$$\begin{array}{lll} d_1 = \{e_i - e_k, \quad -e_j - e_n\}, & d_2 = \{e_i + e_j, \quad e_k + e_n\}, & \Psi(d_1, d_2) = 1, \quad \text{(i)}, \\ d_1 = \{-e_i + e_j, \quad e_k - e_n\}, & d_2 = \{e_i + e_j, \quad e_k + e_n\}, & \Psi(d_1, d_2) = 0, \quad \text{(ii)}. \end{array} \tag{3.4}$$

In case (i) of (3.4), vectors e_k, e_n, e_j enter with different signs into d_1 and d_2. In case (ii), only vectors e_i, e_n have different signs in d_1 and d_2. Then

$$\Psi(d_1, d_2) = 3 \ mod \ 2 = 1 \text{ for case (i)},$$
$$\Psi(d_1, d_2) = 2 \ mod \ 2 = 0 \text{ for case (ii)}.$$

The "evenness" $\Psi(d_1, d_2)$ is invariant under the action of the group $W(D_4)$ since any reflection either changes signs of 2 basis vectors or permutes 2 of them, see §3.1.1.

3.1.3. *Properties of dipoles in D_l.*

Lemma 3.1. *Let us consider dipoles and $D_4(a_1)$-quadruples in D_l.*

(i) *Let d be a certain 2-index (resp. 4-index) dipole. For any $w \in W$, the dipole wd is also a 2-index (resp. 4-index) dipole.*

(ii) *All 2-index dipoles are isomorphic: If $d_1 = \{e_i - e_j, e_i + e_j\}$ and $d_2 = \{e_k - e_n, e_k + e_n\}$ there exists $w \in W$ such that $wd_1 = d_2$.*

(iii) *For the 4-index dipoles on indices $\{i, j, k, n\}$, there exist 2 non-isomorphic classes under the action of the group $W = W(D_4)$ with representatives $d_1 = \{e_i - e_j, e_k - e_n\}$ and $d_2 = \{e_i - e_j, e_k + e_n\}$.*

(iv) *In the root system D_l, any $D_4(a_1)$-quadruple contains a 4-index dipole. It can happen that both dipoles are 4-index.*

(v) *In the root system D_4, dipoles d_1 and d_2 of any $D_4(a_1)$-quadruple are non-isomorphic.*

Proof. (i) It suffices to prove this fact for reflections. Let $d = \{e_k + e_n, e_k - e_n\}$ be a 2-index dipole. By (3.1), for $\alpha = \pm e_i \pm e_j$, where $e_i, e_j \notin \{e_k, e_n\}$, we have $\alpha \perp \{e_k + e_n, e_k - e_n\}$ and s_α acts trivially on d. For $\alpha = \pm(e_k - e_j)$, we have

$$s_\alpha(e_k - e_n) = (e_k - e_n) - (e_k - e_j) = e_j - e_n,$$
$$s_\alpha(e_k + e_n) = (e_k + e_n) - (e_k - e_j) = e_j + e_n.$$

Thus, we get the 2-index dipole $\{e_j + e_n, e_j - e_n\}$, see Fig. 3.12.

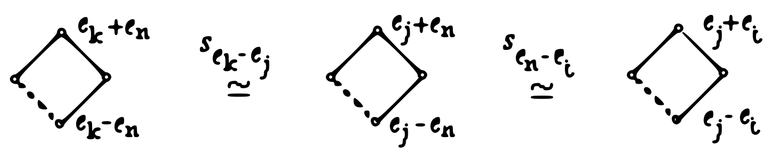

FIG. 3.12. Isomorphism of 2-index dipoles

For $\alpha = \pm(e_k + e_j)$, $s_\alpha d = \{e_n + e_j, e_n - e_j\}$, the 2-index dipole. For $\alpha = \pm(e_n - e_k)$, s_α permutes e_k and e_n. For $\alpha = \pm(e_n + e_k)$, $s_\alpha d = \{-e_n - e_k, e_k - e_n\}$.

It is clear that a 4-index dipole cannot be sent to a 2-index dipole by any $w \in W$, otherwise the inverse element w^{-1} sends a 2-index dipole to a 4-index dipole, contradicting the above.

(ii) The reflection $s_{e_k - e_j}$ sends e_j to e_k, and $s_{e_n - e_i}$ sends e_i to e_n:

$$s_{e_n - e_i} s_{e_k - e_j}\{e_i - e_j, e_i + e_j\} = s_{e_n - e_i}\{e_i - e_k, e_i + e_k\} = \{e_n - e_k, e_n + e_k\},$$

see Fig. 3.12. To change the sign of one of the dipole vertices we use the reflections $s_{e_n \pm e_k}$:

$$s_{e_n - e_k}\{e_n - e_k, e_n + e_k\} = \{e_k - e_n, e_n + e_k\},$$
$$s_{e_n + e_k}\{e_n - e_k, e_n + e_k\} = \{e_n - e_k, -e_n - e_k\}.$$

(iii) In this case, $\Psi(d_1, d_2) = 1$ and it cannot be reduced to zero, see §3.1.2. Any reflection $s_{e_s \pm e_t}$ preserves the "evenness" $\Psi(d_1, d_2)$, see §3.1.1. Thus, there are 2 non-isomorphic classes of dipoles:

$$\text{class } \{d_1\} = \{d \mid \Psi(d, d_1) = 0 \text{ and } \Psi(d, d_2) = 1\},$$
$$\text{class } \{d_2\} = \{d \mid \Psi(d, d_2) = 0 \text{ and } \Psi(d, d_1) = 1\}.$$

(iv) Assume the contrary: $d_1 = \{e_i + e_j, e_i - e_j\}$ and $d_2 = \{e_k + e_n, e_k - e_n\}$ are 2-index dipoles in $D_4(a_1)$-quadruple S. It is clear that every vertex of d_1 and every vertex of d_2 contain exactly one common index: For example, $e_n = e_j$, i.e.,

$$d_1 = \{\alpha_1 = e_i + e_j,\ \alpha_2 = e_i - e_j\}, \quad d_2 = \{\beta_1 = e_k + e_j,\ \beta_2 = e_k - e_j\}.$$

We have $\alpha_1 - \alpha_2 = \beta_1 - \beta_2$ contradicting the linear independence of $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. Therefore, one of dipoles is a 4-index dipole.

(v) By (iv) at least one of dipoles is 4-index. If one of dipoles is 2-index and second is 4-index the statement follows from (i). Suppose, both dipoles are 4-index. Let $\eta[\alpha, \beta]$ be the number of different signs on the common basis vectors of roots α and β. For example,

$$\eta[\alpha, \beta] = \begin{cases} 0 & \text{for} \quad \alpha = e_i \pm e_j, \quad \beta = e_k \pm e_n, \\ 1 & \text{for} \quad \alpha = e_i \pm e_j, \quad \beta = -e_i \pm e_n, \\ 2 & \text{for} \quad \alpha = e_i + e_j, \quad \beta = -e_i - e_j. \end{cases} \tag{3.5}$$

By (3.5)

$$\begin{aligned} (\alpha, \beta) = -1 \quad &\Rightarrow \quad \eta[\alpha, \beta] = 1, \\ (\alpha, \beta) = 1 \quad &\Rightarrow \quad \eta[\alpha, \beta] = 0. \end{aligned} \tag{3.6}$$

Assume $d_1 = \{\delta_1, \delta_2\}$, $d_2 = \{\beta_1, \beta_2\}$ are dipoles of $D_4(a_1)$-quadruple, see Fig. 3.13. Since both dipoles are 4-index, then δ_1 and δ_2 (resp. β_1 and β_2) do not have common summands. We have

$$\Psi(d_1, d_2) = (\ \eta[\delta_1, \beta_1] + \eta[\delta_1, \beta_2] + \eta[\delta_2, \beta_1] + \eta[\delta_2, \beta_2]\ ) \mod 2. \tag{3.7}$$

Since $D_4(a_1)$-quadruple contains 3 solid and 1 dotted edges then by (3.6), (3.7) we have

$$\Psi(d_1, d_2) = (1 + 1 + 1 + 0) \mod 2 = 1, \tag{3.8}$$

i.e., d_1 and d_2 are not isomorphic. □

Corollary 3.2. *Let S_1 and S_2 be $D_4(a_1)$-quadruples in the root system D_l. There exist dipoles $d_1 \in S_1$ and $d_2 \in S_2$, and an element $w \in W$ such that $wd_1 = d_2$.*

Proof. If there exist 2-index dipoles $d_1 \in S_1$ and $d_2 \in S_2$ then, by Lemma 3.1(ii), d_1 and d_2 are isomorphic: $d_1 \simeq d_2$. If S_1 and S_2 contain only 4-index dipoles then, by Lemma 3.1(iii), there exist $d_1 \in S_1$ and $d_2 \in S_2$ which are isomorphic: $d_1 \simeq d_2$. The remaining case: one of $D_4(a_1)$-quadruples contains two 4-index dipoles, while the other contains only one 4-index dipole. Assume that d_1, d'_1 are the 4-index dipoles of S_1, and d_2 is the 4-index dipole in S_2. By Lemma 3.1(v), d_1 and d'_1 are not isomorphic: $d_1 \not\simeq d'_1$. Then either $d_2 \simeq d_1$ or $d_2 \simeq d'_1$. □

