International Journal of Algebra, Vol. 11, 2017, no. 5, 247 - 254
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2017.7314

On Smooth Lie Algebra Bundles

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Dedicates to the memory of R.L.E. Schwarzenberger of Warwick University.

Abstract

The existence of a class of smooth Lie algebra bundles is proved and examples are given. Further we construct smooth vector bundles which have the structure of non trivial smooth Lie algebra bundles.

Mathematics Subject Classification: 58A05, 55R99, 17B66, 22E99, 58E40

Keywords: Differentiable manifold; Lie algebra; Lie group; Lie group action; Smooth Lie algebra bundle; Tangent bundle; Section; Vector field

1 Introduction

The eminent mathematician J.P. Serre posed the question : Does there exist a Hausdorff Lie group bundle whose Lie algebra bundle is isomorphic to a given

Lie algebra bundle.

Douady and Lazard [3] define an analytic family of Lie groups and Lie algebras.They prove that if ξ is an analytic family of Lie algebras parameterized by a finite dimensional analytic manifold M, then there is a nonseparable (i.e., not necessarily separable) family $G(\xi)$ of Lie groups, parametrized by M, whose Lie algebra corresponds to ξ. They ask whether an analogous result still holds locally (around each point of M) for a given analytic family of Lie algebras, if one requires $G(\xi)$ to be separable. Don Coppersmith has constructed an example [1] which provides a negative answer.

In [6] it is proved that if all the fibres ξ_x of a Lie algebra bundle ξ over a topological space M are mutually isomorphic as Lie algebras, then ξ is locally trivial Lie algebra bundle . When the base space and the total space are manifold and all morphisms are $C^r, r \geq 1$, then the above result is found in [4, theorem 1,p 373]. However the techniques there do not generalise to topological spaces and continuous maps. To establish the result in [6], a lemma in algebraic geometry is proved which is of independent interest. It is proved in [7] that every semisimple Lie algebra bundle over a topological space is locally trivial and hence there exist a Hausdorff Lie group bundle $G(\xi)$ over M whose Lie algebra bundle is isomorphic to ξ over the Hausdorff space M [6, remark b].

Bechir Dali has studied Lie algebra bundles and constructed some concrete examples of Lie algebra bundles on X where $X \subseteq M_{m\times n}(\mathbb{R})$ linear space of $m \times n$ matrices over the field of real numbers $\mathbb{R}$ [2].Also Yanovski has defined linear bundles of Lie algebra and given the applications in Physics.[11].

In 1971 during his visit to Ramanujan Institute in India, RLE Schwarzenberger asked the first author 'which vector bundles can have the structure of a non trivial Lie algebra bundles'.

Here we answer the above question, after proving the existence theorem of smooth Lie algebra bundles. Further it is shown that there is a bijective correspondence between the first Čech cohomology group $H^1(M, G)$ and the set of all isomorphism classes of smooth Lie algebra bundles over a manifold M with standard fibre L. Smooth Lie algebra bundles are constructed from tangent bundles using Swan's theorem [9].

2 Preliminary definitions

Definition 2.1. *A smooth vector bundle ξ [12], is a smooth weak Lie algebra bundle [3] if there exist a morphism $[\ ,\ ] : \xi \oplus \xi \to \xi$ inducing a Lie algebra* *structure on each fibre ξ_x.*

Definition 2.2. *A locally trivial smooth Lie algebra bundle [4],for short a smooth Lie algebra bundle , is a smooth vector bundle $\xi = (\xi, \pi, M)$ in which each fibre is a Lie algebra and for each x in M there is an open set U in M containing x, Lie algebra L and a diffeomorphism $\varphi : U \times L \to \pi^{-1}(U)$ such that for each x in U, $\varphi_x : L \to \pi^{-1}(x)$ is a Lie algebra isomorphism.*

Remark 1. *Locally trivial Lie algebra bundle is a weak Lie algebra bundle , but the converse need not be true [8].*

Definition 2.3. *Let ξ, η be two smooth Lie algebra bundles over the same base space M. A smooth Lie algebra bundle morphism $f : \xi \to \eta$ is a smooth vector bundle morphism such that for each x in M, $f_x : \xi_x \to \eta_x$ is a Lie algebra homomorphism.*

Let G be a smooth Lie group and M a smooth manifold.By a system of smooth transition maps in M with values in G is meant an indexed covering $\{U_i\}$ of M by open sets and a collection of smooth transition maps
$g_{ij} : U_i \cap U_j \to G \quad i, j \in I$, $g_{ij}(x)g_{jk}(x) = g_{ik}(x), \quad x \in U_i \cap U_j \cap U_k$,
this condition is known as the Čech cocycle [10].

