International Journal of Algebra, Vol. 11, 2017, no. 6, 255 - 263
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2017.7728

Symmetric Properties for Carlitz's Type (h, q)-Twisted Tangent Polynomials Using Twisted (h, q)-Tangent Zeta Function

Cheon Seoung Ryoo

Department of Mathematics,
Hannam University, Daejeon 34430, Korea

Abstract

Our aim in this paper is to obtain special symmetric properties for Carlitz's type twisted (h, q)-tangent polynomials. We are going to find a symmetric identity for Carlitz's type twisted (h, q)-tangent zeta function. From property of the Carlitz's type twisted (h, q)-tangent zeta function, we derive some symmetric properties of Carlitz's type twisted (h, q)-tangent polynomials.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: Carlitz's type twisted (h, q)-tangent polynomials, Carlitz's type twisted (h, q)-tangent zeta function, symmetric identities of Carlitz's type twisted (h, q)-tangent polynomials, symmetric properties of Carlitz's type twisted (h, q)-tangent zeta function

1 Introduction

The area of the Euler, Bernoulli and Genocchi, tangent polynomials have been studied by many authors. Those polynomials possess many interesting properties and are of great importance in pure mathematics, for example, number theory, mathematical analysis and in the calculus of finite differences. Those polynomials also have various applications in other branches of science(see [1-8]). Throughout this paper, we always make use of the following notations:

$\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. Let q be a complex number with $|q| < 1$ and $h \in \mathbb{Z}$. Then we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-4])} .$$

Note that $\lim_{q\to 1}[x]_q = x$ for any x. The tangent polynomials are defined by

$$F(t,x) = \left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}, \quad |2t| < \pi.$$

The tangent numbers, $T_n = T_n(0)$, implies that they are rational numbers(see [3]).

Definition 1.1 *([4]) Let r be a positive integer, and let ω be rth root of unity. The twisted (h,q)-tangent polynomials are defined as*

$$\sum_{n=0}^{\infty} T_{n,q,\omega}^{(h)}(x)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}(-1)^m\omega^m q^{hm} e^{[2m+x]_q t},$$

where we use the technical method's notation by replacing $\left(T_{q,\omega}^{(h)}\right)^n(x)$ by $T_{n,q,\omega}^{(h)}(x)$, symbolically.

In the special case $x = 0$, $T_{n,q,\omega}^{(h)}(0) = T_{n,q,\omega}^{(h)}$ are called the nth twisted (h,q)-tangent numbers. The following elementary properties of the (h,q)- tangent numbers $T_{n,q,\omega}^{(h)}$ and polynomials $T_{n,q,\omega}^{(h)}(x)$ are readily derived form Definition 1.1(see, for details, [4]). We, therefore, choose to omit details involved.

Theorem 1.2 *Let $q \in \mathbb{C}$ with $|q| < 1$ and ω be the rth root of unity. Then we have*

$$T_{n,q,\omega}^{(h)}(x) = 2\sum_{m=0}^{\infty}(-1)^m\omega^m q^{hm}[x+2m]_q^n.$$

Theorem 1.3 *Let $q \in \mathbb{C}$ with $|q| < 1$ and ω be the rth root of unity. Then we have*

$$\begin{aligned} T_{n,q,\omega}^{(h)}(x) &= \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{xl}T_{l,q,\omega}^{(h)} \\ &= \left(q^x T_{q,\omega}^{(h)} + [x]_q\right)^n . \end{aligned}$$

Theorem 1.4 *Let $x, y \in \mathbb{C}$. Then we have*

$$T_{n,q,\omega}^{(h)}(x+y) = \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{xl}T_{l,q,\omega}^{(h)}(y).$$

Theorem 1.5 *(Property of complement) Let n be a positive integer. Then one has*

$$T^{(h)}_{n,q^{-1},\omega^{-1}}(2-x) = (-1)^n \omega q^{n+h} T^{(h)}_{n,q,\omega}(x)$$

Definition 1.6 *([4]) For $s \in \mathbb{C}$ and $Re(s) > 0$, the Hurwitz-type twisted (h,q)-tangent zeta function is defined by*

$$\zeta^{(h)}_{q,\omega}(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^{hn}\omega^n}{[2n+x]_q^s}.$$

Note that $\zeta^{(h)}_{q,\omega}(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $q \to 1$ and $\omega = 1$, then $\zeta^{(h)}_{q,\omega}(s,x) = \zeta_T(s,x)$ which is the Hurwitz tangent zeta functions(see [3]). Relation between $\zeta^{(h)}_{q,\omega}(s,x)$ and $T^{(h)}_{k,q,\omega}(x)$ is given by the following theorem.