Lemma 3.3. *Let Δ be the root system D_l (with $l > 4$) or E_l. Any two 4-index dipoles in Δ are isomorphic.*

Proof. Any 4-index dipole has a form

$$\{p_i e_i + p_j e_j,\ p_k e_k + p_n e_n\},\ \text{where } p_i, p_j, p_k, p_n = \pm 1.$$

Since dipoles are considered up to similarities, any vertex $p_i e_i + p_j e_j$ can be isomorphically mapped to $-p_i e_i - p_j e_j$. Then we can assume that any 4-index dipole has the following form:

$$\{e_i + p_j e_j,\ e_k + p_n e_n\},\ \text{where } p_j, p_n = \pm 1. \tag{3.9}$$

Since $l > 4$, there exists unit vector e_r orthogonal to vectors e_i, e_j, e_k, e_n. For $q \in \{i, j, k, n\}$, the reflection $s_{rq} = s_{e_r - e_q} s_{e_r + e_q}$ changes signs p_r and p_q, see §3.1.1, and preserves 3 remaining coefficients. Therefore, we can assume that, up to isomorphisms, $p_j = p_n = 1$ in (3.9). In other words, up to isomorphisms, we get 3 possible dipoles

$$d_1 = \{e_i + e_j,\ e_k + e_n\}, \quad d_2 = \{e_i + e_k,\ e_j + e_n\}, \quad d_3 = \{e_i + e_n,\ e_j + e_k\}. \tag{3.10}$$

The reflection $s_{e_k - e_j}$ (resp. $s_{e_n - e_j}$) permutes e_k and e_j (resp. e_n and e_j), see §3.1.1, and realizes isomorphism $d_1 \simeq d_2$ (resp. $d_1 \simeq d_3$). Further, by means of reflections $s_{e_r - e_q}$, where $r \in \{i_1, j_1, k_1, n_1\}$ and $q \in \{i_2, j_2, k_2, n_2\}$, we can map the dipole $\{e_{i_1} + e_{j_1},\ e_{k_1} + e_{n_1}\}$ onto another dipole $\{e_{i_2} + e_{j_2},\ e_{k_2} + e_{n_2}\}$. □

Theorem 3.4. (i) *In any root system, all $D_4(a_1)$-quadruples with two 4-index dipoles are isomorphic.*

(ii) *In the root system D_4, there are exactly three pairwise non-isomorphic classes of $D_4(a_1)$-quadruples: one class contains $D_4(a_1)$-quadruples with two 4-index dipoles as in* (i)*, the remaining two classes contain $D_4(a_1)$-quadruples with one 2-index dipole and one 4-index dipole.*

(iii) *In the root system D_l (with $l > 4$), there are exactly two non-isomorphic classes of $D_4(a_1)$-quadruples: one class contains $D_4(a_1)$-quadruples with two 4-index dipoles as in* (i)*, the other class contains one 2-index dipole and one 4-index dipole.*

Proof. Without loss of generality, instead indices $\{i, j, k, n\}$, we can use indices $\{1, 2, 3, 4\}$.

(i) Let $S = \{\beta_1, \beta_2, \delta_1, \delta_2\}$ be a certain $D_4(a_1)$-quadruple with two 4-index dipoles. Since $\{\delta_1, \delta_2\}$ is a 4-index dipole, without loss of generality, we can assume $\delta_2 = e_1 - e_2$, $\delta_1 = e_3 - e_4$. Then β_1 and β_2 should be out of the following

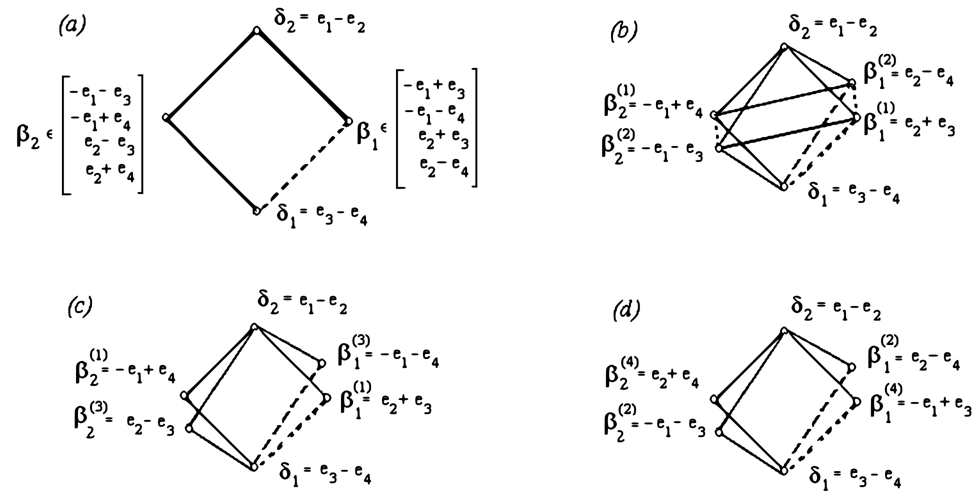

FIG. 3.13. The $D_4(a_1)$-quadruple with two 4-index dipoles

lists:

$$\beta_1 \in \left\{ \begin{array}{c} -e_1 + e_3, \\ -e_1 - e_4, \\ e_2 + e_3, \\ e_2 - e_4, \end{array} \right\}, \qquad \beta_2 \in \left\{ \begin{array}{c} -e_1 - e_3, \\ -e_1 + e_4, \\ e_2 - e_3, \\ e_2 + e_4, \end{array} \right\}, \tag{3.11}$$

see Fig. 3.13(a). Since $\{\beta_1, \beta_2\}$ is also a 4-index dipole, the root β_2 from the list in the right side of (3.13) is uniquely determined. We obtain four possible $D_4(a_1)$-quadruples:

$$\begin{array}{lllll} S_1^{(1)} & = & \{\beta_1^{(1)} = e_2 + e_3, & \beta_2^{(1)} = -e_1 + e_4, & \delta_1 = e_3 - e_4, \quad \delta_2 = e_1 - e_2\}, \\ S_1^{(2)} & = & \{\beta_1^{(2)} = e_2 - e_4, & \beta_2^{(2)} = -e_1 - e_3, & \delta_1 = e_3 - e_4, \quad \delta_2 = e_1 - e_2\}, \\ S_1^{(3)} & = & \{\beta_1^{(3)} = -e_1 - e_4, & \beta_2^{(3)} = e_2 - e_3, & \delta_1 = e_3 - e_4, \quad \delta_2 = e_1 - e_2\}, \\ S_1^{(4)} & = & \{\beta_1^{(4)} = -e_1 + e_3, & \beta_2^{(4)} = e_2 + e_4, & \delta_1 = e_3 - e_4, \quad \delta_2 = e_1 - e_2\}. \end{array} \tag{3.12}$$

In Fig. 3.13(b), (resp. (c), (d))), we depict the pair $\{S_1^{(1)}, S_1^{(2)}\}$ (resp. $\{S_1^{(1)}, S_1^{(3)}\}$, $\{S_1^{(2)}, S_1^{(4)}\}$). The mapping $s_{34}s_{e_3-e_4}$ realizes isomorphism $S_1^{(1)} \simeq S_1^{(2)}$, the mapping $T^{\delta_1}s_{34}$ realizes isomorphisms $S_1^{(1)} \simeq S_1^{(3)}$ and $S_1^{(2)} \simeq S_1^{(4)}$.

(ii) Let $S = \{\varphi_1, \varphi_2, \delta_1, \delta_2\}$ be a certain $D_4(a_1)$-quadruple with the 2-index dipole $\{\delta_1, \delta_2\}$ and the 4-index dipole $\{\varphi_1, \varphi_2\}$. Without loss of generality, we can assume $\delta_1 = e_1 + e_2$, $\delta_2 = e_1 - e_2$. Then e_2 enters with coefficient +1 into φ_1, and e_1 enters with coefficient −1 into φ_2. Therefore, without loss of generality, φ_1 and φ_2 should be out of the following lists:

$$\varphi_1 \in \left\{ \begin{array}{c} e_2 + e_4 \\ e_2 - e_4 \end{array} \right\}, \qquad \varphi_2 \in \left\{ \begin{array}{c} -e_1 - e_3 \\ -e_1 + e_3 \end{array} \right\}, \tag{3.13}$$

see Fig. 3.14(a). Note that transpositions (e_3e_4) or (e_1e_2) in eq. (3.13) do not matter. We have 4 cases for 4-index dipole $\{\varphi_1, \varphi_2\}$, and, therefore, four possible $D_4(a_1)$-quadruples:

$$\begin{array}{lcllll} S_2^{(1)} & = & \{\varphi_1^{(1)} = e_2 + e_4, & \varphi_2^{(1)} = -e_1 + e_3, & \delta_1 = e_1 + e_2, & \delta_2 = e_1 - e_2\}, \\ S_2^{(2)} & = & \{\varphi_1^{(2)} = e_2 - e_4, & \varphi_2^{(2)} = -e_1 - e_3, & \delta_1 = e_1 + e_2, & \delta_2 = e_1 - e_2\}, \\ S_2^{(3)} & = & \{\varphi_1^{(3)} = e_2 - e_4, & \varphi_2^{(3)} = -e_1 + e_3, & \delta_1 = e_1 + e_2, & \delta_2 = e_1 - e_2\}, \\ S_2^{(4)} & = & \{\varphi_1^{(4)} = e_2 + e_4, & \varphi^{(4)} = -e_1 - e_3, & \delta_1 = e_1 + e_2, & \delta_2 = e_1 - e_2\}. \end{array} \tag{3.14}$$