3 Existence theorem

Theorem 3.1. *Let G be a Lie group acting smoothly on a Lie algebra L such that for each fixed g in G, the map $l \mapsto gl$ from L into L is a Lie algebra homomorphism. Suppose M is a smooth manifold with a countable atlas, $\{(U_i, \psi_i)\}$ and $\{g_{ij}\}$ is a system of smooth transition maps in M then there exists a smooth Lie algebra bundle, with the base space M, fibre L, structure group G and the smooth transition maps $\{g_{ij}\}$.*

Proof. Let I be the indexing set with discrete topology for the countable atlas $\mathcal{A} = \{(U_i, \psi_i) | i \in I\}$.Let $T = \{(u, l, i) | u \in U_i, l \in L, i \in I\}$ then $T = \bigsqcup_{i \in I}(U_i \times L \times i)$ is a topological space with quotient topology.

Define equivalence relation in T by $(u, l, i) \sim (u', l', j)$ if and only if $u = u', g_{ji}(u)l = l'$.Let $\xi = \bigsqcup_{i \in I}(U_i \times L \times i)/\sim$ and let $q : T \to \xi$ defined by $(u, l, i) \mapsto [u, l, i]$ be the quotient map and the natural projection map $\pi : \xi \to M, [u, l, i] \mapsto u$. Then for any subset W of $U_j \times L$,

$$q^{-1}(q((W \times i))) = \bigsqcup_{i \in I} h_{ij}(W) \times i, \text{ where } h_{ij}(u, l) = (u, g_{ij}(u)l)$$

and if $W \times j$ is an open subset of $U_j \times L \times j$ and h_{ij} is smooth then $h_{ij}(W \times i)$ is open.Hence q is open being a quotient map. Since its restriction $q_i = q|_{U_i \times L \times i}$ is injective, $(q_i(U_i \times L \times i), q_i^{-1})$ is a smooth chart on ξ and thus $\pi : \xi \to M$ is a smooth vector bundle [12].

Now we shall define the Lie algebra structure on each fibre $\xi_u = \pi^{-1}(u)$ so that h_{iu} is a Lie algebra homomorphism. Let $b_1, b_2 \in \pi^{-1}(u),\ u \in U_i,\ \alpha, \beta \in \mathbb{R}$,

1. Define $\alpha b_1 + \beta b_2 = h_{iu}(\alpha h_{iu}^{-1}(b_1) + \beta h_{iu}^{-1}(b_2))$
2. $[b_1, b_2] = h_{iu}([h_{iu}^{-1}(b_1), h_{iu}^{-1}(b_2)]$

The above definitions are independent of the smooth maps h_i. That ξ is a smooth Lie algebra bundle follows from the methods of [8].

□

If $\{U_i\}_{i \in I}$ and $\{g_{ij} : U_i \cap U_j \to G = \mathrm{Aut}(L) : i, j \in I\}$ are transition maps arising from a smooth Lie algebra bundle (ζ, π', M), then the smooth Lie algebra bundle $\xi = (\xi, \pi, M)$ constructed by previous part is isomorphic to ζ. Let $\{(U_i, h_i)\}$ be the trivialisations as above , giving rise to the transition maps $\{g_{ij}\}$. We define

$$\tilde{f} : \zeta \to \xi \quad \text{by} \quad \tilde{f}(v) = [i, h_i(v)] \quad \text{if } \pi(v) \in U_i$$

If $\pi(v) \in U_i \cap U_j$,then

$$[j, h_j(v)] = [i, h_{ij}(h_j(v))] = [i, h_i(v)] \in \xi$$

and the map $\tilde{f}$ is well defined as it depends only on v, not on i.It is immediate that $\pi \circ \tilde{f} = \pi'$. Since the map

$$q_i^{-1} \circ \tilde{f} \circ h_i^{-1} : U_i \times L \to U_i \times L$$

is smooth (being the identity), $\tilde{f}$ is a smooth map and $q_i = q|_{U_i \times L \times i}$.Since the restrictions of q_i and h_i to every fibre are Lie algebra isomorphisms, it follows that so is $\tilde{f}$.Hence we conclude that $\tilde{f}$ is a Lie algebra bundle isomorphism.

