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An Algebraic Proof of the Fundamental Theorem of Algebra

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Dedicated to Lic. Jessica Páez, my kinesiologist, by her unconditional support.

Abstract

We prove the fundamental theorem of algebra (FTA on brief) by using linear algebra. The proof which arises from a new equivalent reformulation of FTA also works for any infinite field, having root for all polynomials of degree 2, and thus, sufficient condition for that field being algebraically closed.

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1. Introduction

Many proofs of FTA have been proposed from the 18th century until today, e. g., [3], [4], among the most recent ones. The proof presented in this paper only uses linear algebra together with some basic concepts of general algebra or affine geometry (see, e.g., [7],[5], or [2],[6]), being therefore appropriate for undergraduate students.

Let us now introduce the necessary notation. We represent by $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of positive integers, rationals, real and complex numbers, respectively. We denote by $\mathbb{C}[x]$ the ring of polynomials and by $\mathcal{P}_n(\mathbb{C})$ the space of all polynomials of degree at most $n \in \mathbb{Z}^+$, with coefficients in $\mathbb{C}$. The degree of

$p(x) \in \mathbb{C}[x]$ is denoted by $\deg(p(x))$ and the linear hull of a set $A \subset \mathcal{P}_n(\mathbb{C})$ by $\mathrm{span}(A)$, and by $\mathrm{aff}(A)$, the affine hull of A. By $\pi p(x)$ we denote the projection of $p(x) \in \mathcal{P}_n(\mathbb{C})$ on some coordinate subspace, depending on context. Linearly independent, on brief, l.i. and, without lost of generality, by wlog.

We will work always with finite dimensional linear spaces (a com-parison may be see in, Finite Versus Infinite Dimensions, [1]). In particular, $\mathcal{P}_n(\mathbb{C})$, equipped with the ordered basis $< x, x^2, ...x^n, 1 >$.

Definition 1.1. For any $a \in \mathbb{C}$, let $A_a^n := \{p(x) \in \mathcal{P}_n(\mathbb{C}) \mid p(a) = 0\}$.

For all $a \in \mathbb{C}$, $A_a^n \subset \mathcal{P}_n(\mathbb{C})$, is a subspace, n-dimensional (an hyperplane). We denote by $\mathcal{A}_n := \bigcup_{a \in \mathbb{C}} A_a^n$.
We describe the family $\mathcal{A}_n$ on more useful way. First off

$$B_w^n := \{(1/w^{m-1})x^m - w \,|\, 1 \leq m \leq n\} \subset A_w^n \,, \tag{1.1}$$

since these n vectors in B_w^n are l.i., it is clear that, $\mathrm{span}(B_w^n) = A_w^n$.

The keynote is the following equivalent reformulation of FTA, which puts the problem on the area of affine geometry.

Theorem 1.2 (FTA). *For any $n \in \mathbb{Z}^+$, $\mathcal{P}_n(\mathbb{C}) = span(1) \cup \mathcal{A}_n$.*

Lemma 1.3. *On an infinite field F, $r(x) \in F[x]$ of $\deg(r(x)) \geq 2$, $r(x) = \sum_{m=0}^n r_m\, x^m$, is transformed by a variable change, for some $a \in F$, on $s(y) = \sum_{m=0}^n s_m\, y^m \in F[y]$, for that both $s_1 \neq 0$ and $s_2 \neq 0$ by the change $x = y + a$.*

Proof. (i) The two interesting, coefficient, after the change, remain

$$s_1 = \sum{}_{m=1}^{n} m r_m a^{m-1} \quad , \quad s_2 = \sum{}_{m=2}^{n} \binom{m}{2} r_m a^{m-2} \,.$$