The mapping $s_{34} = s_{e_3-e_4}s_{e_3+e_4}$ from §3.1.1 acts as follows: $e_3 \to -e_3$, $e_4 \to -e_4$. Then s_{34} realizes isomorphisms $S_2^{(1)} \simeq S_2^{(2)}$ and $S_2^{(3)} \simeq S_2^{(4)}$, see Fig. 3.14($b$),($c$). Consider dipoles $\{\varphi_1^{(1)}, \varphi_2^{(1)}\}$ and $\{\varphi_1^{(3)}, \varphi_2^{(3)}\}$ in the pair of $D_4(a_1)$-quadruples $\{S_2^{(1)}, S_2^{(3)}\}$:

$$\begin{array}{lcl} \{\varphi_1^{(1)}, \varphi_2^{(1)}\} & = & \{e_2 + e_4, \ -e_1 + e_3\}, \\ \{\varphi_1^{(3)}, \varphi_2^{(3)}\} & = & \{e_2 - e_4, \ -e_1 + e_3\}, \end{array} \tag{3.15}$$

Here, $\varphi_2^{(3)} = \varphi_2^{(1)}$. The pair of $D_4(a_1)$-quadruples $\{S_2^{(1)}, S_2^{(3)}\}$ has the common triple $\{\delta_1, \varphi_2^{(1)}, \delta_2\}$, see Fig. 3.14($d$). By Lemma 3.1(iii), the dipoles (3.15) are non-isomorphic.

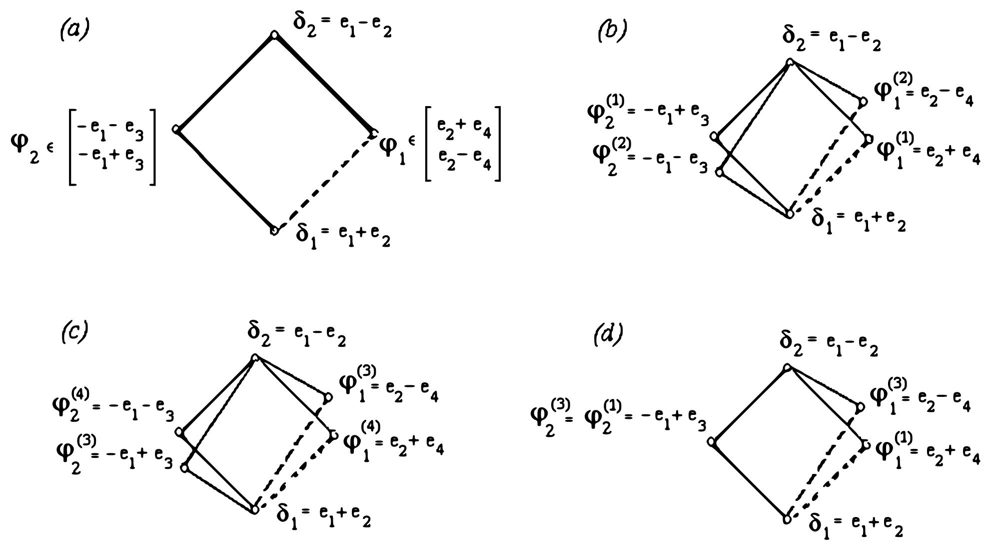

FIG. 3.14. The $D_4(a_1)$-quadruple with one 2-index and one 4-index dipole

(iii) By §3.1.1 any reflection in $W(D_4)$ changes the signs exactly of two basis vectors, so we do not have the element $w \in W(D_4)$ changing only the sign of e_4. However, in the root systems D_l (with $l > 4$) there exists, at least, one additional basis vector e_5. The mapping s_{45} from §3.1.1 maps $e_4 \to -e_4$ and $e_5 \to -e_5$ and preserves signs of basis vectors e_1, e_2, e_3. Thus, s_{45} realizes the isomorphism of dipoles (3.15). The 2 non-isomorphic classes of $D_4(a_1)$-quadruples in D_4 are merged. □

3.2. **The roots and dipoles in E_l.**

3.2.1. *The* 8*-index and* 2-8*-index dipoles in* $D_4(a_1)$*-quadruples.* The roots (3.1), 2-index dipoles (3.2) and 4-index dipoles (3.3) exist also in the root systems E_l with $l = 6, 7, 8$. However, there exist additional roots in E_l, namely

$$\begin{array}{ll} \text{positive roots:} & \frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{\nu(i)}e_i), \text{ where the sum } \sum_{i=1}^{7}\nu(i) \text{ is even,} \\ \text{negative roots:} & \frac{1}{2}(-e_8 + \sum_{i=1}^{7}(-1)^{\nu(i)}e_i), \text{ where the sum } \sum_{i=1}^{7}\nu(i) \text{ is odd,} \end{array} \quad (3.16)$$

see [1, Table VII].

Set $p_{i_k} = (-1)^{\nu(i_k)}$. The roots in eq. (3.16) are said to be 8*-index roots.* For the 8-index root δ, the sum of $\nu(i_k)$ is denoted as follows:

$$\Theta(\delta) \coloneqq \sum_{k=1}^{7}\nu(i_k). \quad (3.17)$$

If both δ_1 and δ_2 are 8-index roots, we can assume, without loss of generality, that

$$\delta_1 = \frac{1}{2}(\sum_{k=1}^{8} p_{i_k}e_{i_k}), \quad \delta_2 = \frac{1}{2}(\sum_{k=1}^{4} p_{i_k}e_{i_k} - \sum_{k=5}^{8} p_{i_k}e_{i_k}), \text{ where } \quad i_k \in \{1, \dots, 8\}. \quad (3.18)$$

For the same indices i_k in δ_1 and δ_2, the coefficients p_{i_k} coincide. A dipole $\{\delta_1, \delta_2\}$ consisting of two 8-index roots is said to be an 8*-index dipole.* For the case of 8-index dipole $\{\delta_1, \delta_2\}$, we denote sums of $\nu(i_k)$ by $\Theta_i \coloneqq \Theta(\delta_i)$, where $i = 1, 2$. The orthogonality $\delta_1 \perp \delta_2$ is equivalent to the fact that exactly four p_{i_k} have different signs:

$$(\delta_1, \delta_2) = \frac{1}{4}(\sum_{k=1}^{4} e_{i_k}^2 - \sum_{k=5}^{8} e_{i_k}^2) = \frac{1}{4}(4 - 4) = 0.$$

If dipole $\{\delta_1, \delta_2\}$ consists of a 2-index root and an 8-index root, we can assume, without loss of generality, that

$$\delta_1 = \frac{1}{2}(\sum_{\substack{k=1 \\ i_k \neq i,j}}^{8} p_{i_k}e_{i_k} + p_i(e_i - e_j)), \quad \delta_2 = \pm(e_i + e_j), \quad (3.19)$$

or

$$\delta_1 = \frac{1}{2}(\sum_{\substack{k=1 \\ i_k \neq i,j}}^{8} p_{i_k}e_{i_k} + p_i(e_i + e_j)), \quad \delta_2 = \pm(e_i - e_j). \quad (3.20)$$

Such dipoles are said to be 2-8*-index dipoles.* The orthogonality $\delta_1 \perp \delta_2$ is immediately checked.

Lemma 3.5. (i) *The* $D_4(a_1)$*-quadruple containing a* 2*-index dipole cannot contain any* 8*-index or* 2-8*-index dipole.*

(ii) *By one reflection, an* 8*-index dipole can be transformed into some* 4*-index dipole.*

(iii) *By one reflection, a* 2-8*-index dipole can be transformed into some* 2*-index dipole.*

Proof. (i) Let $\{\beta_1, \beta_2\}$ be the 2-index dipole $\{e_i + e_j, e_i - e_j\}$ and $\{\delta_1, \delta_2\}$ be any 8-index or 2-8-index dipole. At least, one of δ_1 or δ_2 is the 8-index root, see (3.16). Suppose, δ_1 is such a root. Then e_i and e_j are summands of δ_1. If e_i and e_j enter with the same sign (resp. different signs) into δ_1 then $(\beta_2, \delta_1) = 0$ (resp. $(\beta_1, \delta_1) = 0$), which is contradiction.

(ii) Let α be as follows:

$$\alpha = \frac{1}{2}(p_{i_1}e_{i_1} - \sum_{k=2}^{3} p_{i_k}e_{i_k} + \sum_{k=4}^{8} p_{i_k}e_{i_k}). \tag{3.21}$$

We have the following inner products:

$$\begin{aligned}
&(\alpha, \delta_1) = \frac{1}{4}(e_{i_1}^2 - \sum_{k=2}^{3} e_{i_k}^2 + \sum_{k=5}^{8} e_{i_k}^2) = \frac{-2+6}{4} = 1, \quad s_\alpha(\delta_1) = \delta_1 - \alpha = p_{i_2}e_{i_2} + p_{i_3}e_{i_3}.\\
&(\alpha, \delta_2) = \frac{1}{4}(e_{i_1}^2 - \sum_{k=2}^{3} e_{i_k}^2 + e_{i_4}^2 - \sum_{k=5}^{8} e_{i_k}^2) = \frac{2-6}{4} = -1, \quad s_\alpha(\delta_2) = \alpha + \delta_2 = p_{i_1}e_{i_1} + p_{i_4}e_{i_4}.
\end{aligned} \tag{3.22}$$

The pair $\{s_\alpha(\delta_1), s_\alpha(\delta_2)\} = \{p_{i_2}e_{i_2} + p_{i_3}e_{i_3},\ p_{i_1}e_{i_1} + p_{i_4}e_{i_4}\}$ constitutes the 4-index dipole.