Two Lie algebra bundles are defined to be equivalent if they are isomorphic as Lie algebra bundles over M.Two sets of transition data

$$\{g_{ij}\}_{i,j \in I} \quad \text{and } \{g'_{ij}\}_{i,j \in I}$$

with I consisting of all sufficiently small open subsets of M, are said to be equivalent if there exists a collection of smooth functions
$\{f_1 : U_i \to G = \mathrm{Aut}(L)\}_{i \in I}$ such that $g'_{ij} = f_i g_{ij} f_j^{-1}$, for all $i, j \in I$

That is two sets of transition maps differ by a Čech 0-chain.Along with the cocycle condition on the gluing data, this means that isomorphism classes of Lie algebra bundles over M can be identified with $\check{H}^1(M;G)$, the quotient of the space of Čech cocycles of degree one by the subspace of Čech boundaries [10]. Thus we have

Theorem 3.2. *There is a bijective correspondence between the first Čech cohomology group $H^1(M,G)$ and the set of all isomorphism classes of smooth Lie algebra bundles over M with standard fibre L.*

Example using existence theorem: Special orthogonal Lie group $SO(n)$ acts smoothly on the Lie algebra $o(n)$ of the orthogonal Lie group $O(n)$,

$$o(n) \times SO(n) \to o(n), \quad (X,U) \mapsto UXU^{-1}$$

and the map $\phi : o(n) \to o(n), \quad X \mapsto UXU^{-1}$ is a Lie algebra automorphism.$SO(n)$ being a closed subgroup of $O(n)$ there exists locally trivial principal bundle having $SO(n)$ as a structure group.So the existence of smooth transition maps $g_{ij} : U_i \cap U_j \to SO(n)$ is assured, where $\{U_i\}$ is an open covering of $M = O(n)/SO(n)$. Then by the Existence theorem there exists a smooth Lie algebra bundle $\xi = (\xi, \pi, M, o(n), SO(n))$.

4 Bundles with Lie algebra bundle structures

We construct smooth Lie algebra bundles following [5, 8].

Fibre bundle associated to a principal bundle : Let M be a smooth manifold.Suppose that $\pi : P \to M$ is a smooth principal $G-$ bundle and $\rho : G \to Aut(L)$ be a smooth action on Lie algebra L. Define a right action of G on $P \times L$, $(p,l).g = (p.g, \rho(g^{-1})l)$, and equivalence relation in $P \times L$, by $(p_1, l_1) \sim (p_2, l_2)$ if and only if there exists $g \in G$ such that $p_2 = p_1 g$ and $l_2 = \rho(g^{-1})l_1$. Let $E = P \times_\rho L = (P \times L)/G$ be the collection of equivalence classes $\{[p,l]\}$ identified by the action of G. Clearly $\{[p.g,l]\} = \{[p, \rho(g)l]\}$,for all $g \in G$ as $(p, \rho(g)l)g = (p.g, \rho(g^{-1}g)l) = (p.g, l)$, for all $g \in G$ and hence projection map $\pi_\rho : E \to M$, $\pi_\rho(\{[p,l]\}) = \pi(p)$ is well defined. Thus $\pi_\rho : E \to M$ is a fibre bundle with fibre L and structure group G.The transition functions are given by $\rho(t_{ij})$ where t_{ij} are the transition functions of the principal bundle P.

Adjoint bundle : Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Define $\psi : G \to Aut(G)$ by $g \mapsto \psi_g$ where $\psi_g : G \to G$, $\psi_g(h) = ghg^{-1}$ is an automorphism. The differential of ψ_g at the identity e,denoted by Ad_g is an automorphism of the Lie algebra $\mathfrak{g}$.

The map,$Ad : G \to Aut(\mathfrak{g}),\ g \mapsto Ad_g$ is called the adjoint action of G on $\mathfrak{g}$ [8, page 40].

Let $\pi : P \to M$ be a principal G bundle over a smooth manifold M. The **adjoint bundle** of P is the fibre bundle associated with P [5, Page 96]. It is denoted by, $adP = P \times^{Ad} \mathfrak{g}$, where $P \times^{Ad} \mathfrak{g} = (P \times \mathfrak{g})/G$, the collection of equivalence classes obtained on identifying the elements in $P \times \mathfrak{g}$ by the relation,

$$(p_1, l_1) \sim (p_2, l_2) \text{ if and only if there exists } g \in G \text{ such that}$$
$$p_2 = p_1 \cdot g \text{ and } l_2 = Ad_{g^{-1}}(l_1)$$

where $p \cdot g$ in first component is the action of G on P.

Explicitly, elements of the adjoint bundle are equivalence classes of pairs $\{[p, l]\}$ for $p \in P$ and $x \in \mathfrak{g}$ such that

$$\{[p.g, x]\} = \{[p, Ad_g(x)]\}, \text{ for all } g \in G$$

Define $\pi_{Ad} : adP \to M$ by $\pi_{Ad}(\{[p, l]\}) = \pi(p)$. Clearly π_{Ad} is an onto smooth projection such that each fibre $\pi_{Ad}^{-1}(x)$ carries Lie algebra structure of $\mathfrak{g}$.

