International Journal of Algebra, Vol. 11, 2017, no. 8, 349 - 356
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2017.7942

Quadrics Defined by Skew-Symmetric Matrices

Joydip Saha[1], **Indranath Sengupta**[1,*] and **Gaurab Tripathi**[2]

[1] Discipline of Mathematics, IIT Gandhinagar
Palaj, Gandhinagar, Gujarat 382355, India
[*] Corresponding author

[2] Department of Mathematics, Jadavpur University
Kolkata, WB 700 032, India

Abstract

In this paper we propose a model for computing a minimal free resolution for ideals of the form $I_1(X_nY_n)$, where X_n is an $n \times n$ skew-symmetric matrix with indeterminate entries x_{ij} and Y_n is a generic column matrix with indeterminate entries y_j. We verify that the model works for $n = 3$ and $n = 4$ and pose some statements as conjectures. Answering the conjectures in affirmative would enable us to compute a minimal free resolution for general n.

Mathematics Subject Classification: Primary 13D02; Secondary 13C40, 13P10, 13D07

Keywords: Gröbner basis, Betti numbers, determinantal ideals, skew-symmetric matrix, mapping cone

1. Introduction

Let K be a field. Let $\{x_{ij};\, 1 \leq i \leq m,\, 1 \leq j \leq n\}$, $\{y_j;\, 1 \leq j \leq n\}$ be indeterminates over K, so that $R = K[x_{ij}, y_j]$ denotes the polynomial algebra over K. Let X_n denote an $n \times n$ skew-symmetric matrix such that its entries are the indeterminates $\pm x_{ij}$ and 0. We call such a matrix a generic skew-symmetric matrix. Let $Y_n = (y_j)_{n\times 1}$ be the generic $n \times 1$ matrix. It is very hard to compute a graded minimal free resolution of the ideal $I_1(X_nY_n)$.

Ideals of the form $I_1(X_nY_n)$ has been studied by [2] and they appear in some of our recent works; see [4], [5], [6], [7]. We described its Gröbner bases, primary decompositions and Betti numbers through computational techniques, mostly under the assumption that X_n is either a generic or a generic symmetric matrix. It is in deed the case that these ideals are far more difficult to understand when X_n is a generic skew-symmetric matrix.

In this paper, we present a scheme for computing a graded minimal free resolution of the ideal $I_1(X_nY_n)$, where X_n is a $n \times n$ generic skew-symmetric matrix and and Y_n is a generic $n \times 1$ matrix. We show that if we assume the truth of two statements then the scheme works for a general n. These two statements which have been proposed as conjectures appear to be correct as seen from symbolic computation using the computer algebra software *Singular* [1]. We finally verify the validity of these conjectures for $n = 3$ and $n = 4$. We refer to [3] for basic knowledge on the techniques used by us.

2. General Scheme and Conjectures

$$\text{Let } X_n = \begin{bmatrix} 0 & x_{12} & x_{13} & \dots & x_{1n} \\ -x_{12} & 0 & x_{23} & \dots & x_{2n} \\ x_{13} & x_{23} & & & \\ \vdots & \vdots & & & \\ -x_{1n} & -x_{2n} & \dots & & 0 \end{bmatrix} \text{ and } Y_n = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Our aim is to find a minimal free resolution of $I_1(X_nY_n)$. Assuming $x_{ij} = -x_{ji}$, if $i > j$ and $x_{ii} = 0$, let $g_{ki} = \Sigma_{j=1}^{i} x_{kj}y_j$. Therefore the generators of $I_1(X_nY_n)$ are $\langle g_{1i}, g_{2i}, \cdots, g_{ni}\rangle$. Let $\Delta_{(i)n}$ denote the Pffafian of the skew symmetric matrix X_n with the i-th row and the i-th column deleted.

Lemma 2.1. *Assuming $x_{ij} = -x_{ji}$, if $i > j$ and $x_{ii} = 0$*

(i) $y_n g_{nn} = -\big(y_1 g_{1n} + y_2 g_{2n} + \cdots + y_{n-1} g_{(n-1)n}\big)$.

(ii) $g_{k(n-1)} g_{nn} = x_{kn} y_1 g_{1n} + x_{kn} y_2 g_{2n} + \cdots + g_{nn} + x_{kn} y_k g_{kn} + \cdots + x_{kn} y_n g_{n-1n}$.