Theorem 1.7 *For $k \in \mathbb{N}$, we have*

$$\zeta^{(h)}_{q,\omega}(-k,x) = T^{(h)}_{k,q,\omega}(x).$$

Observe that $\zeta^{(h)}_{q,\omega}(-k,x)$ function interpolates $T^{(h)}_{k,q,\omega}(x)$ numbers at non-negative integers.

2 Symmetric properties about Hurwitz-type twisted (h,q)-tangent zeta function

In this section, we establish some interesting symmetric identities for Carlitz's type twisted (h,q)-tangent polynomials by using Carlitz's type twisted (h,q)-tangent zeta function. Let w_1, w_2 be any positive odd integers. Our main result of symmetry of Carlitz's type twisted (h,q)-tangent zeta function is given the following theorem, which is symmetric in w_1 and w_2.

Theorem 2.1 *Let $s \in \mathbb{C}$ with $Re(s) > 0$. Then we have*

$$[w_1]_q^s \sum_{i=0}^{w_2-1} (-1)^i q^{hw_1 i}\omega^{w_1 i}\zeta^{(h)}_{q^{w_2},\omega^{w_2}}\left(s, w_1 x + \frac{2w_1 i}{w_2}\right)$$
$$= [w_2]_q^s \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j}\omega^{w_2 j}\zeta^{(h)}_{q^{w_1},\omega^{w_1}}\left(s, w_2 x + \frac{2w_2 j}{w_1}\right).$$

Proof. Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. By substitute $w_1x + \frac{2w_1i}{w_2}$ for x in Definition 1.6, replace q by q^{w_2} and replace ω by ω^{w_2}, respectively, we derive

$$\begin{aligned}\zeta^{(h)}_{q^{w_2},\omega^{w_2}}\left(s, w_1x + \frac{2w_1i}{w_2}\right) &= 2\sum_{n=0}^{\infty}\frac{(-1)^n q^{hw_2n}\omega^{w_2n}}{[w_1x + \frac{2w_1i}{w_2} + 2n]^s_{q^{w_2}}} \\ &= 2[w_2]^s_q\sum_{n=0}^{\infty}\frac{(-1)^n q^{hw_2n}\omega^{w_2n}}{[w_1w_2x + 2w_1i + 2w_2n]^s_q}.\end{aligned}$$

Since for any non-negative integer m and odd positive integer w_1, there exist unique non-negative integer r such that $m = w_1r + j$ with $0 \le j \le w_1 - 1$. Hence, this can be written as

$$\begin{aligned}\zeta^{(h)}_{q^{w_2},\omega^{w_2}}\left(s, w_1x + \frac{2w_1i}{w_2}\right) &= 2[w_2]^s_q\sum_{\substack{w_1r+j=0\\0\le j\le w_1-1}}^{\infty}\frac{(-1)^{w_1r+j}q^{hw_2(w_1r+j)}\omega^{w_2(w_1r+j)}}{[2w_2(w_1r+j) + w_1w_2x + 2w_1i]^s_q} \\ &= 2[w_2]^s_q\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}\frac{(-1)^{w_1r+j}q^{hw_2(w_1r+j)}\omega^{w_2(w_1r+j)}}{[w_1w_2(2r+x) + 2w_1i + 2w_2j]^s_q}.\end{aligned}$$

It follows from the above equation that

$$\begin{aligned}&[w_1]^s_q\sum_{i=0}^{w_2-1}(-1)^i q^{hw_1i}\omega^{w_1i}\zeta^{(h)}_{q^{w_2},\omega^{w_2}}\left(s, w_1x + \frac{2w_1i}{w_2}\right) \\ &= 2[w_1]^s_q[w_2]^s_q\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}\frac{(-1)^{r+i+j}q^{h(w_1w_2r+w_1i+w_2j)}\omega^{w_1w_2r+w_1i+w_2j}}{[w_1w_2(2r+x) + 2w_1i + 2w_2j]^s_q}.\end{aligned} \tag{2.1}$$

From the similar method, we can have that

$$\begin{aligned}\zeta^{(h)}_{q^{w_1},\omega^{w_1}}\left(s, w_2x + \frac{2w_2j}{w_1}\right) &= 2\sum_{n=0}^{\infty}\frac{(-1)^n q^{hw_1n}\omega^{w_1n}}{[w_2x + \frac{2w_2j}{w_1} + 2n]^s_{q^{w_1}}} \\ &= 2[w_1]^s_q\sum_{n=0}^{\infty}\frac{(-1)^n q^{hw_1n}\omega^{w_1n}}{[w_1w_2x + 2w_2j + 2w_1n]^s_q}.\end{aligned}$$