Thus, the adjoint bundle $(adP, \pi_{Ad}, M, \mathfrak{g}, G)$ is a smooth Lie algebra bundle.

Example : The special unitary group $SU(n)$ is a closed subgroup of the unitary Lie group $U(n)$ and thus $U(n)$ is a smooth principal $SU(n)$-bundle over the quotient space, $U(n)/SU(n)$. Then $\pi : U(n) \to U(n)/SU(n)$ is smooth, fibres are left cosets of $SU(n)$, which are diffeomorphic to $SU(n)$ and hence carry the Lie group structure of $SU(n)$.

Let $su(n) := \{X \mid e^{tX} \in SU(n), \text{ for all } t \in \mathbb{R}\}$ be the Lie algebra of G.

Define a right action of $SU(n)$ on $U(n) \times su(n)$ by

$$(A, X) \cdot B = (AB, Ad_{B^{-1}}(X))$$

where, $Ad_{B^{-1}}$ is an automorphism on $su(n)$ defined by, $Ad_{B^{-1}}(X) = B^{-1}XB$, for all $X \in su(n), B \in SU(n), A \in U(n)$.

Thus $\pi_{Ad} : U(n) \times^{Ad} su(n) \to U(n)/SU(n)$ is a smooth Lie algebra bundle.

Tangent Bundle: A smooth vector field S over a smooth manifold M is a smooth section of the smooth tangent bundle TM. It is clear that S is smooth if and only if its components are smooth for all charts in some atlas for M. Let $\Gamma(\xi)$ denote the set of all smooth vector fields on M. Then $\Gamma(\xi)$ is a module over $C^\infty(M)$ as every smooth vector field is a map $S : C^\infty(M) \to C^\infty(M),\ f \mapsto Sf$ where $(Sf)(p) = S_p(f)$.

Theorem 4.1. *Let $\xi = (TM, \pi, M)$ be a smooth tangent bundle over a smooth compact manifold M. Then there exists a smooth weak Lie algebra bundle.*

Proof. All smooth sections $\Gamma(\xi)$ of a tangent bundle ξ is a finitely generated projective module [9] over $C^\infty(M)$. They form a Lie algebra over $C^\infty(M)$ with Lie product $[\ ,\ ] : \Gamma(\xi) \times \Gamma(\xi) \to \Gamma(\xi),\ (S,T) \mapsto [S,T]$ where $[S,T](f) = T(S(f)) - S(T(f))$ implies $[\ ,\ ] : \Gamma(\xi)/I_x\Gamma(\xi) \times \Gamma(\xi)/I_x\Gamma(\xi) \to \Gamma(\xi)/I_x\Gamma(\xi)$ a Lie algebra structure, $x \in M$ where $I_x = \{f \mid f(x) = 0, f \in C^\infty(M)\}$ is a maximal ideal in $C^\infty(M)$ and $I_x\Gamma(\xi) = \{T \mid T = \sum_i f_i s_i,\ f_i \in I_x,\ s_i \in \Gamma(\xi)\}$is an ideal in $\Gamma(\xi)$.

Then $\theta_x : \Gamma(\xi)/I_x\Gamma(\xi) \to \xi_x, \quad S + I_x\Gamma(\xi) \mapsto S(x)$ is a Lie algebra isomorphism over the field of real numbers $\mathbb{R}$ as $C^\infty(M)/I_x$ is isomorphic to $\mathbb{R}$ [8].

Thus we have the following commutative diagram

$$\begin{array}{ccc}
\Gamma(\xi)/I_x\Gamma(\xi) \times \Gamma(\xi)/I_x\Gamma(\xi) & \xrightarrow{[\ ,\ ]} & \Gamma(\xi)/I_x\Gamma(\xi) \\
\Big\downarrow{\scriptstyle \theta_x \times \theta_x} & & \Big\downarrow{\scriptstyle \theta_x} \\
\xi_x \times \xi_x & \xrightarrow{\quad\Theta_x\quad} & \xi_x
\end{array}$$

$(\theta_x \times \theta_x) \circ \Theta_x = \theta_x \circ [\ ,\ ]$.

Then obviously on each fibre ξ_x, the map $\Theta_x : \xi_x \times \xi_x \to \xi_x$ induces a Lie algebra structure. The totality of this yield a smooth morphism $\Theta : \xi \oplus \xi \to \xi$ inducing a Lie algebra structure on each fibre ξ_x Lie algebraby following the methods as in [8]. Thus ξ is a smooth weak Lie algebra. □

Acknowledgements. The first author is thankful to the SERB/DST, New Delhi, India for the financial assistance SR/S4/MS:856/13. We thank R. Rangarajan for his support.

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Received: April 6, 2017; Published: July 24, 2017