(iii) $\Delta_{(n)n} y_n = (-\Delta_{(1)n}) g_{1n} + (\Delta_{(2)n}) g_{2n} + \cdots + ((-1)^{n-1} \Delta_{(n-1)n}) g_{(n-1)n}$.

Proof. A simple calculation gives the proof. □

Lemma 2.2. $\{g_{1n}, g_{2n}, \cdots, g_{(n-1)n}\}$ *forms a regular sequence for $n \geq 2$.*

Proof. See part (ii) of Theorem 2.2 in [5]. □

Notations.

(i) Let $I_n = \langle g_{1n}, g_{2n}, \cdots, g_{(n-1)n}\rangle$. By Lemma 2.2, the ideal I_n is minimally resolved by the Koszul complex

$$0 \longrightarrow R \longrightarrow \cdots \longrightarrow R^{\binom{n-1}{2}} \xrightarrow{\psi_{2n}} R^{\binom{n-1}{1}} \xrightarrow{\psi_{1n}} R \xrightarrow{\psi_{0n}} R/I_n \longrightarrow 0;$$

where $\psi_{kn} : R^{\binom{n-1}{k}} \longrightarrow R^{\binom{n-1}{k-1}}$ and $k \in \{0, 1, 2, \cdots, n-1\}$.

(ii) Let $J_n = \langle g_{nn}\rangle$ and $L_n = I_n + J_n = I_1(X_nY_n)$.

Computations with Singular give us enough evidence in support of the Conjectures proposed below:

Conjecture 1. $C_n := (I_n : J_n) = \langle g_{1(n-1)}, g_{2(n-2)}, \cdots, g_{n-1(n-1)}, y_n, \Delta_{(n)n} \rangle$. If n is even then $\Delta_{(n)n} = 0$ and $C_n = \langle g_{1(n-1)}, g_{2(n-2)}, \cdots, g_{n-1(n-1)}, y_n \rangle$, for every $n \geq 4$.

Conjecture 2. If n is odd then $\Delta_{(n)n} \neq 0$. For every $n \geq 4$,

$$P_n := (\langle g_{1(n-1)}, g_{2(n-2)}, \cdots, g_{n-1(n-1)} \rangle : \Delta_{(n)n}) = (y_1, \cdots, y_{n-1}).$$

Assuming the validity of these conjectures we can construct a minimal free resolution for $I_1(X_n Y_n)$ through the following steps.

We proceed by induction on $n \geq 3$. We first compute a resolution of L_3, which is not difficult. For $3 \leq i-1 < n$, let a resolution of L_{i-1} be

$$0 \longrightarrow \cdots \longrightarrow R^{\beta_{1(i)}} \xrightarrow{d_{1i}} R^{\beta_{0(i)}} \xrightarrow{d_{0i}} R/L_{i-1} \longrightarrow 0$$

where $d_{0i} : R \longrightarrow R/L_{i-1}$ is the projection map.

A resolution of P_n is the Koszul complex, which is the following:

$$0 \longrightarrow R \longrightarrow \cdots \longrightarrow R^{\binom{n-1}{2}} \xrightarrow{\phi_{2n}} R^{\binom{n-1}{1}} \xrightarrow{\phi_{1n}} R \xrightarrow{\phi_{0n}} R/P_n \longrightarrow 0$$

where $\phi_{kn} : R^{\binom{n-1}{k}} \longrightarrow R^{\binom{n-1}{k-1}}$, for $k \in \{0, 1, 2, \cdots, n-1\}$.

Case 1. For $i < n$ and i is odd, let $T_i := L_{i-1} + \langle \Delta_{(i)i} \rangle$. Using mapping cone we get,

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & R^{\binom{i-1}{3}} & \longrightarrow & R^{\binom{i-1}{2}} & \xrightarrow{\phi_{2i}} & R^{\binom{i-1}{1}} & \xrightarrow{\phi_{1i}} & R & \xrightarrow{\phi_{0i}} & R/P_i & \longrightarrow 0 \\
& \downarrow \delta_{3i} & & \downarrow \delta_{2i} & & \downarrow \delta_{1i} & & \downarrow \delta_{0i}=\Delta_{i(i)} & & \downarrow & \\
\cdots \longrightarrow & R^{\beta_{3(i-1)}} & \longrightarrow & R^{\beta_{2(i-1)}} & \xrightarrow[d_2 i-2]{} & R^{\beta_{1(i-1)}} & \xrightarrow[d_1 i-1]{} & R & \xrightarrow[d_{0i}]{} & R/L_{i-1} & \longrightarrow 0
\end{array}$$