After some calculations in the above, we have

$$\begin{aligned}&[w_2]^s_q\sum_{j=0}^{w_1-1}(-1)^j q^{hw_2j}\omega^{w_2j}\zeta^{(h)}_{q^{w_1},\omega^{w_1}}\left(s, w_2x + \frac{2w_2j}{w_1}\right) \\ &= 2[w_1]^s_q[w_2]^s_q\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}\frac{(-1)^{r+i+j}q^{h(w_1w_2r+w_1i+w_2j)}\omega^{w_1w_2r+w_1i+w_2j}}{[w_1w_2(2r+x) + 2w_1i + 2w_2j]^s_q}.\end{aligned} \tag{2.2}$$

Thus, we complete the proof of the theorem from (2.1) and (2.2).

In Theorem 2.1, we get the following formulas for the twisted (h,q)-tangent zeta function.

Corollary 2.2 *Let $w_2 = 1$ in Theorem 2.1. Then we get*

$$\zeta_{q,\omega}^{(h)}(s,x) = [w_1]_q^{-s} \sum_{j=0}^{w_1-1} (-1)^j q^{hj} \omega^j \zeta_{q^{w_1},\omega^{w_1}}^{(h)} \left(s, \frac{x+2j}{w_1}\right).$$

Corollary 2.3 *Let $w_1 = 2, w_2 = 1$ in Theorem 2.1. Then we have*

$$\zeta_{q^2,\omega^2}^{(h)} \left(s, \frac{x}{2}\right) - q^h \omega \zeta_{q^2,\omega^2}^{(h)} \left(s, \frac{x+1}{2}\right) = [2]_q^s \zeta_{q,\omega}^{(h)}(s,x).$$

Theorem 2.4 *Let w_1, w_2 be any odd positive integer. Then for non-negative integers n, one has*

$$\begin{aligned}
&[w_2]_q^n \sum_{i=0}^{w_2-1} (-1)^i q^{hw_1 i} \omega^{w_1 i} T_{n,q^{w_2},\omega^{w_2}}^{(h)} \left(w_1 x + \frac{2w_1 i}{w_2}\right) \\
&= [w_1]_q^n \sum_{i=0}^{w_1-1} (-1)^i q^{hw_2 i} \omega^{w_2 i} T_{n,q^{w_1},\omega^{w_1}}^{(h)} \left(w_2 x + \frac{2w_2 i}{w_1}\right).
\end{aligned}$$

Proof. By substituting $T_{n,q,\omega}^{(h)}(x)$ for $\zeta_{q,\omega}^{(h)}(s,x)$ in Theorem 2.1 and Theorem 1.7, we can derive that

$$\begin{aligned}
&[w_1]_q^{-n} \sum_{i=0}^{w_2-1} (-1)^i q^{hw_1 i} \omega^{w_1 i} \zeta_{q^{w_2},\omega^{w_2}}^{(h)} \left(-n, w_1 x + \frac{2w_1 i}{w_2}\right) \\
&= [w_1]_q^{-n} \sum_{i=0}^{w_2-1} (-1)^i q^{hw_1 i} \omega^{w_1 i} T_{n,q^{w_2},\omega^{w_2}}^{(h)} \left(w_1 x + \frac{2w_1 i}{w_2}\right),
\end{aligned}$$

and

$$\begin{aligned}
&[w_2]_q^{-n} \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j} \omega^{w_2 j} \zeta_{q^{w_1},\omega^{w_1}}^{(h)} \left(-n, w_2 x + \frac{2w_2 j}{w_1}\right) \\
&= [w_2]_q^{-n} \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j} \omega^{w_2 j} T_{n,q^{w_1},\omega^{w_1}}^{(h)} \left(w_2 x + \frac{2w_2 j}{w_1}\right).
\end{aligned}$$

Thus, we can complete the proof of the theorem from Theorem 2.1.

3 Some symmetric identities about twisted (h,q)-tangent polynomials

In this section, we derive the symmetric results by using definition and theorem of twisted (h,q)-tangent polynomials. We obtain another result by applying the addition theorem for the twisted (h,q)-tangent polynomials(Theorem 1.3).