(iii) For 2-8-index dipole $\{\delta_1, \delta_2\}$ in (3.19) (resp. (3.20)), we take γ (resp. τ) as follows:

$$\gamma = \frac{1}{2}(-\sum_{\substack{k=1\\ i_k \neq i,j}}^{8} p_{i_k}e_{i_k} + p_i(e_i - e_j)), \quad \tau = \frac{1}{2}(-\sum_{\substack{k=1\\ i_k \neq i,j}}^{8} p_{i_k}e_{i_k} + p_i(e_i + e_j)). \tag{3.23}$$

Then,

$$\begin{aligned}
&(\delta_1, \gamma) = \frac{1}{4}(-6+2) = -1, \quad s_\gamma(\delta_1) = \delta_1 + \gamma = p_i(e_i - e_j), \quad s_\gamma(\delta_2) = \delta_2 = \pm(e_i + e_j),\\
&(\delta_1, \tau) = \frac{1}{4}(-6+2) = -1, \quad s_\tau(\delta_1) = \delta_1 + \tau = p_i(e_i + e_j), \quad s_\tau(\delta_2) = \delta_2 = \pm(e_i - e_j).
\end{aligned} \tag{3.24}$$

The pair $\{s_\gamma(\delta_1), s_\gamma(\delta_2)\} = \{p_i(e_i - e_j),\ \pm(e_i + e_j)\}$ (resp. $\{s_\tau(\delta_1), s_\tau(\delta_2)\} = \{p_i(e_i + e_j),\ \pm(e_i - e_j)\}$) constitutes the 2-index dipole. □

Lemma 3.6. *Let S be a $D_4(a_1)$-quadruple with a 4-index dipole $\{\delta_1, \delta_2\}$ and a certain dipole $\{\beta_1, \beta_2\}$.*

(i) *Suppose $\{\beta_1, \beta_2\}$ is an 8-index dipole. Then, there exists the root $\alpha \perp \beta_1, \delta_1, \delta_2$, such that s_α preserves $\{\delta_1, \delta_2\}$ and transforms $\{\beta_1, \beta_2\}$ into 2-8-index dipole $\{\beta_1,\ s_\alpha(\beta_2)\}$, where $s_\alpha(\beta_2)$ is a 2-index root.*

(ii) *Suppose $\{\beta_1, \beta_2\}$ is a 2-8-index dipole, where β_1 is an 8-index root and β_2 is a 2-index root. Then, there exists the root $\gamma \perp \beta_2, \delta_1, \delta_2$, such that s_γ preserves $\{\delta_1, \delta_2\}$ and transforms $\{\beta_1, \beta_2\}$ into 4-index dipole $\{s_\gamma(\beta_1),\ \beta_2\}$, where $s_\gamma(\beta_1)$ is a 2-index root.*

Proof. (i) Without loss of generality, we can assume that the 4-index dipole $\{\delta_1, \delta_2\}$ is as follows:

$$\delta_1 = e_{i_1} + e_{i_2}, \quad \delta_2 = e_{i_3} + e_{i_4}, \quad \text{where} \quad i_1, i_2, i_3, i_4 \in \{1, 2, \dots, 8\}. \tag{3.25}$$

Since β_1 is an 8-index root and $(\beta_1, \delta_1) = 1$, then $\beta_1 = \frac{1}{2}(e_{i_1} + e_{i_2} + \dots)$. Since $(\beta_1, \delta_2) = -1$, then $\beta_1 = \frac{1}{2}(e_{i_1} + e_{i_2} - e_{i_3} - e_{i_4} \dots)$. Without loss of generality,

$$\beta_1 = \frac{1}{2}(e_{i_1} + e_{i_2} - e_{i_3} - e_{i_4} + \sum_{k=5}^{8} p_{i_k} e_{i_k}), \quad \text{where} \quad p_{i_k} = \pm 1, \tag{3.26}$$

and coefficients p_{i_k} are chosen in such a way that $\Theta(\beta_1)$ from (3.17) satisfies to conditions of (3.16).

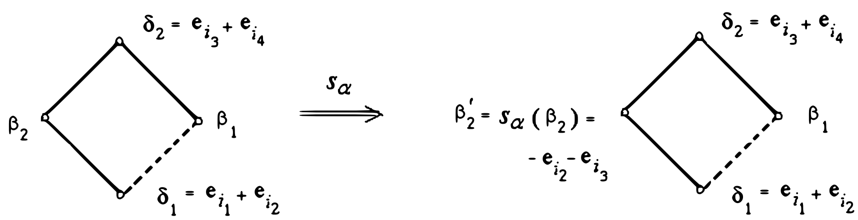

FIG. 3.15. Here, β_1, β_2, are 8-index roots, see (3.26), (3.27); the root α determining the reflection s_α is given by (3.28)

Similarly, since β_2 is an 8-index root and $(\beta_2, \delta_1) = -1$, $(\beta_2, \delta_2) = -1$, we have $\beta_2 = \frac{1}{2}(-e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} \dots)$. At last, since $(\beta_1, \beta_2) = 0$, without loss of generality, the root β_2 is as follows:

$$\beta_2 = \frac{1}{2}(-e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} + \sum_{k=5}^{6} p_{i_k} e_{i_k} - \sum_{k=7}^{8} p_{i_k} e_{i_k}), \tag{3.27}$$

where coefficients p_{i_k} coincide with the same coefficients in (3.26). Since β_2 differs from β_1 in 4 signs, then β_2 also satisfies to conditions of (3.16). Let α be the following root:

$$\alpha = \frac{1}{2}(-e_{i_1} + e_{i_2} + e_{i_3} - e_{i_4} + \sum_{k=5}^{6} p_{i_k} e_{i_k} - \sum_{k=7}^{8} p_{i_k} e_{i_k}), \tag{3.28}$$

where coefficients p_{i_k} coincide with the coefficients in (3.26) and (3.27). We have

$$\alpha \perp \delta_1, \delta_2, \beta_1, \text{ and } (\alpha, \beta_2) = 1,$$

and $s_\alpha(\beta_2) = \beta_2 - \alpha = -e_{i_2} - e_{i_3}$, see Fig. 3.15.

(ii) As in (i), we can assume that the 4-index dipole $\{\delta_1, \delta_2\}$ is given by eq. (3.25) and β_1 is given by eq. (3.26). Since $(\beta_1, \beta_2) = 0$, then the 2-index root β_2 can be chosen from the following 4 roots:

$$-e_{i_1} - e_{i_3}, \quad -e_{i_1} - e_{i_4}, \quad -e_{i_2} - e_{i_3}, \quad -e_{i_2} - e_{i_4}.$$

We suppose that

$$\beta_2 = -e_{i_2} - e_{i_3}. \tag{3.29}$$

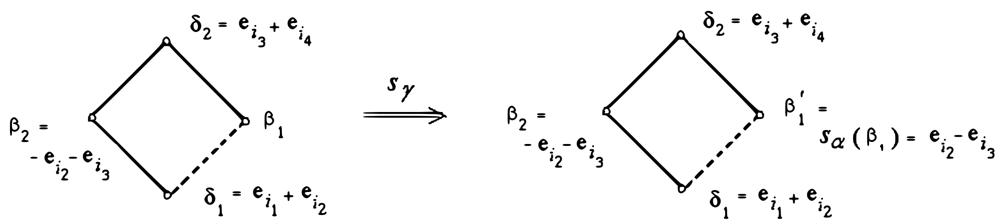

FIG. 3.16. Here, β_1 is the 2-index root, β_2 is the 8-index root, see (3.26), (3.29); the root γ determining the reflection s_γ is given by (3.30)

Let us take the root γ as follows:

$$\gamma = \frac{1}{2}(e_{i_1} - e_{i_2} + e_{i_3} - e_{i_4} + \sum_{k=5}^{8} p_{i_k} e_{i_k}). \tag{3.30}$$

We have

$$\gamma \perp \beta_2, \delta_1, \delta_2, \text{ and } (\beta_1, \gamma) = 1. \tag{3.31}$$

Then $s_\gamma(\beta_1) = \beta_1 - \gamma = e_{i_2} - e_{i_3}$, see Fig. 3.16. □

Corollary 3.7. *Any $D_4(a_1)$-quadruple containing 8-index roots can be isomorphically mapped onto a certain $D_4(a_1)$-quadruple without 8-index roots.*

Proof. Let S be $D_4(a_1)$-quadruple containing 8-index roots.

1) By Lemma 3.5(i) S can contain only 4-index, 8-index or 2-8-index dipoles.