Therefore a resolution of T_i is

$$\longrightarrow R^{\binom{i-1}{1}} \oplus R^{\beta_{2(i-1)}} \xrightarrow{\delta'_{2i}} R \oplus R^{\beta_{1(i-1)}} \xrightarrow{\delta'_{1i}} R \longrightarrow R/T_i \longrightarrow 0;$$

where $\delta'_{ki} : R^{\binom{i-1}{k-1}} \oplus R^{\beta_{k(k_1-1)}} \longrightarrow R^{\binom{i-1}{k-2}} \oplus R^{\beta_{k(k_1-1)}}$ and $\delta'_{ki} = \begin{bmatrix} -\phi_{k-1i} & 0 \\ \delta_{k-1i} & d_{ki-1} \end{bmatrix}$.

The resolution of T_i obtained above may not be minimal. Assuming that we can extract a minimal free resolution from it by identifying matching of degrees and cancelling them (see the computations for some special values of n in the next section) let $\mathbb{T}_i$ be the minimal free resolution of T_i, whose differentials are δ_{ki}, i.e.

$$\mathbb{T}_i \cdots \longrightarrow R^{\gamma_{2i}} \xrightarrow{\delta_{2i}} R^{\gamma_{1i}} \xrightarrow{\delta_{1i}} R \longrightarrow R/T_i \longrightarrow 0$$

To find the resolution of C_i, we need to tensor $0 \longrightarrow R \xrightarrow{y_i} R \longrightarrow 0$ with the complex $\mathbb{T}_i$, which gives us
$\cdots \longrightarrow (R^{\gamma_{3i}} \otimes R) \oplus (R^{\gamma_{2i}} \otimes R) \xrightarrow{\eta_{3i}} (R^{\gamma_{2i}} \otimes R) \oplus (R^{\gamma_{1i}} \otimes R) \xrightarrow{\eta_{2i}} (R^{\gamma_{1i}} \otimes R) \oplus (R \otimes R) \xrightarrow{\eta_{1i}} R \otimes R \longrightarrow R/C_i \longrightarrow 0;$
where $\eta_{ki} = \begin{bmatrix} \delta_{ki} & -y_i I \\ 0 & \delta_{k-1i} \end{bmatrix}$. (*)

We first rewrite complex(*), which gives us a minimal free resolution of C_i. Then, we construct the mapping cone of the following complexes with respect to the following connecting maps:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & R^{(\gamma_{i3}+\gamma_{i2})} & \longrightarrow & R^{(\gamma_{i2}+\gamma_{i1})} & \xrightarrow{\eta_{2i}} & R^{(\gamma_{i1}+1)} & \xrightarrow{\eta_{1i}} & R & \xrightarrow{\eta_{0i}} & R/C_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \xi_{3i}} & & \downarrow{\scriptstyle \xi_{2i}} & & \downarrow{\scriptstyle \xi_{1i}} & & \downarrow{\scriptstyle \xi_{0i}=g_{(i)i}} & & \downarrow & & \\
\cdots & \longrightarrow & R^{\binom{i-1}{3}} & \longrightarrow & R^{\binom{i-1}{2}} & \xrightarrow[\psi_{2(i)}]{} & R^{\binom{i-1}{1}} & \xrightarrow[\psi_{1(i)}]{} & R & \xrightarrow[\psi_{0i}]{} & R/I_i & \longrightarrow & 0
\end{array}$$

We create a minimal resolution out after the mapping cone construction by suitable cancellation of matched degrees.

Case 2. Let i be even and $i < n$. Then, $\Delta_{(i)i} = 0$. Therefore the ideal $T_i = L_{i-1} + \langle \Delta_{(i)i} \rangle = L_{i-1}$. We proceed in a similar way as Case 1.

