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On the Symmetric Identities for the Second Kind Generalized q-Euler Polynomials

Cheon Seoung Ryoo

Department of Mathematics
Hannam University, Daejeon 34430, Korea

Abstract

In this paper, we study the symmetry for the second kind generalized q-Euler numbers $E_{n,\chi,q}$ and polynomials $E_{n,\chi,q}(x)$. We obtain some interesting identities of the power odd sums and the second kind generalized polynomials $E_{n,\chi,q}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

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1 Introduction

The area of the Bernoulli, Euler, Genocchi, and tangent polynomials have been studied by many authors. Those polynomials possess many interesting properties and are of great importance in pure mathematics, for example, number theory, mathematical analysis and in the calculus of finite differences. Those polynomials also have various applications in other branches of science(see [1-11]).

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

Ryoo [10] introduced the second kind generalized q-Euler numbers $E_{n,\chi,q}$ and polynomials $E_{n,\chi,q}(x)$ attached to χ. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. The second kind generalized q-Euler numbers $E_{n,\chi,q}$ attached to χ are defined by the generating function:

$$\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}. \tag{1.1}$$

We consider the second kind generalized q-Euler polynomials $E_{n,\chi,q}(x)$ attached to χ as follows:

$$\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,q}(x) \frac{t^n}{n!}. \tag{1.2}$$

We recall that the second kind generalized q-Euler numbers $E_{n,\chi,q}$ and generalized q-Euler polynomials $E_{n,\chi,q}(x)$ are given by Ryoo [10]

$$E_{n,\chi,q}(x) = \int_X \chi(y) q^y (2y+1+x)^n d\mu_{-1}(y) \tag{1.3}$$

and

$$E_{n,\chi,q} = \int_X \chi(y) q^y (2y+1)^n d\mu_{-1}(y) \tag{1.4}$$

respectively.

By (1.3) and (1.4), we have the following theorem.

Theorem 1.1 *For positive integers n, we have*

$$E_{n,\chi,q}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,\chi,q}.$$

2 Symmetry for the second kind generalized q-Euler polynomials

In this section, we assume that $q \in \mathbb{C}_p$. We obtain some interesting identities of the power odd sums and the second kind generalized polynomials $E_{n,\chi,q}(x)$

using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$. If n is odd, we obtain

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^k g(k), \text{ (see [1, 4]).} \tag{2.1}$$

Substituting $g(x) = \chi(x)q^x e^{(2x+1)t}$ into the above, we obtain

$$\begin{aligned}&\int_X \chi(x+n)q^{x+n}e^{(2(x+n)+1)t}d\mu_{-1}(x) + \int_X \chi(x)q^x e^{(2x+1)t}d\mu_{-1}(x)\\ &= 2\sum_{j=0}^{n-1}(-1)^j\chi(j)q^j e^{(2j+1)t}.\end{aligned} \tag{2.2}$$

For $k \in \mathbb{Z}_+$, let us define the power odd sums $T_{k,\chi,q}(n)$ as follows:

$$T_{k,\chi,q}(n) = \sum_{l=0}^{n}(-1)^l\chi(l)q^l(2l+1)^k. \tag{2.3}$$

After some calculations, we have

$$\begin{aligned}&\int_X \chi(x)q^x e^{(2x+1)t}d\mu_{-1}(x) = \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1},\\ &\int_X \chi(x)q^{x+n}e^{(2(x+n)+1)t}d\mu_{-1}(x) = q^n e^{2nt}\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1}.\end{aligned} \tag{2.4}$$

By using (2.4), we have

$$\begin{aligned}&\int_X \chi(x)q^{x+nd}e^{(2(x+nd)+1)t}d\mu_{-1}(x) + \int_X \chi(x)q^x e^{(2x+1)t}d\mu_{-1}(x)\\ &= \left(1+q^{nd}e^{2ndt}\right)\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1}.\end{aligned}$$

From the above, we have

$$\begin{aligned}&\int_X \chi(x)q^{x+nd}e^{(2(x+nd)+1)t}d\mu_{-1}(x) + \int_X \chi(x)q^x e^{(2x+1)t}d\mu_{-1}(x)\\ &= \frac{2\int_X \chi(x)q^x e^{(2x+1)t}d\mu_{-1}(x)}{\int_X q^{ndx}e^{2ndtx}d\mu_{-1}(x)}.\end{aligned} \tag{2.5}$$

