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The Symmetric Identity on $M_n(E)$ in Characteristic $p > 2$

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Abstract

In this note we obtain one necessary condition for that symmetric polynomial degree m let an polynomial identity for the algebra $M_n(E)$ when $\operatorname{char} K = p > 2$.

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1 Introduction

Verbally prime algebras play a prominent role in the PI theory. Recall that an algebra A is verbally prime if its T-ideal is prime in the class of all T-ideals in the free associative algebra. The vast majority of the known results about verbally prime algebras concern the case when the base field is of characteristic 0. The structure theory of T-ideals developed by Kemer classified the verbally prime algebras over such fields. Furthermore Kemer showed that verbally semiprime T-ideals are finite intersections of verbally prime ones, and finally that if I is a T-ideal then $J^n \subseteq I \subseteq J$ for appropriate positive integer n and verbally semiprime T-ideal J.

All algebras and vector spaces here are over a field K, $\operatorname{char} K = p \neq 2$.

Let V be an vector space over K of countable infinite dimension with basis $e_1, e_2, \dots$. The Grassmann algebra $E(K) = E$ of V is the associative algebra with K-basis consisting of 1 and all products of the form

$$\{e_{i_1} e_{i_2} \dots e_{i_m} \mid i_1 < i_2 < \dots < i_m \ ; \ m = 1, 2, \dots\}$$

and with multiplication induced by $e_i^2 = 0$ and $e_i e_j = -e_j e_i$.

When $\operatorname{char} K = p = 2$, then obviously E is commutative and hence they are not very "interesting" from the PI point of view. Therefore, we restrict our attention of the case $p > 2$.

Let $M_n(E)$ the $n \times n$ matrix algebras over E.

Let $X = \{x_1, x_2, \dots\}$ be a countable infinite set of symbols (variables) no commuting, and let $K\langle X \rangle$ be the algebra free associative algebra with 1, over k. If A is an associative algebra over K, we denote by $T(A)$ the T-ideal of A. The symmetric polynomial of degree m

$$w_m(x_1, \dots, x_m) = \sum_{\sigma \in S_m} x_{\sigma(1)} \dots x_{\sigma(m)} \in K\langle X \rangle.$$

Here S_m is the symmetric group of order m.

The following results can be found in [2] , [1] and [3], respectively.

Theorem 1.1 *Let $P(\mathcal{I})$ be the set of multilinear polynomials of the $\mathcal{I}$ and let K be a field $\operatorname{char} K = p \neq 2$. Them:*

1. $P(T(M_{a,b}(E) \otimes E)) = P(T(M_{a+b}(E)))$;
2. $P(T(M_{a,b}(E) \otimes M_{c,d}(E))) = P(T(M_{ac+bd,ad+bc}(E)))$;
3. $P(T(M_{1,1}(E))) = P(T(E \otimes E))$.

Theorem 1.2 *The algebras $A_{k,k}$ e $M_{k,k}(E) \otimes E$ are PI-equivalents.*

Lemma 1.3 *If the algebra A satisfies the symmetric polynomial w_m then $M_k(A)$ satisfies the symmetric polynomial w_{km}.*

2 The main result

We obtain one necessary condition for that symmetric polynomial w_m let an polynomial identity for the algebra $M_n(E)$. Recall that one theorem debit the Kemer [4] to assure that such m there exists.

Lemma 2.1 *Let* $\mathit{char}\, K = p$. *The symmetric polynomial* w_m *is a polynomial identity of* E *if and only if* $m \geq p$.

Proof. Initially, suppose that $m \geq p$ and let $a_1, a_2, \ldots, a_m$ elements of the base of E. Let's consider two cases:

Case 1: Suppose that at most one of the elements $a_1, a_2, \ldots, a_m$ has odd length. Them,

$$w_m(a_1, \ldots, a_m) = \sum_{\sigma \in S_m} a_{\sigma(1)} \ldots a_{\sigma(m)} = m!\, a_1 \ldots a_m,$$

because base elements of even length are in the center of the E. As $char(K) = p$,

$$w_m(a_1, \ldots, a_p) = m!\, a_1 \ldots a_m = 0$$

Case 2: Suppose that a_i e a_j, $1 \leq i, j \leq m$ has odd length. Them, for every b in the base of the E,

$$a_i b a_j = -a_j b a_i,$$

So,

$$w_m(a_1, \ldots, a_m) = \sum_{\sigma \in S_m} a_{\sigma(1)} \ldots a_{\sigma(m)} = 0.$$

and therefore, $w_m \in T(E)$.