3. Computation for $n = 3$

Let $X_3 = \begin{pmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & x_{23} \\ -x_{13} & -x_{23} & 0 \end{pmatrix}$ and $Y_3 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. We write

$$I_1(X_3 Y_3) = \langle g_{13}, g_{23}, g_{33} \rangle,$$

such that

$$\begin{aligned} g_{13} &= x_{12}y_2 + x_{13}y_3 \\ g_{23} &= -x_{12}y_1 + x_{23}y_3 \\ g_{33} &= -x_{13}y_1 - x_{23}y_2 \end{aligned}$$

and $I_3 = \langle g_{13}, g_{23} \rangle$; $J_3 = \langle g_{33} \rangle$.

We claim that $(I_3 : J_3) = \langle x_{12}, y_3 \rangle$. We first compute a Gröbner basis of I_3. Let us fix the lexicographic monomial order induced by the ordering among the variables $y_1 > y_2 > y_3 > x_{12} > x_{13} > x_{23}$ on R. Then $h = s(g_{13}, g_{23}) = x_{13}y_3y_1 + x_{23}y_3y_2$. We have $\mathrm{Lt}(h) = x_{13}y_3y_1$ and that it is not divisible by $\mathrm{Lt}(g_{13})$ and $\mathrm{Lt}(g_{23})$. We therefore take the enlarged set $\{g_{13}, g_{23}, h\}$. It is clear that $\gcd(\mathrm{LT}(g_{13}), \mathrm{LT}(h)) = 1$, therefore we need to examine only $s(h, g_{23})$. Now $s(h, g_{23}) = x_{13}x_{23}y_3^2 + x_{12}x_{23}y_3y_2 = x_{23}y_3(g_{13}) \longrightarrow 0$; therefore the set $\{g_{13}, g_{23}, h\}$ forms a Gröbner basis of I_3. We observe that $x_{12}g_{33} = x_{13}g_{23} - x_{23}g_{13}$ and $y_3g_{33} = -(y_1g_{13} + y_2g_{23})$. Therefore, $\langle x_{12}, y_3 \rangle \subset (I_3 : J_3)$

Let $pg_{33} \in I_3$, and let r be the remainder term upon division of p by x_{12}, y_3. We know that $\langle x_{12}, y_3\rangle \subset (I_3 : J_3)$. Therefore, $rg_{33} \in I_3$. The set $\{g_{13}, g_{23}, h\}$ is a Gröbner basis for I_3, therefore one of the following must hold: $x_{12}y_2 \mid \mathrm{Lt}(r)(x_{13}y_1)$ or $x_{12}y_1 \mid \mathrm{Lt}(r)(x_{13}y_1)$ or $x_{13}y_1y_3 \mid \mathrm{LT}(r)(x_{13}y_1)$. This gives us $x_{12} \mid \mathrm{Lt}(r)$ or $y_3 \mid \mathrm{Lt}(r)$, which leads to a contradiction if $r \neq 0$. Therefore $r = 0$ and $p \in \langle x_{12}, y_3\rangle$, and hence $(I_3 : J_3) = \langle x_{12}, y_3\rangle$.

Let $L_3 = I_3 + J_3 = \langle g_{13}, g_{23}, g_{33}\rangle$. A minimal free resolution of L_3 is

$$0 \longrightarrow R^2 \xrightarrow{d_{23}} R^3 \xrightarrow{d_{13}} R \xrightarrow{d_{03}} R/L_3 \longrightarrow 0$$

where

$$d_{13} = \begin{pmatrix} g_{13} & g_{23} & g_{33} \end{pmatrix}, d_{23} = \begin{pmatrix} x_{23} & y_1 \\ -x_{13} & y_2 \\ x_{12} & y_3 \end{pmatrix}$$

4. Computation for $n = 4$

$$X_4 = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix} \text{ and } Y_4 = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

By our notation we have,

$$\begin{aligned} g_{14} &= x_{12}y_2 + x_{13}y_3 + x_{14}y_4, \\ g_{24} &= -x_{12}y_1 + x_{23}y_3 + x_{24}y_4, \\ g_{34} &= -x_{13}y_1 - x_{23}y_2 + x_{34}y_4, \\ g_{44} &= -x_{14}y_1 - x_{24}y_2 - x_{34}y_3 \end{aligned}$$

and $I_4 = \langle g_{14}, g_{24}, g_{34}\rangle$, $J_4 = \langle g_{44}\rangle$, $L_4 = I_4 + J_4$,

We claim that, $C_4 = (I_4 : J_4) = \langle g_{13}, g_{23}, g_{33}, y_4\rangle$. We first find a Gröbner basis of I_4. Let us fix the lexicographic monomial order induced by $y_1 > y_2 > y_3 > y_4 > x_{12} > x_{13} > x_{14} > x_{23} > x_{24} > x_{34}$ on R.