Theorem 3.1 *Let w_1, w_2 be any odd positive integer. Then we have*

$$\begin{aligned}&\sum_{l=0}^{n}\binom{n}{l}[w_1]_q^l[w_2]_q^{n-l}T_{n-l,q^{w_2},\omega^{w_2}}^{(h)}(w_1x)\mathcal{T}_{n,l,q^{w_1},\omega^{w_1}}^{(h)}(w_2)\\&=\sum_{l=0}^{n}\binom{n}{l}[w_2]_q^l[w_1]_q^{n-l}T_{n-l,q^{w_1},\omega^{w_1}}^{(h)}(w_2x)\mathcal{T}_{n,l,q^{w_2},\omega^{w_2}}^{(h)}(w_1),\end{aligned}$$

where $\mathcal{T}_{n,l,q,\omega}^{(h)}(k)=\sum_{i=0}^{k-1}(-1)^iq^{(h+2n-2l)i}\omega^i[2i]_q^l$ is called as the sums of powers.

Theorem 3.2 *Let n,m be any non-negative integers. Then we obtain that*

$$\sum_{k=0}^{n}\binom{n}{k}T_{m+k,q,\omega}^{(h)}(x+y)q^{(n-k)x}[-x]_q^{n-k}=\sum_{k=0}^{m}\binom{m}{k}T_{k+n,q,\omega}^{(h)}(y)q^{(n+k)x}[x]_q^{m-k}.$$

Proof. Observe that

$$[x]_qu+q^x[y+2m]_q(u+v)=[x+y+2m]_q(u+v)-[x]_qv. \tag{3.1}$$

From Definition 1.1, we easily see that

$$2\sum_{m=0}^{\infty}(-1)^mq^{hm}\omega^me^{[x+y+2m]_q(u+v)}=2e^{[x]_q(u+v)}\sum_{m=0}^{\infty}(-1)^mq^{hm}\omega^me^{q^x[y+2m]_q(u+v)}.$$

Since $[x+y]_q=[x]_q+q^x[y]_q$, we can find out

$$\begin{aligned}&e^{-[x]_qv}2\sum_{m=0}^{\infty}(-1)^mq^{hm}\omega^me^{[x+y+2m]_q(u+v)}\\&=e^{[x]_qu}2\sum_{m=0}^{\infty}(-1)^mq^{hm}\omega^me^{q^x[y+2m]_q(u+v)}.\end{aligned} \tag{3.2}$$

Note that

$$\begin{aligned}2\sum_{m=0}^{\infty}(-1)^mq^{hm}\omega^me^{[x+y+2m]_q(u+v)}&=\sum_{n=0}^{\infty}T_{n,q,\omega}^{(h)}(x+y)\frac{(u+v)^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}T_{n+m,q,\omega}^{(h)}(x+y)\frac{u^m}{m!}\frac{v^n}{n!}.\end{aligned}$$

From the above equation, the left-hand side of (3.2) can be expressed as

$$\begin{aligned} & e^{-[x]_q v} 2\sum_{m=0}^{\infty}(-1)^m q^{hm}\omega^m e^{[x+y+2m]_q(u+v)} \\ &= \left(\sum_{n=0}^{\infty}(-[x]_q)^n\frac{v^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}T^{(h)}_{n+m,q,\omega}(x+y)\frac{u^m}{m!}\frac{v^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}q^{(n-k)x}T^{(h)}_{k+m,q,\omega}(x+y)[-x]_q^{n-k}\right)\frac{u^m}{m!}\frac{v^n}{n!}. \end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned} 2\sum_{m=0}^{\infty}(-1)^m q^{hm}\omega^m e^{q^x[y+2m]_q(u+v)} &= \sum_{n=0}^{\infty}q^{nx}T^{(h)}_{n,q,\omega}(y)\frac{(u+v)^n}{n!} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q^{(n+m)x}T^{(h)}_{n+m,q,\omega}(y)\frac{v^n}{n!}\frac{u^m}{m!}. \end{aligned}$$

From the above equation, the right-hand side of (3.2) can be expressed as follows:

$$\begin{aligned} & e^{[x]_q u} 2\sum_{m=0}^{\infty}(-1)^m q^{hm}\omega^m e^{q^x[y+2m]_q(u+v)} \\ &= \left(\sum_{m=0}^{\infty}[x]_q^m\frac{u^m}{m!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q^{(n+m)x}T^{(h)}_{n+m,q,\omega}(y)\frac{v^n}{n!}\frac{u^m}{m!}\right) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}q^{(k+n)x}T^{(h)}_{n+k,q,\omega}(y)[x]_q^{m-k}\right)\frac{v^n}{n!}\frac{u^m}{m!}. \end{aligned} \tag{3.4}$$