2) Suppose, S contains 4-index dipole. Then, by Lemma 3.6(i),(ii) S can be transformed into a $D_4(a_1)$-quadruple containing one 2-index and one 4-index dipole.

3) Suppose, S contains 2-8-index dipole d. By Lemma 3.5(iii), we can transform $d \in S$ into 2-index dipole in a $D_4(a_1)$-quadruple S'. Then, by Lemma 3.5(i) S' does not contain 8-index roots what was to be shown.

4) Suppose, S contains a 8-index dipole. By Lemma 3.5(i), we can transform this 8-index dipole into a $D_4(a_1)$-quadruple S' with a 4-index dipole. If the second dipole still contain 8-index roots, we apply Lemma 3.6(i),(ii). Then S' can be transformed into a $D_4(a_1)$-quadruple S'' containing one 2-index and one 4-index dipole. □

Theorem 3.8 (On isomorphic dipoles)**.** *Let Δ be the root system D_l or E_l, and $\{S_1, S_2\}$ be a pair of $D_4(a_1)$-quadruples in Δ. There exist dipoles $d_1 \in S_1$ and $d_2 \in S_2$ which are isomorphic.*

Proof. First, any $D_4(a_1)$-quadruple in D_l or E_l contains a 4-index dipole: For D_l, it follows from Lemma 3.1(iv); for E_l, it follows from Corollary 3.7. Further, for root systems $\Delta = D_l$ (with $l > 4$) or $\Delta = E_l$, by Lemma 3.3 the 4-index dipoles in S_1 and S_2 are isomorphic. For $\Delta = D_4$, the statement follows from Corollary 3.2. □

4. Classification of $D_4(a_1)$-configurations

4.1. Reduction in the number of $D_4(a_1)$-configurations.

4.1.1. *$D_4(a_1)$-quadruples S_1 and S_2.* Let S_1, S_2 be two $D_4(a_1)$-quadruples. By Theorem 3.8, there exists the element $T \in W$ such that S_1 and TS_2 have a common dipole. In what follows we shall assume that from the very beginning S_1 and S_2 have a common dipole $\{\delta_1, \delta_2\}$:

$$S_1 = \{\beta_1, \delta_1, \beta_2, \delta_2\}, \quad S_2 = \{\varphi_1, \delta_1, \varphi_2, \delta_2\},$$

and $\{\beta_1, \delta_1\}$, $\{\varphi_1, \delta_1\}$ are the only dotted edges. (except for possible dotted edges connecting S_1 and S_2, see list (1.10)). We consider valid $D_4(a_1)$-configurations $\{S_1, S_2\}$ up to isomorphisms.

4.1.2. *Lemma on reduction.* The rectangle $R = \{\beta_1, \varphi_1 \beta_2, \varphi_2\}$ is crucial in the classification of $D_4(a_1)$-configurations, see Table 1.1. A priori, up to similarities, there are $3^4 = 81$ possible $D_4(a_1)$-configurations: each of four pairs $\{\beta_i, \varphi_j\}$, where $i, j = 1, 2$, can be dotted or solid edge or not forming any edge. The following lemma reduces the number of configurations from 81 to 12.

Lemma 4.1. *Let $D_4(a_1)$-quadruples S_1 and S_2 satisfy the conditions of §4.1.1.*

(i) *Each of two pairs $\{\beta_1, \varphi_1\}$ and $\{\beta_2, \varphi_2\}$ is a dotted edge or does not form any edge, see Table 1.1.*

(ii) *If one of pairs $\{\beta_1, \varphi_2\}$ or $\{\beta_2, \varphi_1\}$ is a dotted or solid edge then pairs $\{\beta_1, \varphi_1\}$ and $\{\beta_2, \varphi_2\}$ constitute dotted edges, see Table 1.1(a),(b),(e)-(j).*

Proof. (i) If $\{\beta_1, \varphi_1\}$ is solid then by Lemma 2.3 we have

$$\varphi_1 + \beta_1 + \delta_2 = 0, \quad \varphi_1 + \beta_1 - \delta_1 = 0.$$

Then $\delta_1 = -\delta_2$ which is contradiction. Similarly, for $\{\beta_2, \varphi_2\}$.

(ii) Consider cases, where $\{\beta_1, \varphi_2\}$ or $\{\beta_2, \varphi_1\}$ is dotted or solid.

$\{\beta_2, \varphi_1\}$ is dotted, see Table 1.1(a),(e),(i). By Lemma 2.3 we have $\varphi_1 = \beta_2 + \delta_1$, So, we get the following equalities:

$$\begin{aligned}
(\varphi_2, \varphi_1) &= (\varphi_2, \beta_2 + \delta_1) = 0 \Rightarrow (\varphi_2, \beta_2) = -(\varphi_2, \delta_1), \\
(\beta_2, \beta_1) &= (\varphi_1 - \delta_1, \beta_1) = 0 \Rightarrow (\varphi_1, \beta_1) = (\delta_1, \beta_1).
\end{aligned}$$

Then $\{\beta_2, \varphi_2\}$ and $\{\beta_1, \varphi_1\}$ are dotted edges.

$\underline{\{\beta_2, \varphi_1\} \text{ is solid}}$, see Table 1.1$(b)$,$(f)$,$(j)$. By Lemma 2.3, $\varphi_1 = -(\beta_2 + \delta_2)$. We get as follows:

$$\begin{aligned}(\varphi_2, \varphi_1) &= -(\varphi_2, \beta_2 + \delta_2) = 0 \Rightarrow (\varphi_2, \beta_2) = -(\varphi_2, \delta_2),\\ (\beta_2, \beta_1) &= -(\varphi_1 + \delta_2, \beta_1) = 0 \Rightarrow (\varphi_1, \beta_1) = -(\delta_2, \beta_1).\end{aligned}$$

Once more, $\{\beta_2, \varphi_2\}$ and $\{\beta_1, \varphi_1\}$ are dotted edges.

$\underline{\{\beta_1, \varphi_2\} \text{ is dotted}}$, see Table 1.1$(a)$,$(j)$,$(g)$. By Lemma 2.3, $\beta_1 = \varphi_2 + \delta_1$, we get:

$$\begin{aligned}(\beta_2, \beta_1) &= (\beta_2, \varphi_2 + \delta_1) = 0 \Rightarrow (\beta_2, \varphi_2) = -(\beta_2, \delta_1),\\ (\varphi_2, \varphi_1) &= (\beta_1 - \delta_1, \varphi_1) = 0 \Rightarrow (\beta_1, \varphi_1) = (\delta_1, \varphi_1).\end{aligned}$$

Again, $\{\beta_2, \varphi_2\}$ and $\{\beta_1, \varphi_1\}$ are dotted edges.

$\underline{\{\beta_1, \varphi_2\} \text{ is solid}}$, see Table 1.1$(b)$,$(i)$,$(h)$. By Lemma 2.3, $\beta_1 = -(\varphi_2 + \delta_2)$, then we get:

$$\begin{aligned}(\beta_2, \beta_1) &= -(\beta_2, \varphi_2 + \delta_2) = 0 \Rightarrow (\beta_2, \varphi_2) = -(\beta_2, \delta_2),\\ (\varphi_2, \varphi_1) &= -(\beta_1 + \delta_2, \varphi_1) = 0 \Rightarrow (\beta_1, \varphi_1) = -(\delta_2, \varphi_1).\end{aligned}$$

The same as above, $\{\beta_2, \varphi_2\}$, $\{\beta_1, \varphi_1\}$ are dotted edges. □

Corollary 4.2. *$D_4(a_1)$-configurations (1)-(4) in Fig. 4.17 are incomplete. They are fixed to configurations (a),(b),(e)-(j) in Table 1.1. Diagrams (a)-(l) in Table 1.1, up to similarities, constitute all possible valid $D_4(a_1)$-configurations.*

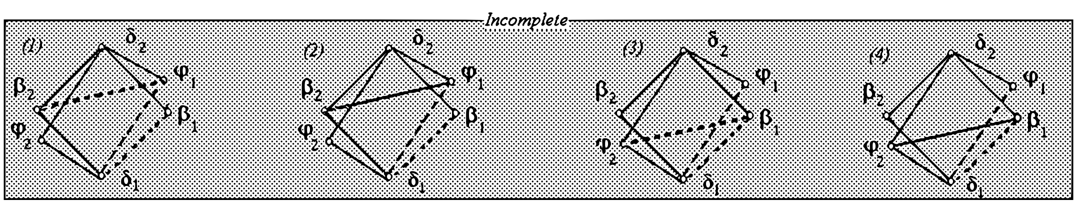

FIG. 4.17. Incomplete $D_4(a_1)$-configurations $\{S_1, S_2\}$ with a common dipole $\{\delta_1, \delta_2\}$

Proof. For example, in (1), we have $\varphi_1 = \delta_1 + \beta_2$. Since $(\varphi_1, \varphi_2) = 0$, we get $(\delta_1, \varphi_2) = -(\beta_2, \varphi_2)$, i.e., $\{\beta_2, \varphi_2\}$ forms the dotted edge. □

4.2. **Three lemmas on $D_4(a_1)$-configurations.**

Lemma 4.3. *Let $D_4(a_1)$-quadruples S_1 and S_2 satisfy the conditions of §4.1.1. For $D_4(a_1)$-configurations $\{S_1, S_2\}$ in Fig. 4.18(a)-(d), the $D_4(a_1)$-quadruples S_1 and S_2 are isomorphic. The isomorphism is realized by means of the element $T \in W$ defined in Table 4.3.*