Consider the s-polynomials:

$$\begin{aligned} s(g_{14}, g_{24}) &= y_1y_3x_{13} + y_1y_4x_{14} + y_2y_3x_{23} + y_2y_4x_{23} \\ &= -y_3g_{34} + y_1y_4x_{14} + y_2y_4x_{24} + y_3y_4x_{34} \end{aligned}$$

$$\begin{aligned} s(g_{24}, g_{34}) &= y_2x_{12}x_{23} + y_3x_{13}x_{23} - y_4x_{12}x_{34} + y_4x_{13}x_{24} \\ &= x_{23}g_{14} + y_4x_{12}x_{34} - y_4x_{13}x_{24} + y_4x_{14}x_{23} \end{aligned}$$

We have $\gcd(\mathrm{Lt}(g_{14}), \mathrm{Lt}(g_{24}) = 1$, therefore $s(g_{14}, g_{24}) \longrightarrow 0$. Let us take $p_1 = y_1y_4x_{14} + y_2y_4x_{24} + y_3y_4x_{34}$ and $p_2 = y_4x_{12}x_{34} - y_4x_{13}x_{24} + y_4x_{14}x_{23}$ and consider the bigger set $\{g_{14}, g_{24}, g_{34}, p_1, p_2\}$. We now compute

$$p_3 = s(g_{14}, p_2) = y_2y_4x_{13}x_{24} - y_2y_4x_{14}x_{23} + y_3y_4x_{13}x_{34} + y_4^2x_{14}x_{34}.$$

It is evident that $\mathrm{Lt}(p_3)$ is not divisible by any element of the set

$$\{\mathrm{Lt}(g_{14}), \mathrm{Lt}(g_{24}), \mathrm{Lt}(g_{34}), \mathrm{Lt}(p_1), \mathrm{Lt}(p_2)\}.$$

Therefore we add p_3 in the list and get the set $\mathcal{G} = \{g_{14}, g_{24}, g_{34}, p_1, p_2, p_3\}$. It is now straightforward to check that every s polynomial reduces to zero. Hence $\mathcal{G}$ is a a Gröbner basis for the ideal I_4.

We now compute a Gröbner basis for the ideal $\langle g_{13}, g_{23}, g_{33}\rangle$. Consider the s-polynomials,

$$\begin{aligned} s(g_{13}, g_{23}) &= x_{13}y_3y_1 + x_{23}y_3y_2 &= -y_3g_{33} \longrightarrow 0 \\ s(g_{23}, g_{33}) &= -y_2x_{12}x_{23} - y_3x_{13}x_{23} &= -x_{23}g_{13} \longrightarrow 0. \end{aligned}$$

Also, we have $\gcd(\mathrm{Lt}(g_{13}), \mathrm{Lt}(g_{33}) = 1$. Therefore, the set $\{g_{13}, g_{23}, g_{33}\}$ itself is a Gröbner basis. Hence it follows easily that $\{g_{13}, g_{23}, g_{33}, y_4\}$ is a Gröbner basis for the ideal $\langle g_{13}, g_{23}, g_{33}, y_4\rangle$.