By substituting Taylor series of $e^{(2x+1)t}$ into (2.2), we obtain

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\int_X \chi(x)q^{x+nd}(2x+1+2nd)^m d\mu_{-1}(x) + \int_X \chi(x)q^x(2x+1)^m d\mu_{-1}(x)\right)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(2\sum_{j=0}^{nd-1}(-1)^j\chi(j)q^j(2j+1)^m\right)\frac{t^m}{m!}\end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$q^{nd}\sum_{k=0}^{m}\binom{m}{k}(2nd)^{m-k}\int_X \chi(x)q^x(2x+1)^k d\mu_{-1}(x)+\int_X \chi(x)q^x(2x+1)^m d\mu_{-1}(x)$$
$$=2\sum_{j=0}^{nd-1}(-1)^j\chi(j)q^j(2j+1)^m$$

By using (2.3), we have

$$q^{nd}\sum_{k=0}^{m}\binom{m}{k}(2nd)^{m-k}\int_X \chi(x)q^x(2x+1)^k d\mu_{-1}(x)+\int_X \chi(x)q^x(2x+1)^m d\mu_{-1}(x)$$
$$=2T_{m,\chi,q}(nd-1). \tag{2.6}$$

By using (2.5) and (2.6), we arrive at the following theorem:

Theorem 2.1 *Let n be odd positive integer. Then we obtain*

$$\frac{2\int_X \chi(x)q^x e^{(2x+1)t}d\mu_{-1}(x)}{\int_X q^{ndx}e^{2ndtx}d\mu_{-1}(x)}=\sum_{m=0}^{\infty}\left(2T_{m,\chi,q}(nd-1)\right)\frac{t^m}{m!}. \tag{2.7}$$

Let w_1 and w_2 be odd positive integers. Then we set

$$S(w_1,w_2)=$$
$$\frac{\int_X\int_X \chi(x_1)\chi(x_2)q^{(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_X q^{w_1w_2dx}e^{2w_1w_2dxt}d\mu_{-1}(x)}. \tag{2.8}$$

By (2.7) and (2.8), after calculations, we obtain

$$\begin{aligned}S(w_1,w_2)&=\left(\frac{1}{2}\int_X \chi(x_1)q^{w_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_{-1}(x_1)\right)\\&\quad\times\left(\frac{2\int_X\chi(x_2)q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_{-1}(x_2)}{\int_X q^{w_1w_2dx}e^{2w_1w_2dtx}d\mu_{-1}(x)}\right)\\&=\left(\frac{1}{2}\sum_{m=0}^{\infty}E_{m,\chi,q^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}T_{m,\chi,q^{w_2}}(w_1d-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \tag{2.9}$$

By using Cauchy product in the above, we have

$$S(w_1,w_2)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}E_{j,\chi,q^{w_1}}(w_2x)w_1^jT_{m-j,\chi,q^{w_2}}(w_1d-1)w_2^{m-j}\right)\frac{t^m}{m!}. \tag{2.10}$$

From the symmetry of $S(w_1, w_2)$ in w_1 and w_2, we also see that

$$\begin{aligned}S(w_1, w_2) &= \left(\frac{1}{2}\int_X \chi(x_2) q^{w_2 x_2} e^{(w_2(2x_2+1)+w_1 w_2 x)t} d\mu_{-1}(x_2)\right) \\ &\quad\times \left(\frac{2\int_X \chi(x_1) q^{w_1 x_1} e^{(2x_1+1)(w_1 t)} d\mu_{-1}(x_1)}{\int_X q^{w_1 w_2 dx} e^{2w_1 w_2 dtx} d\mu_{-1}(x)}\right) \\ &= \left(\frac{1}{2}\sum_{m=0}^{\infty} E_{m,\chi,q^{w_2}}(w_1 x) w_2^m \frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty} T_{m,\chi,q^{w_1}}(w_2 d - 1) w_1^m \frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$S(w_1, w_2) = \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j} E_{j,\chi,q^{w_2}}(w_1 x) w_2^j T_{m-j,\chi,q^{w_1}}(w_2 d - 1) w_1^{m-j}\right)\frac{t^m}{m!} \tag{2.11}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.10) and (2.11), we arrive at the following theorem:

Theorem 2.2 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j} w_1^{m-j} w_2^j E_{j,\chi,q^{w_2}}(w_1 x) T_{m-j,\chi,q^{w_1}}(w_2 d - 1) \\ &= \sum_{j=0}^{m}\binom{m}{j} w_1^j w_2^{m-j} E_{j,\chi,q^{w_1}}(w_2 x) T_{m-j,\chi,q^{w_2}}(w_1 d - 1),\end{aligned}$$

where $E_{k,\chi,q}(x)$ and $T_{m,\chi,q}(k)$ denote the second kind generalized q-Euler polynomials and the alternating sums of powers of consecutive q-odd integers, respectively.