Proof. (a) By Table 4.3 we take $T = s_{\beta_1 - \varphi_1}$, then

$$T\varphi_1 = \beta_1, \quad T\beta_1 = \varphi_1, \quad T\beta_2 = \beta_2 - \varphi_1 + \beta_1. \tag{4.1}$$

a	$T = s_{\beta_1-\varphi_1}$	
b	$T = s_{\beta_1-\varphi_1}$	
c	$T = T_2T_1$	where $T_1 = s_{\beta_1-\varphi_1}$, $T_2 = s_{\beta_2-\varphi_2}$
d	$T = T_2T_1$	where $T_1 = s_{\beta_1-\delta_1-\beta_2}$, $T_2 = s_{\beta_2+\delta_2+\beta_1}$

TABLE 4.3. The element T isomorphically mapping S_2 on S_1, $T^2 = 1$

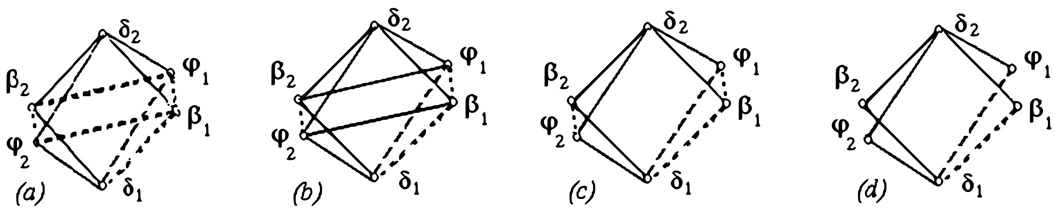

FIG. 4.18. $D_4(a_1)$-configurations $\{S_1, S_2\}$, where S_1 and S_2 are isomorphic $D_4(a_1)$-quadruples

From the rectangle R, by Lemma 2.5(g), we have $\beta_2 - \varphi_1 = \varphi_2 - \beta_1$. Since $(\varphi_2, \beta_1 - \varphi_1) = 1$ we have

$$T\varphi_2 = \varphi_2 - 2\frac{(\varphi_2, \beta_1 - \varphi_1)}{(\beta_1 - \varphi_1, \beta_1 - \varphi_1)}(\beta_1 - \varphi_1) = \varphi_2 - \beta_1 + \varphi_1 = \beta_2.$$

We have $T\varphi_i = \beta_i$ for $i = 1, 2$. In addition, T preserves δ_i, where $i = 1, 2$, Thus, $TS_2 = S_1$. By (4.1) we have also $T\beta_i = \varphi_i$ for $i = 1, 2$, i.e., $TS_1 = S_2$ as well.

(b) As above, taking $T = s_{\beta_1-\varphi_1}$, we obtain

$$T\varphi_1 = \beta_1, \quad T\beta_1 = \varphi_1, \quad T\beta_2 = \beta_2 + \varphi_1 - \beta_1. \tag{4.2}$$

By Lemma 2.5(f), we have $\beta_2 + \varphi_1 = \beta_1 + \varphi_2$. Since $(\varphi_2, \beta_1 - \varphi_1) = -1$, we get

$$T\varphi_2 = \varphi_2 - 2\frac{(\varphi_2, \beta_1 - \varphi_1)}{(\beta_1 - \varphi_1, \beta_1 - \varphi_1)}(\beta_1 - \varphi_1) = \varphi_2 + \beta_1 - \varphi_1 = \beta_2.$$

Thus, $T\varphi_i = \beta_i$ for $i = 1, 2$. As above, in case (a), we have $TS_2 = S_1$ and $TS_1 = S_2$.

(c) Let us take $T_1 = s_{\beta_1-\varphi_1}$, $T_2 = s_{\beta_2-\varphi_2}$. The reflection T_1 (resp. T_2) permutes β_1 and φ_1 (resp. β_2 and φ_2) and preservers all remaining roots in Fig. 4.18(c). Then, $T = T_2T_1$ realizes the isomorphism of S_2 onto S_1. In addition, T_1 and T_2 commute, i.e., $TS_1 = S_2$ and $TS_2 = S_1$.

(d) By Lemma 2.5(c),(d) we have, $\varphi_1 = \delta_1 - \beta_1 - \delta_2$ and $\varphi_2 = -(\delta_2 + \delta_1 + \beta_2)$. Let us take $T_1 = s_{\beta_1-\delta_1-\beta_2}$, $T_2 = s_{\beta_2+\delta_2+\beta_1}$ and $T = T_2T_1$. We will show that $TS_2 = S_1$. First of all, we have

$$T_1 : \begin{cases} \delta_1 \to \delta_1, \\ \delta_2 \to \delta_2, \\ \beta_1 \to \delta_1 + \beta_2, \\ \beta_2 \to \beta_1 - \delta_1, \end{cases} \qquad T_2 : \begin{cases} \delta_1 \to \delta_1, \\ \delta_2 \to \delta_2, \\ \beta_1 \to -(\beta_2 + \delta_2), \\ \beta_2 \to -(\beta_1 + \delta_2), \end{cases} \tag{4.3}$$

Further, by (4.3) we get:

$$\begin{aligned} T_1 S_2 &= T_1\{\varphi_1, \delta_1, \varphi_2, \delta_2\} = T_1\{\delta_1 - \beta_1 - \delta_2,\ \delta_1,\ -(\delta_2 + \delta_1 + \beta_2),\ \delta_2\} = \\ &\{\delta_1 - (\delta_1 + \beta_2) - \delta_2,\ \delta_1,\ -(\delta_2 + \delta_1 + (\beta_1 - \delta_1),\ \delta_2\} = \\ &\{-\beta_2 - \delta_2,\ \delta_1,\ -\delta_2 - \beta_1,\ \delta_2\},\ \text{and} \end{aligned}$$

$$\begin{aligned} T_2 T_1 S_2 &= T_2\{-\beta_2 - \delta_2,\ \delta_1,\ -\delta_2 - \beta_1,\ \delta_2\} = \\ &\{(\beta_1 + \delta_2) - \delta_2,\ \delta_1,\ -\delta_2 + (\beta_2 + \delta_2),\ \delta_2\} = \{\beta_1,\ \delta_1,\ \beta_2,\ \delta_2\} = S_1. \end{aligned}$$

Since $(\beta_2 + \delta_2 + \beta_1, \beta_1 - \delta_1 - \beta_2) = 0$ reflections T_1 and T_2 commute. Then we have also $T_2 T_1 S_1 = S_2$. □

Lemma 4.4. *Let $D_4(a_1)$-quadruples S_1 and S_2 satisfy the conditions of §4.1.1.*

(i) *For $D_4(a_1)$-configurations $\{S_1, S_2\}$ in Fig. 4.19(e)-(h), we use reflection $T_1 = s_{\varphi_1 - \beta_1}$. Then, we obtain the basic $D_4(a_1)$-configuration $\{S_1, T_1 S_2\}$ with the common triple of roots $\{\beta_1, \delta_1, \delta_2\}$ and the dotted edge $\{\beta_2, \varphi'_2\}$, where $\varphi'_2 = T_1\varphi_2$, see Fig. 4.19(Ψ).*

(ii) *For $D_4(a_1)$-configuration $\{S_1, S_2\}$ in Fig. 4.19(e)-(h), the mapping $T = T_2 T_1$ realizes the isomorphism S_2 onto S_1:*

$$S_1 = T S_2, \ \textit{where}\ T_1 = s_{\varphi_1 - \beta_1},\ T_2 = s_{\beta_2 - T_1\varphi_2}.$$

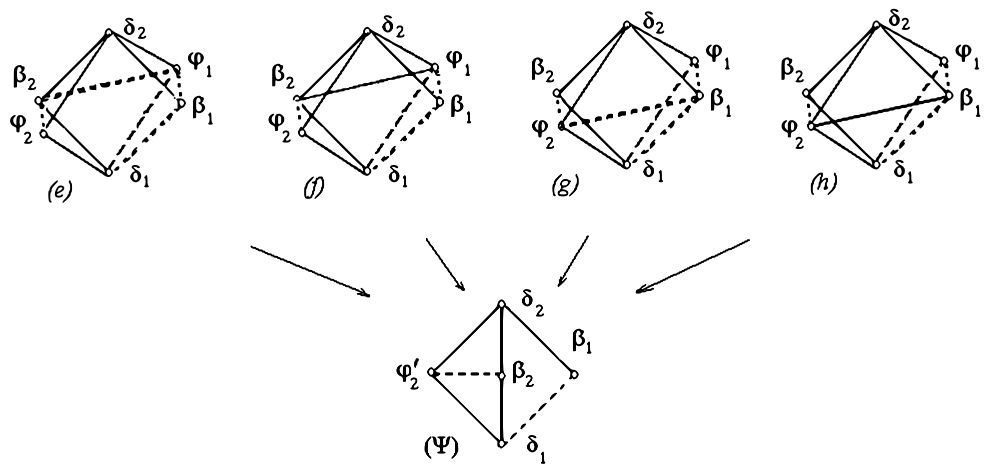

FIG. 4.19. More $D_4(a_1)$-configurations $\{S_1, S_2\}$ consisting of two isomorphic $D_4(a_1)$-quadruples

Proof. (i) In all cases, $T_1\delta_i = \delta_i$ for $i = 1, 2$ and $T_1\varphi_1 = \beta_1$. We need to determine the root $\varphi'_2 = T_1\varphi_2$.