Using proposition 2.1 we obtain $\{g_{13}, g_{23}, g_{33}, y_4\} \subset (I_4 : J_4)$, Let $pg_{44} \in I_4$ and assume that r is the remainder upon division of p by $\{g_{14}, g_{24}, g_{34}, p_1, p_2\}$. Suppose that $r \neq 0$. We have $rg_{44} \in I_4$. Moreover, $\mathrm{Lt}(rg_{44}) = \mathrm{Lt}(r)x_{14}y_1$ is divisible by one of the leading terms $\mathrm{Lt}(g_{14}) = x_{12}y_2$, $\mathrm{Lt}(g_{24}) = x_{12}y_1$, $\mathrm{Lt}(g_{34}) = x_{13}y_1$, $\mathrm{Lt}(p_1) = y_1y_4x_{14}$, $\mathrm{Lt}(p_2) = y_4x_{12}x_{34}$, $\mathrm{Lt}(p_3) = y_2y_4x_{13}x_{24}$. If $\mathrm{Lt}(rg_{44})$ is divisible by any one of the leading terms $\mathrm{Lt}(g_{14}) = x_{12}y_2$, $\mathrm{Lt}(p_1) = y_1y_4x_{14}$, $\mathrm{Lt}(p_2) = y_4x_{12}x_{34}$, $\mathrm{Lt}(p_3) = y_2y_4x_{13}x_{24}$, then we get a contradiction. If $\mathrm{Lt}(g_{24}) = x_{12}y_1 \mid \mathrm{LT}(rg_{44})$, then $x_{12} \mid \mathrm{Lt}(r)$. Let $r = x_{12}m + l$. Therefore, $r.g_{44} = (x_{12}m + l)(-x_{14}y_1 - x_{24}y_2 - x_{34}y_3)$ and after division we get

$$q = (-x_{34}x_{12}y_3 - x_{14}x_{23}y_3 + x_{24}x_{13}y_3)m + lg_{44} \in I_4.$$

We have $\mathrm{Lt}(q) = x_{34}x_{12}y_3m$ and it must be divisible by one of the leading terms $\mathrm{Lt}(g_{14}) = x_{12}y_2$, $\mathrm{Lt}(g_{24}) = x_{12}y_1$, $\mathrm{Lt}(g_{34}) = x_{13}y_1$, $\mathrm{Lt}(p_1) = y_1y_4x_{14}$, $\mathrm{Lt}(p_2) = y_4x_{12}x_{34}$, $\mathrm{Lt}(p_3) = y_2y_4x_{13}x_{24}$. This implies that $\mathrm{Lt}(r)$ must be divisible by one of the leading terms $\mathrm{Lt}(g_{14}) = x_{12}y_2$, $\mathrm{Lt}(g_{24}) = x_{12}y_1$, $\mathrm{Lt}(g_{34}) = x_{13}y_1$, $\mathrm{Lt}(p_1) = y_1y_4x_{14}$, $\mathrm{Lt}(p_2) = y_4x_{12}x_{34}$, $\mathrm{Lt}(p_3) = y_2y_4x_{13}x_{24}$, which is a contradiction. Similarly, if $\mathrm{Lt}(g_{34}) = x_{13}y_1 \mid \mathrm{Lt}(rg_{44})$, we get a contradiction. Therefore $r = 0$ and our claim is proved.

To find the resolution of C_4, we take the tensor product of the complexes:

$$0 \longrightarrow R^2 \xrightarrow{d_{23}} R^3 \xrightarrow{d_{13}} R \xrightarrow{d_{03}} R/L_3 \longrightarrow 0$$

and

$$0 \longrightarrow R \xrightarrow{y_4} R \longrightarrow R/y_4R \longrightarrow 0$$

and obtain a resolution of C_4 as

$$0 \longrightarrow R^2 \xrightarrow{\eta_{34}} R^5 \xrightarrow{\eta_{24}} R^4 \xrightarrow{\eta_{14}} R \xrightarrow{\eta_{04}} R/C_4 \longrightarrow 0$$

where

- $\eta_{14} = [d_{13}|y_4] = [g_{13}, g_{23}, g_{33}, y_4]$,

- $\eta_{24} = \left[\begin{array}{c|c} d_{23} & -y_4 I_3 \\ \hline 0 & d_{13} \end{array}\right] = \begin{pmatrix} x_{23} & y_1 & -y_4 & 0 & 0 \\ -x_{13} & y_2 & 0 & -y_4 & 0 \\ x_{12} & y_3 & 0 & 0 & -y_4 \\ 0 & 0 & g_{13} & g_{23} & g_{33} \end{pmatrix}$,
- $\eta_{34} = \begin{pmatrix} y_4 & 0 \\ 0 & y_4 \\ x_{23} & y_1 \\ -x_{13} & y_2 \\ x_{12} & y_3 \end{pmatrix}$.