By comparing the coefficients of $\frac{v^n u^m}{n!m!}$ in (3.3) and (3.4), we assert that the theorem is right. As another special case, we discover that for any non-negative integers m, n,

$$(-1)^m\sum_{k=0}^{m}\binom{m}{k}q^{n+k}T^{(h)}_{n+k,q,\omega}(x) = (-1)^n\sum_{k=0}^{n}\binom{n}{k}q^{1-m-k-h}\omega^{-1}T^{(h)}_{m+k,q^{-1},\omega^{-1}}(1-x). \tag{3.5}$$

Theorem 3.3 *Let k, n, m be non-negative integers. Then we have*

$$\begin{aligned} & (-1)^m\sum_{k=0}^{m+1}\binom{m+1}{k}(k+n+1)q^{k+n}T^{(h)}_{k+n,q,\omega}(x) \\ & + (-1)^n\sum_{k=0}^{n+1}\binom{n+1}{k}(k+m+1)q^{-(k+m+h-1)}w^{-1}T^{(h)}_{k+m,q^{-1},\omega^{-1}}(1-x) = 0. \end{aligned}$$

Proof. Since $k\binom{m+1}{k} = (m+1)\binom{m}{k-1}$ for any non-negative integers k and m then

$$\begin{aligned}
&(-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k}(k+n+1)q^{k+n}T^{(h)}_{k+n,q,\omega}(x) \\
&= (-1)^m (n+1) \sum_{k=0}^{m+1} \binom{m+1}{k} q^{k+n}T^{(h)}_{k+n,q,\omega}(x) \\
&\quad + (-1)^m (m+1) \sum_{k=0}^{m+1} \binom{m}{k-1} q^{k+n}T^{(h)}_{k+n,q,\omega}(x).
\end{aligned} \tag{3.6}$$

It follows from (3,5) that

$$\begin{aligned}
&(-1)^m (n+1) \sum_{k=0}^{m+1} \binom{m+1}{k} q^{k+n}T^{(h)}_{k+n,q,\omega}(x) \\
&= (-1)^{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} k q^{1-k-m-h}\omega^{-1}T^{(h)}_{k+m,q^{-1},\omega^{-1}}(1-x),
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&(-1)^m (m+1) \sum_{k=0}^{m+1} \binom{m}{k-1} q^{k+n}T^{(h)}_{k+n,q,\omega}(x) \\
&= (-1)^{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} (m+1) q^{1-k-m-h}\omega^{-1}T^{(h)}_{k+m,q^{-1},\omega^{-1}}(1-x),
\end{aligned} \tag{3.8}$$

Thus, putting (3.7) and (3.8) to the right hand side of (3.6) gives the desired result and this completes the proof.

References

[1] J. Y. Kang, C. S. Ryoo, On symmetric property for q-Genocchi polynomials and zeta function, *Int. Journal of Math. Analysis*, **8** (2014), 9-16. https://doi.org/10.12988/ijma.2014.311275

[2] Y. He, Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials, *Advances in Difference Equations*, **2013** (2013), 246. https://doi.org/10.1186/1687-1847-2013-246

[3] C. S. Ryoo, A note on the tangent numbers and polynomials, *Adv. Studies Theor. Phys.*, **7** (2013), 447-454. https://doi.org/10.12988/astp.2013.13042

[4] C. S. Ryoo, Carlitz's type twisted (h, q)-tangent numbers and polynomials, *Applied Mathematical Sciences*, **9** (2015), 1475-1482. https://doi.org/10.12988/ams.2015.5112

[5] C.S. Ryoo, A numerical investigation on the zeros of the tangent polynomials, *J. Appl. Math. & Informatics*, **32** (2014), 315-322. https://doi.org/10.14317/jami.2014.315

[6] C.S. Ryoo, Differential equations associated with tangent numbers, *J. Appl. Math. & Informatics*, **34** (2016), 487-494. https://doi.org/10.14317/jami.2016.487

[7] C.S. Ryoo, A Note on the Zeros of the q-Bernoulli Polynomials, *J. Appl. Math. & Informatics*, **28** (2010), 805-811.

[8] C.S. Ryoo, Reflection Symmetries of the q-Genocchi Polynomials, *J. Appl. Math. & Informatics*, **28** (2010), 1277-1284.

Received: July 17, 2017; Published: August 2, 2017