(e), (f) Since $(\varphi_2, \varphi_1 - \beta_1) = 0$, in both cases, we have $\varphi'_2 = \varphi_2$, and φ'_2 is connected to β_2 by dotted edge, see Fig. 4.19(Ψ).

(g) Since φ_2 is connected to β_1, we have $\varphi'_2 = \varphi_2 - \beta_1 + \varphi_1$, then

$$\begin{aligned}
&(\varphi'_2, \delta_i) = (\varphi_2, \delta_i) \text{ for } i = 1, 2,\\
&(\varphi'_2, \beta_1) = (\varphi_2, \beta_1) - (\beta_1, \beta_1) + (\varphi_1, \beta_1) = 1 - 2 + 1 = 0,\\
&(\varphi'_2, \beta_2) = (\varphi_2, \beta_2) + (\varphi_1, \beta_2) = 1 + 0 = 1,
\end{aligned} \tag{4.4}$$

i.e., $\{\varphi'_2, \beta_2\}$ is the dotted edge, see Fig. 4.19(Ψ).

(h) Here, $\varphi'_2 = \varphi_2 + \beta_1 - \varphi_1$. In eq. (4.4)

$$(\varphi'_2, \beta_1) = (\beta_1, \beta_1) + (\varphi_2, \beta_1) - (\varphi_1, \beta_1) = 2 - 1 - 1 = 0,$$

the remaining relations are not changed. Again, $\{\varphi'_2, \beta_2\}$ is the dotted edge as shown in Fig. 4.19(Ψ).

(ii) The reflection $T_2 = s_{\beta_2 - \varphi'_2}$, where $\varphi'_2 = T\varphi_2$ is the corrective map for every $D_4(a_1)$-configuration in Fig. 4.19(Ψ), see Corollary 2.10. Then, T_2 preserves δ_1, δ_2, β_1 in Fig. 4.19(Ψ), and T_2 sends φ'_2 into β_2. □

Lemma 4.5. *Let $D_4(a_1)$-quadruples S_1 and S_2 satisfy the conditions of §4.1.1.*

(i) *Let*

$$T = \begin{cases} s_{\varphi_1 - \beta_1} & \textit{for Fig. } 4.20(i), (j), (l),\\ \rho^{\beta_1\beta_2} T^{\delta_1} s_{\varphi_2 - \beta_2} & \textit{for Fig. } 4.20(k). \end{cases} \tag{4.5}$$

We get $\{S_1, TS_2\}$, the $D_4(a_1)$-configuration with a common triple $\{\beta_1, \delta_1, \delta_2\}$, see Table 1.1($\Theta$).

(ii) *Consider $D_4(a_1)$-configurations $\{S_1, S_2\}$ in Fig. 4.20*(Θ)*, where*

$$S_1 = \{\beta_1, \delta_1, \beta_2, \delta_2\}, \quad S_2 = \{\beta_1, \delta_1, \varphi'_2, \delta_2\},$$

There can be at most one $D_4(a_1)$-quadruple S_2 not isomorphic to any given $D_4(a_1)$-quadruple S_1. An example of two non-isomorphic $D_4(a_1)$-quadruples is shown in Fig. 4.21(1).

Proof. For all cases, set $\varphi'_2 := T\varphi_2$.

(i) We have $T\varphi_1 = \beta_1$. Further, $\varphi'_2 = \varphi_2 + \beta_1 - \varphi_1$, and

$$\begin{aligned}
&(\varphi'_2, \delta_i) = (\varphi_2, \delta_i) \text{ for } i = 1, 2,\\
&(\varphi'_2, \beta_1) = (\beta_1, \beta_1) + (\varphi_2, \beta_1) - (\varphi_1, \beta_1) = 2 - 1 - 1 = 0,\\
&(\varphi'_2, \beta_2) = (\varphi_2, \beta_2) - (\varphi_1, \beta_2) = 1 - 1 = 0.
\end{aligned}$$

Thus, φ'_2 is not connected with β_1 and β_2, the $D_4(a_1)$-configuration is mapped to the configuration in Table 1.1(Θ).

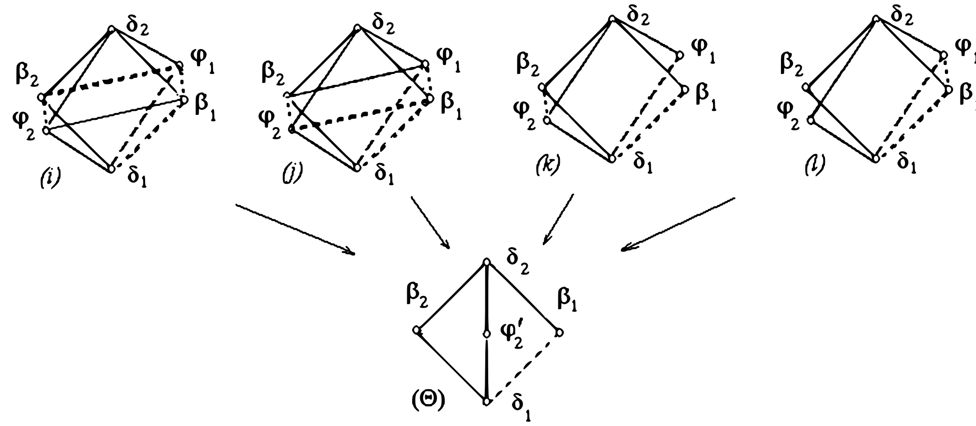

FIG. 4.20. $D_4(a_1)$-configurations with not necessarily isomorphic $D_4(a_1)$-quadruples $\{S_1, S_2\}$

(j) Here, $\varphi_2' = \varphi_2 - \beta_1 + \varphi_1$, and

$$(\varphi_2', \delta_i) = (\varphi_2, \delta_i) \text{ for } i = 1, 2,$$
$$(\varphi_2', \beta_1) = -(\beta_1, \beta_1) + (\varphi_2, \beta_1) + (\varphi_1, \beta_1) = -2 + 1 + 1 = 0,$$
$$(\varphi_2', \beta_2) = (\varphi_2, \beta_2) + (\varphi_1, \beta_2) = -1 + 1 = 0.$$

Again, $\varphi_2' \perp \beta_1$ and $\varphi_2' \perp \beta_2$, we get the $D_4(a_1)$-configuration in Table 1.1(Θ).

(k), (l) In case (k) (resp. (l)), the reflection $s_{\varphi_2-\beta_2}$ (resp. $s_{\varphi_1-\beta_1}$) is the corrective map, see Corollary 2.10. This reflection maps φ_2 to β_2 (resp. φ_1 to β_1) and preserves δ_1, δ_2 and β_1 (resp. β_2). In case (l), we get the $D_4(a_1)$-configuration in Table 1.1(Θ). In case (k), after $s_{\varphi_2-\beta_2}$ we apply T^{δ_1} to get only one dotted edge, after that we apply the rotation $\rho^{\beta_1\beta_2}$ to get the dotted edge in the lower right position as in Table 1.1(Θ).

(ii) By Lemma (2.3) $\varphi_2' = -(\delta_1 + \delta_2 + \beta_2)$, i.e., S_2 is uniquely determined by S_1. In Fig. 4.21(1), S_1 and S_2 are non-isomorphic because the dipole $\{\beta_1, \beta_2\}$ is 2-index, and the dipole $\{\beta_1, \varphi_2'\}$ is 4-index, see Lemma 3.1(i). In Fig. 4.21(2), we have $\varphi_2'' = -(\beta_1 + \beta_2 + \delta_2)$, however S_1 and S_2 are isomorphic in D_l (with $l > 4$) and E_l, since by 4-index dipoles $\{\delta_1, \delta_2\}$ and $\{\delta_1, \varphi_2''\}$ are isomorphic in D_l (with $l > 4$) and E_l, see Lemma 3.3. □

Remark 4.6. Consider the $D_4(a_1)$-configuration in the root system D_4 depicted in Fig. 4.21(1). There exist 2 non-isomorphic $D_4(a_1)$-quadruples: one of them is represented by 2-index dipole $\{\beta_1, \beta_2\}$, the other is represented by 4-index dipole $\{\beta_1, \varphi_2'\}$, see Table 4.4(a),(b). The third $D_4(a_1)$-quadruple non-isomorphic to $D_4(a_1)$-quadruples of Table 4.4(a),(b) is depicted in Fig. 4.21(2) and is presented in Table 4.4(c).