We can see $s_1 = (r_1, 2r_2, \dots, nr_n) \bullet (1, a, a^2, \dots, a^{n-1})$ as a bilinear product, $F^n \times F^n \to F$. Analogously for s_2, but on $(n-1)$ dimension. Taking a suit of n distinct values for a, the n resulting vectors are l.i. ($n \times n$ Vandermonde determinant), and thus, can not be always $s_1 = 0$, being $r(x) \neq 0$. Let α_1^1 be the value of a for which $s_1 \neq 0$. Now we choice another suit of n values for a, distinct between them and also distinct of the previous, and let α_1^2 the value that produces the inequality. Repeating the procedure $(n-1)$ times, we obtain $(n-1)$ distinct values, $\{\alpha_1^m \,|\, m = 1, \dots, n-1\}$, whichever results $s_1 \neq 0$. Now, using these $(n-1)$ values for the product on s_2, for at least one of them, $s_2 \neq 0$ (again by Vandermonde determinant, but now $(n-1) \times (n-1)$). Therefore, there is $a \in F$ such that both, $s_1 \neq 0$ and $s_2 \neq 0$ simultaneously. ☐

2. The fundamental theorem

Theorem 2.1. *Each $r(x) \in \mathbb{C}[x]$, $\deg(r(x)) \geq 1$, has one root in $\mathbb{C}$.*

Proof. For $n = 1$ it is trivial and for $n = 2$ it is well known. Suppose $r(x)$, with $\deg(r(x)) = n \geq 3$, without any root in $\mathbb{C}$ and $r_j \neq 0$, for $j = 0, 1, 2, n$. (wlog. by lemma 1.3). Let be, $R := \mathrm{span}(r(x)) \subset \mathcal{P}_n(\mathbb{C})$, it is clear that all polynomials in R does not have any root in $\mathbb{C}$, neither.

Because $r(x) \notin \operatorname{span}(\{x^m \mid 1 \le m \le n\})$ (since $r_0 \neq 0$), for each $z \in (\mathbb{C}\backslash\{0\})$, there is a unique $q_z \in R \cap (\operatorname{span}(\{x^m \mid 1 \le m \le n\} + z)$.

Moreover, $q_z \notin (\operatorname{span}(\{x^m \mid 0 \le m \le n, m \neq 1\}) + 1)$, except for one z, that **we denote** by $\zeta \in \mathbb{C} \setminus \{0\}$, because the linear 1-dimensional subspace R cross once this affine subspace (since $r_1 \neq 0$).

On the projection of the slices $\pi(\operatorname{span}(\{x^m \mid 1 \le m \le n\}) + z)$, over the coordinates hyperplane $\operatorname{span}(\{x^m \mid 1 \le m \le n\}$, we observe, besides πq_z, the following projections of points from B^n_w (see (1.1)), the point x and a set of points in the coordinates axes, in particular, covering whole the $(\operatorname{span}((x^2) \setminus \{0\})$ axis, because the coefficient function of x^2, $1/w$ is onto over $\mathbb{C} \setminus \{0\}$. Now projecting again, over the $(\operatorname{span}(x, x^2))$ plane, for each $z \in (\mathbb{C} \setminus \{0, \zeta\})$ there is a unique $p\,x^2$ $(p \neq 0)$ (into this plane), which corresponds to the $w \in (\mathbb{C} \setminus \{0\})$, such that $1/w = p$, all that because $r_1 \neq 0$, $r_2 \neq 0$ and $z \neq \zeta$. Thus, $p\,x^2 \in \operatorname{aff}(x, \pi q_z(x))$, also, $\pi q_z(x) \in \operatorname{aff}(\{(1/w^{m-1}x^m \mid 1 \le m \le n\})$ at 0 level. We want to work on the same level of either, q_z and $((x^m/w^{m-1}) - w)$. For that, it should be

$$-w = z \quad . \tag{2.1}$$

By the following lemma 2.2, there is a that $\overline{z}$, then $\pi q_{\overline{z}} \in \operatorname{aff}(x + \overline{z}, px^2 + \overline{z}) \subset A_{-\overline{z}}$. In general, if $\pi R \not\parallel \operatorname{aff}(p^{2j-1}x^{2j}, p^{2j-2}x^{2j-1})$, there are $z_j \in \mathbb{C}$ for $j = 1, \ldots, n/2$, $(z_1 = \overline{z})$ such that

$$\begin{aligned} &\pi q_{z_j} \in \operatorname{aff}(p^{2j-1}x^{2j} + z_j, p^{2j-2}x^{2j-1} + z_j) \subset A_{-z_j} \quad \Rightarrow \\ &({}^j s)(w_j) = 0 \quad \wedge \quad w_j = 1/p^{2j-1} \quad \wedge \quad z_j = -w_j = \overline{z}/p^{2j-2}\,. \end{aligned} \tag{2.2}$$