By Theorem 1.1, we have the following corollary.

Corollary 2.3 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k} w_1^{m-k} w_2^j x^{j-k} E_{k,\chi,q^{w_2}} T_{m-j,\chi,q^{w_1}}(w_2 d - 1) \\ &= \sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k} w_1^j w_2^{m-k} x^{j-k} E_{k,\chi,q^{w_1}} T_{m-j,\chi,q^{w_2}}(w_1 d - 1).\end{aligned}$$

Now we will derive another interesting identities for the second kind generalized q-Euler polynomials using the symmetric property of $S(w_1, w_2)$.

$$
\begin{aligned}
S(w_1,w_2) &= \left(\frac{1}{2}\int_X \chi(x_1) q^{w_1x_1} e^{(w_1(2x_1+1)+w_1w_2x)t} d\mu_{-1}(x_1)\right) \\
&\quad\times\left(\frac{2\int_X \chi(x_2) q^{w_2x_2} e^{(2x_2+1)(w_2t)} d\mu_{-1}(x_2)}{\int_X q^{w_1w_2dx} e^{2w_1w_2dtx} d\mu_{-1}(x)}\right) \\
&= \left(\frac{1}{2} e^{w_1w_2xt}\int_X \chi(x_1) q^{w_1x_1} e^{(2x_1+1)w_1t} d\mu_{-1}(x_1)\right) \\
&\quad\times\left(2\sum_{j=0}^{w_1d-1}(-1)^j\chi(j) q^{w_2j} e^{(2j+1)(w_2t)}\right) \\
&= \sum_{j=0}^{w_1d-1}(-1)^j\chi(j) q^{w_2j}\int_X \chi(x_1) q^{w_1x_1} e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)} d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1d-1}(-1)^j\chi(j) q^{w_2j} E_{n,\chi,q^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right) w_1^n\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.12}
$$

By using the symmetry property in (2.12), we also obtain

$$
\begin{aligned}
S(w_1,w_2) &= \left(\frac{1}{2} e^{w_1w_2xt}\int_X \chi(x_2) q^{w_2x_2} e^{(2x_2+1)w_2t} d\mu_{-1}(x_2)\right) \\
&\quad\times\left(\frac{2\int_X \chi(x_1) q^{w_1x_1} e^{(2x_1+1)(w_1t)} d\mu_{-1}(x_1)}{\int_X q^{w_1w_2dx} e^{2w_1w_2dtx} d\mu_{-1}(x)}\right) \\
&= \left(\frac{1}{2} e^{w_1w_2xt}\int_X \chi(x_2) q^{w_2x_2} e^{(2x_2+1)w_2t} d\mu_{-1}(x_2)\right) \\
&\quad\times\left(2\sum_{j=0}^{w_2d-1}(-1)^j\chi(j) q^{w_1j} e^{(2j+1)(w_1t)}\right) \\
&= \sum_{j=0}^{w_2d-1}(-1)^j\chi(j) q^{w_1j}\int_X \chi(x_2) q^{w_2x_2} e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)} d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j\chi(j) q^{w_1j} E_{n,\chi,q^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right) w_2^n\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.13}
$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.12) and (2.13), we have the following theorem.

Theorem 2.4 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)q^{w_2j}E_{n,\chi,q^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^n\\&=\sum_{j=0}^{w_2d-1}(-1)^j\chi(j)q^{w_1j}E_{n,\chi,q^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^n.\end{aligned}\tag{2.14}$$

Observe that if $q\to 1$, then (2.14) reduces to Theorem 2.4 in [5].

If we take $x=0$ in Theorem 2.4, we also derive the interesting identity for the second kind generalized q-Euler numbers as follows:

$$\begin{aligned}&\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)q^{w_2j}E_{n,\chi,q^{w_1}}\left((2j+1)\frac{w_2}{w_1}\right)w_1^n\\&=\sum_{j=0}^{w_2d-1}(-1)^j\chi(j)q^{w_1j}E_{n,\chi,q^{w_2}}\left((2j+1)\frac{w_1}{w_2}\right)w_2^n.\end{aligned}$$

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