4.3. **Theorem on $D_4(a_1)$-configurations.**

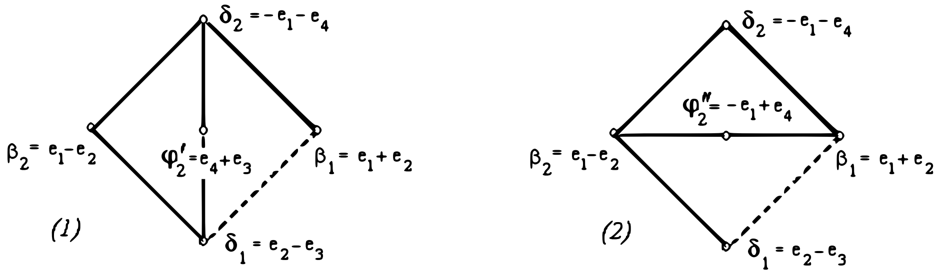

FIG. 4.21. Non-isomorphic $D_4(a_1)$-quadruples $S_1 = \{\beta_1, \delta_1, \beta_2, \delta_2\}$, $S_2 = \{\beta_1, \delta_1, \varphi'_2, \delta_2\}$ and $S_3 = \{\beta_1, \delta_1, \beta_2, \varphi''_2\}$

	Dipole 1	Dipole 2	$D_4(a_1)$-quadruple
a	$\{\beta_1,\ \beta_2\} = \{e_1 + e_2,\ e_1 - e_2\}$	$\{\delta_1,\ \delta_2\} = \{e_2 - e_3,\ -e_1 - e_4\}$	$\{\beta_1,\ \delta_1,\ \beta_2,\ \delta_2\}$
b	$\{\beta_1,\ \varphi'_2\} = \{e_1 + e_2,\ e_4 + e_3\}$	$\{\delta_1,\ \delta_2\} = \{e_2 - e_3,\ -e_1 - e_4\}$	$\{\beta_1,\ \delta_1,\ \varphi'_2,\ \delta_2\}$
c	$\{\beta_1,\ \beta_2\} = \{e_1 + e_2,\ e_1 - e_2\}$	$\{\delta_1,\ \varphi''_2\} = \{e_2 - e_3,\ -e_1 + e_4\}$	$\{\beta_1,\ \delta_1,\ \beta_2,\ \varphi''_2\}$

TABLE 4.4. Three pairwise non-isomorphic quadruples in D_4

4.3.1. *Common dipole.* There are 12 different $D_4(a_1)$-configurations $\{S_1, S_2\}$ such that $D_4(a_1)$-quadruples S_1 and S_2 have a common dipole, see Corollary 4.2. In Theorem 4.7 we show there are only two non-isomorphic classes of $D_4(a_1)$-quadruples having a common dipole. However, all $D_4(a_1)$-quadruples are conjugate.

Theorem 4.7 (On $D_4(a_1)$-configurations)**.** (i) *For $D_4(a_1)$-configurations in Table 1.1(a)-(h), the $D_4(a_1)$-quadruples S_1 and S_2 are isomorphic.*

(ii) *$D_4(a_1)$-configurations in Table 1.1(i)-(l) are isomorphically mapped to Table 1.1(Θ). In all these cases $D_4(a_1)$-quadruples S_1 and S_2 are not necessarily isomorphic.*

(iii) *Let $\{S_1, S_2\}$ be any $D_4(a_1)$-configuration in Table 1.1, where*

$$\begin{cases} S_1 = \{\beta_1, \delta_1, \beta_2, \delta_2\}, \\ S_2 = \{\varphi_1, \delta_1, \varphi_2, \delta_2\ \}, \end{cases} \quad \textit{resp.} \quad \begin{cases} w_1 = s_{\beta_1} s_{\beta_2} s_{\delta_1} s_{\delta_2}, \\ w_2 = s_{\varphi_1} s_{\varphi_2} s_{\delta_1} s_{\delta_2}. \end{cases} \tag{4.6}$$

For any case in Table 1.1, the S_1-associated element w_1 and S_2-associated element w_2 are conjugate.

Proof. (i) Follows from Lemmas 4.3 and 4.4.

(ii) Follows from Lemma 4.5.

(iii) For cases (a)-(h) in Table 1.1, the statement holds because the isomorphism $S_1 = TS_2$ leads the conjugacy $w_1 = T^{-1} w_2 T$. For cases (i)-(l) in Table 1.1, by (ii) we can consider the case (Θ) in Table 1.1, where

$$\begin{cases} w_1 = s_{\beta_1} s_{\beta_2} s_{\delta_1} s_{\delta_2}, \\ w_2 = s_{\beta_1} s_{\varphi'_2} s_{\delta_1} s_{\delta_2}. \end{cases}$$

By Lemma 2.5, we have $\varphi'_2 = -(\beta_2 + \delta_1 + \delta_2)$. Then

$$w_2 = s_{\beta_1} s_{\varphi'_2} s_{\delta_1} s_{\delta_2} = s_{\beta_1} s_{\beta_2+\delta_1+\delta_2} s_{\delta_1} s_{\delta_2} = s_{\beta_1} s_{\delta_1} s_{\delta_2} s_{\beta_2}.$$

Taking $T = s_{\beta_2}$, we get $T^{-1} w_2 T = w_1$. □

4.3.2. *Common triple.* We show that all $D_4(a_1)$-configurations with a common triple are also reduced to the $D_4(a_1)$-configuration with isomorphic $D_4(a_1)$-quadruples or to the case Table 1.1(Θ). Then, all further considerations are similar to Theorem 4.7.

Proposition 4.8. *Let $\{S_1, S_2\}$ be an $D_4(a_1)$-configuration with a common triple $\{\delta_1, \beta_2, \delta_2\}$, see* (4.7). *Suppose w_i, where $i = 1, 2$, is the S_i-associated element:*

$$\begin{cases} S_1 = \{\beta_1, \delta_1, \beta_2, \delta_2\}, \\ S_2 = \{\varphi_1, \delta_1, \beta_2, \delta_2\ \}, \end{cases} \quad \textit{resp.} \quad \begin{cases} w_1 = s_{\beta_1} s_{\beta_2} s_{\delta_1} s_{\delta_2}, \\ w_2 = s_{\varphi_1} s_{\beta_2} s_{\delta_1} s_{\delta_2}, \end{cases} \tag{4.7}$$

(i) *In Fig. 4.22, $D_4(a_1)$-configurations* (1)-(3) *are inconsistent.*

(ii) *Any $D_4(a_1)$-configuration with a common triple $\{\delta_1, \beta_2, \delta_2\}$ containing solid or dotted edge $\{\beta_2, \varphi_1\}$ or solid edge $\{\beta_1, \varphi_1\}$ is inconsistent, see Fig. 4.22.*

(iii) *There are only two valid $D_4(a_1)$-configurations up to similarities, see Fig. 4.22*(*a*),(*b*).

(iv) *For case* (*a*) *in Fig. 4.22, S_1 and S_2 are conjugate, not necessarily isomorphic. For case* (*b*) *in Fig. 4.22, S_1 and S_2 are isomorphic and conjugate.*

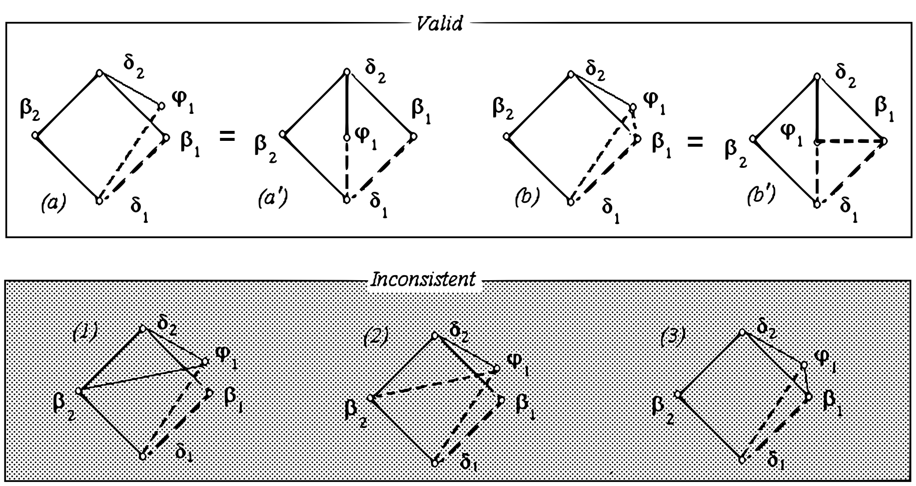

FIG. 4.22. $D_4(a_1)$-configurations with a common triple $\{\delta_1, \beta_2, \delta_2\}$: Valid (a),(b) and inconsistent (1),(2),(3)

Proof. (i) Inconsistency of cases (1)-(3) in Fig. 4.22 follows from the linear dependence of roots in $S_2 = \{\varphi_1, \beta_2, \delta_1, \delta_2\}$. By Lemma 2.3 for case (1), we have

$\beta_2 + \delta_2 + \varphi_1 = 0$. Inconsistency of case (2) follows from the 3-cycle $\{\beta_2, \delta_1, \varphi_1\}$. For case (3), we have $\beta_1 + \varphi_1 + \delta_2 = 0$ and $\beta_1 + \varphi_1 - \delta_1 = 0$, i.e., $\delta_2 = -\delta_1$.

(ii) The inconsistent cases (1)-(3) cannot be corrected by adding any edge.

(iii) By (ii) either $\{\beta_1, \varphi_1\}$ is the dotted edge, or does not form any edge.

(iv) The case (a) follows from Lemma 4.5. For case (b) the reflection $T = s_{\varphi_1 - \beta_1}$ is the corrective mapping which isomorphically maps S_2 to S_1, see Lemma 2.9 and Corollary 2.10. □

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