The other case, $\pi R \parallel \operatorname{aff}(p^{2j-1}x^{2j}, p^{2j-2}x^{2j-1})$. That is, $r_{2j}x^{2j} + r_{2j-1}x^{2j-1} \parallel \operatorname{aff}(p^{2j-1}x^{2j}, p^{2j-2}x^{2j-1})$, which implies, $(r_{2j} = \alpha p^{2j-1} \wedge r_{2j-1} = -\alpha p^{2j-2})$ then $r_{2j} = -pr_{2j-1}$. But we can supposse, wlog. (by a variable change $x = ay$), $r_{2j} \neq -pr_{2j-1}$ for $j = 1, \ldots, n/2$. Therefore, all equations on (2.2) are true.

On other hand the polynomial $({}^j s)(x)$ made with the pair of coefficients $(2j, 2j-1)$ of q_{z_j}, verifies $({}^j s)(x) + 2z_j \in A_{-z_j}$, also $\pi({}^j s) = \pi(q_{z_j})$ for these two coefficients. Therefore,

$$\begin{aligned} ({}^j s)(w_k) + 2z_k &= (z_j/r_0)(r_{2j}(w_k)^{2j} + r_{2j-1}(w_k)^{2j-1}) + 2z_k \\ &= (z_j/r_0)(r_{2j}(1/p)^{2j(2k-1)} + r_{2j-1}(1/p)^{(2j-1)(2k-1)}) + 2(1/p)^{2k-2}\overline{z} \\ &= (1/p)^{2k-1}((z_j/r_0)(r_{2j}(1/p)^{2j} + r_{2j-1}(1/p)^{(2j-1)}) + 2p\overline{z}) \\ &= (1/p)^{2k-1}(({}^j s)(1/p) + 2p\overline{z}) \\ &= (1/p)^{2k-1}(({}^j s)(w_1) + 2pz_1)\,, \end{aligned}$$

in particular, if $j = k$, for $j = 1, \ldots, n/2$,

$$({}^j s)(w_j) + 2z_j = (1/p)^{2j-1}(({}^j s)(w_1) + 2pz_1)\,. \tag{2.3}$$

Moreover,

$$\sum_{j=1}^{n/2}((^j s)(x)+2z_j)=\sum_{j=1}^{n/2}((z_j/r_0)(r_{2j}x^{2j}+r_{2j-1}x^{2j-1})+2z_j)$$
$$=(\overline{z}/p^2r_0)\sum_{j=1}^{n/2}p^{2j}(r_{2j}x^{2j}+r_{2j-1}x^{2j-1})+2\sum_{j=1}^{n/2}z_j\,.$$

Let be $z* := 2p\sum_{j=1}^{n/2}z_j$. If we modify $r(x)$ by the change $x = py$, the coefficients of $a(y) := (z*/pr_0)r(py)$, remain $a_m = z*p^{m-1}r_m/r_0$, for $m = 1,\ldots,n.$, $a_0 = z*/p$, thus

$$a(y)=\sum_{j=1}^{n/2}z*(r_{2j}/r_0)y^{2j}+(r_{2j-1}/r_0)y^{2j-1})\ +z*/p$$
$$=\sum_{j=1}^{n/2}(z*(1/z_j)(^j s)(y)+2z_j)\,.$$