Besides that, if $m < p$,

$$w_m(e_1e_2, e_3e_4, \ldots, e_{2m-1}e_{2m}) = m! e_1e_2e_3 \ldots e_{2m-1}e_{2m} \neq 0.$$

Which implies that $w_m \notin T(E)$ ∎

Theorem 2.2 *Let* w_m *an polynomial identity for the algebra* $M_n(E)$. *Then,* $m \geq 2n + p - 2$.

Proof. First, observe that the case $n = 1$ is art of the Lemma 2.1. Then, we assume that $n > 1$ and that w_m is polynomial identity for the algebra $M_n(E)$.

Write $w_m(x_1, \ldots, x_m) = w_{m-2}(x_1, \ldots, x_{m-2})x_{m-1}x_m + h(x_1, \ldots, x_m)$. We may suppose that $w_{m-2}(x_1, \ldots, x_{m-2}) \notin T(M_n(E))$. We now intend to show that $w_{m-2}(x_1, \ldots, x_{m-2})$ is an polynomial identity for $M_{n-1}(E)$. Since w_{m-2} is multilinear it will suffice to show that $w_{m-2}(a_1E_1, \ldots, a_{m-2}E_{m-2}) = 0$, for every substitution of $a_i \in E$ and E_i matrix units of order $(n-1) \times (n-1)$.

The matrices E_i will also be considered as matrices E_i' of order $n \times n$, by adding n-th row and column of zeros.

Since w_m is polynomial identity for $M_n(E)$, we have:

$$\begin{aligned} 0 &= w_m(a_1E_1', \ldots, a_{m-2}E_{m-2}', e_{ln}, e_{nn}) \\ &= w_{m-2}(a_1E_1', \ldots, a_{m-2}E_{m-2}')e_{ln}e_{nn} + h(a_1E_1', \ldots, a_{m-2}E_{m-2}', e_{ln}, e_{nn}), \end{aligned}$$

where $1 \leq l < n$ is arbitrary, and e_{ij} are matrix units of order $n \times n$.

Since in each monomial of $h(x_1, \ldots, x_m)$ at least one of the variables $x_1, \ldots, x_{m-2}$ appear to the right of x_{m-1} or x_m, it clear that $h(a_1E_1', \ldots, a_{m-2}E_{m-2}', e_{ln}, e_{nn}) = 0$.

Thus we have that $w_{m-2}(a_1E_1', \ldots, a_{m-2}E_{m-2}')e_{ln}e_{nn} = 0$, $\forall 1 \leq l < n$. Therefore $w_{m-2}(a_1E_1, \ldots, a_{m-2}E_{m-2}) = 0$.

Which prove that,

$$w_{m-2}(x_1, \ldots, x_{m-2}) \in T(M_{n-1}(E)).$$

Now we trivially obtain that $w_{m-2(n-1)} \in T(E)$. By Lemma 2.1, we have that,

$$m - 2(n-1) \geq p$$

or

$$m \geq 2n + p - 2$$

which completes the proof of the theorem. ∎

By Lemma 2.1 E satisfies w_p then, by Lemma 1.3, $M_n(E)$ satisfies w_{np}.

Consider the set:

$$S = \{m \in \mathbb{N} \ / \ 2n + p - 2 \leq m < np \ \textit{and} \ w_m \in T(M_n(E))\}$$

With respect to this set we have two open questions:

1. $S \neq \emptyset$?

2. In the case of S is not empty, which is its minimum?

Using the above theorem together whith 1.1 e 1.2, we get the following results:

Corollary 2.3 *Let $a \geq b$. If w_m is a polynomial identity for algebra $M_{a,b}(E)$, them $m \geq 2b + p - 2$.*

Proof. It is knowm that:

$$P(T(M_{a,b}(E))) \subseteq P(T(M_{b,b}(E))) \subseteq P(T(M_b(E) \otimes E)) \subseteq P(T(M_b(E)))$$

By hypothesis, $w_m \in T(M_{a,b}(E))$, and as w_m is multilinear, $w_m \in T(M_b(E))$ and therefore, $m \geq 2b + p - 2$. ∎

Corollary 2.4 *Let $a \geq b$. If w_m is polynomial identity for algebra $A_{a,b}$ them, $m \geq 4b + p - 2$.*

Proof. It is known that:

$$P(T(A_{a,b})) \subseteq P(T(A_{b,b})) = P(T(M_{b,b}(E) \otimes E)) = P(T(M_{2b}(E)))$$

Soon, if $w_m \in T(A_{a,b})$, follow that $w_m \in T(M_{2b}(E))$, and therefore, $m \geq 4b + p - 2$.

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