Using the mapping cone between these complexes we get

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & R^2 & \xrightarrow{\eta 34} & R^5 & \xrightarrow{\eta_{24}} & R^4 & \xrightarrow{\eta_{14}} & R & \xrightarrow{\eta_{04}} & R/C_4 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi 34} & & \downarrow{\scriptstyle \xi_{24}} & & \downarrow{\scriptstyle \xi_{14}} & & \downarrow{\scriptstyle \xi_{04}=g_{44}} & & \downarrow & & \\ 0 & \longrightarrow & R & \xrightarrow[\psi_{34}]{} & R^3 & \xrightarrow[\psi_{24}]{} & R^3 & \xrightarrow[\psi_{14}]{} & R & \xrightarrow[\psi_{04}]{} & R/I_4 & \longrightarrow & 0 \end{array}$$

where

- $\psi_{14} = [g_{14}, g_{24}, g_{34}]$, $\psi_{24} = \begin{pmatrix} g_{24} & 0 & -g_{34} \\ -g_{14} & -g_{34} & 0 \\ 0 & -g_{24} & g_{14} \end{pmatrix}$,
- $\psi_{34} = \begin{bmatrix} g_{34} \\ g_{14} \\ g_{24} \end{bmatrix}$
- $\xi_{04} = [g_{44}]$,
- $\xi_{14} = \begin{pmatrix} g_{44} + x_{14}y_1 & x_{24}y_1 & x_{34}y_1 & -y_1 \\ x_{14}y_2 & g_{44} + x_{24}y_2 & x_{34}y_2 & -y_2 \\ x_{14}y_3 & x_{24}y_3 & g_{44} + x_{34}y_3 & -y_3 \end{pmatrix}$,
- $\xi_{24} = \begin{pmatrix} -x_{34} & 0 & y_2 & -y_1 & 0 \\ -x_{14} & 0 & 0 & y_3 & -y_2 \\ -x_{24} & 0 & -y_3 & 0 & y_1 \end{pmatrix}$,
- $\xi_{24} = \begin{pmatrix} -1 & 0 \end{pmatrix}$.

Therefore a non-minimal resolution of L_4 is

$$0 \longrightarrow R^2 \xrightarrow{\widetilde{d_{44}}} R^6 \xrightarrow{\widetilde{d_{34}}} R^7 \xrightarrow{\widetilde{d_{24}}} R^4 \xrightarrow{\widetilde{d_{14}}} R \xrightarrow{\widetilde{d_{04}}} R/L_4 \longrightarrow 0,$$

where $\widetilde{d_{14}} = [g_{14}, g_{24}, g_{34}, g_{44}]$, $\quad \widetilde{d_{24}} = \left[\begin{array}{c|c} \xi_{14} & \psi_{24} \\ \hline -\eta_{14} & 0 \end{array}\right]$,

$$\widetilde{d_{34}} = \left[\begin{array}{c|c} -\eta_{24} & 0 \\ \hline \xi_{24} & \psi_{34} \end{array}\right], \quad \widetilde{d_{44}} = \left[\begin{array}{c} -\eta_{34} \\ \hline \xi_{34} \end{array}\right].$$

Therefore a minimal free resolution of $L_4 = I_1(X_4Y_4)$ is

$$0 \longrightarrow R \xrightarrow{d_{44}} R^5 \xrightarrow{d_{34}} R^7 \xrightarrow{d_{24}} R^4 \xrightarrow{d_{14}} R \xrightarrow{d_{04}} R/L_4 \longrightarrow 0$$

where $d_{14} = \widetilde{d_{14}}, \quad d_{24} = \widetilde{d_{24}}, \quad d_{34} = \begin{bmatrix} -\eta_{24} \\ \hline \xi_{24} \end{bmatrix}, \quad d_{44} = \begin{bmatrix} 0 \\ y_4 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}.$

Acknowledgements. Joydip Saha is a research associate supported by the research project EMR/2015/000776 sponsored by the SERB, Government of India. Indranath Sengupta is supported by the research project EMR/2015/000776 sponsored by the SERB, Government of India. Gaurab Tripathi thanks CSIR for the Senior Research Fellowship.

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Received: October 12, 2017; Published: November 21, 2017