Therefore, by using also (2.3),

$$a(w_k)=\sum_{j=1}^{n/2}(z*/z_j)(^j s)(w_k)+2z_j)$$
$$=\sum_{j=1}^{n/2}(z*/z_j)(1/p)^{2j-1}((^j s)(w_1)+2pz_1)\,. \tag{2.4}$$

This is a contradiction, because replacing $r(x)$ by $\alpha r(x)$, for any α such that $0\neq\alpha\neq1$, all terms on (2.4) stay equal, whereas $\alpha r(pw_k)\neq r(pw_k)$, otherwise $r(pw_k)=0$, in contradicion as well. □

Lemma 2.2 (relationship $w\sim z$)**.** *Let be $f:\mathbb{C}\setminus\{0,\zeta\}\to\mathbb{C}$ the function on the TFA proof, such that, $w=f(z)$. If all polynomials of degree 2 have a root in the field, then there exists $\overline{z}$, such that, $f(\overline{z})=-\overline{z}$, and also, $0\neq(\overline{z})\neq\zeta$.*

Proof. First of we expres $q_z(x)$ on coordinates, $R=\{\lambda(\sum_{m=0}^{n}r_mx^m)\}$, $\lambda\in\mathbb{C}$, then $q_z(x)=\sum_{m=0}^{n}\lambda r_mx^m$, and, $q_z(x)=\sum_{m=0}^{n}(r_m/r_0)\,z$. When $z=\lambda r_0=\zeta$, $\lambda=\zeta/r_0$, and by using $\pi q_\zeta\in(\mathrm{span}(x^n)+1)$,

$$\zeta=r_0/r_1\ . \tag{2.5}$$

Also, $q_z(x)=\sum_{m=0}^{n}z(r_m/r_0)\,x^m$.

Now the proof. On the coordinates plane, span$\{x,x^2)$, consider the non homogeneous linear equations of a right line L, $A_1u_1+A_2u_2=1$. Since L cross $(1,0)$, then $A_1=1$, likewise cross $\pi q_{\lambda r_0}=z(r_1/r_0,r_2/r_0)$,

$$(zr_1/r_0)+zA_2r_2/r_0=1\quad\Rightarrow A_2=(1/zr_2)(r_0-r_1z)\,,$$

moreover, because L cross $(0,p)$ too, $A_2=1/p$. And, by using (2.5), $1/p=(r_1/zr_2)(\zeta-z)$. To finish, since $1/p=w$, it remains explicit,

$$\underline{relationship\ w\sim z:}\qquad w=(r_1/zr_2)(\zeta-z)\quad. \tag{2.6}$$

Furthermore, we want $-w=z$ (see equation (2.1)), replacing $z=\overline{z}$, and also, $w=-\overline{z}$, the relation (2.6) is rewritten, as,

$$(\overline{z})^2-(r_1/r_2)\overline{z}+(r_0/r_2)=0\,, \tag{2.7}$$

by hypothesis, it has solution. Therefore there is a pair $(\overline{z}, w)$, on the relationship (2.6) such that, $-w = \overline{z}$, and also $\overline{z} \neq \zeta$, because, if were $\overline{z} = \zeta$, then $\overline{z} = 0$, by (2.6), thus, $0 = \zeta = r_0/r_1$, but, $r_0 \neq 0$. Neither, can be $(\overline{z}) = 0$, since $\zeta r_1 = r_0 \neq 0$. □

In fact, we have proven a new result.

Theorem 2.3. *An infinite field, is algebraically closed if, and only if, all polynomials of degree* 2 *have a root in the field.*

3. Final remarks

The Cauchy completeness results irrelevant on our proof. Both fields $\mathbb{Q}$ and $\mathbb{R}$ fail because they lack of many square roots. In fact an ordered field F can not be algebraically closed, since necessarily $\sqrt[2]{-1} \notin F$. The interesting example is the field of, algebraic on $\mathbb{Q}$, complex numbers, which passes our proof.

On $\mathbb{Z}_2[x]$, the conclusion of lemma 1.3 fails, e. g., for the polynomial $x^3 + x^2 + 1$, which do not pass the first paragraph on our proof.

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