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Lie Triple Derivations of the Nilpotent Subalgebra of D_m

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Abstract

Let g be a complex simple Lie algebra with Cartan subalgebra h and standard Borel subalgebra b. Put $n = [b, b]$. In this paper, we describe Lie triple derivations of the nilpotent subalgebras n for the classical Lie algebra $D_m (m \geq 6)$ over the complex number field C.

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1 Introduction

Let C be the complex number field and g a simple Lie algebra over C with Cartan subalgebra h and root system Δ. Fix a basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of Δ, let Δ^+ (Δ^-) denote the positive root set (negative root set). Then $b = h \oplus \sum_{\alpha \in \Delta^+} g_\alpha$ is the standard Borel subalgebra of g and $n = [b, b] = \sum_{\alpha \in \Delta^+} g_\alpha$ is a nilpotent subalgebra of g. Denote the height of the root α by $\mathrm{ht}\alpha$.

Definition 1.1.[1] A linear mapping $\phi : n \to n$ is called a Lie triple derivation if it satisfies

$$\phi([[x, y], z]) = [[\phi(x), y], z] + [[x, \phi(y)], z] + [[x, y], \phi(z)], \quad \text{for all } x, y, z \in n.$$

Clearly, Lie derivations are all Lie triple derivations, while the converse may not be true.

Recently, many studies have been done in Lie derivations of matrix algebras and their subalgebras (see [1]). And they use matrices commutation to obtain many equalities. However, it is difficult to find such equalities for D_m. As $0 = [g_\alpha, g_\beta]$ for $\alpha+\beta \notin \Delta$, in this paper, we use root system to obtain some equalities and simplify the image of ϕ on n of the classical Lie algebra D_m, which generalize the arithmetic of [2].

It is obvious that the nonzero root vectors in g_α with $\operatorname{ht}\alpha = 1$ are generators of n, if a Lie triple derivation $\phi : n \to n$ satisfies $\phi(g_\alpha) = 0$ with $\operatorname{ht}\alpha = 1, 2$, then $\phi(g_\alpha) = 0$ for all $\alpha \in \Delta^+$. Our main idea arises from this.

2 Main results

For $g = D_n = \mathrm{SO}(2m, C)$, it is well known that

$$\begin{aligned}
&h = \{\mathrm{diag}(x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n) \,|\, x_i \in C\},\\
&\Delta = \{\pm(\lambda_i - \lambda_j), \pm(\lambda_i + \lambda_j) \,|\, 1 \le i < j \le n\},\\
&\Pi = \{\lambda_i - \lambda_{i+1}, \lambda_{n-1} + \lambda_n \,|\, 1 \le i \le n-1\},\\
&\Delta^+ = \{\lambda_i - \lambda_j, \lambda_i + \lambda_j \,|\, 1 \le i < j \le n\},\\
&g_{\lambda_i - \lambda_j} = CA_{ij}, \quad A_{ij} = E_{ij} - E_{n+j,n+i},\\
&g_{\lambda_i + \lambda_j} = CB_{ij}, \quad B_{ij} = E_{i,n+j} - E_{j,n+i},
\end{aligned}$$

where $\lambda_i\left(\mathrm{diag}(x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n)\right) = x_i,\ 1 \le i \le n$. Then

$$n = \sum_{1 \le i < j \le n} g_{\lambda_i - \lambda_j} + \sum_{1 \le i < j \le n} g_{\lambda_i + \lambda_j}.$$

Firstly, we give four types standard Lie triple derivations of n, which describe any Lie triple derivation of n. They are defined as follows:

(1) Inner triple derivations:
Let $x \in n$, then $\mathrm{ad}x : n \to n, y \mapsto [x, y]$ is a Lie triple derivation.

(2) Diagonal triple derivations:
Let $y \in h$, then $\eta_y : n \to n,\ x \mapsto [y, x]$ is a Lie triple derivation.

(3) Central triple derivations:
If $m \ge 6$, let $\xi = (\xi_4, \ldots, \xi_{n-1})$, where $\xi_i = (k_i, l_i),\ k_i, l_i \in C,\ 4 \le i \le n-1$,

$$\zeta = (\zeta_1, \ldots, \zeta_{n-2}), \text{ where } \zeta_j = (k'_j, l'_j),\ k'_j, l'_j \in C,\ 1 \le j \le n-2,$$
$$\theta = (p_2, p_3), \text{ where } p_2, p_3 \in C,$$

$\vartheta = (q_2, q_3),$ where $q_2, q_3 \in C$.

Define a linear mapping $\mu_{(\xi,\theta,\zeta,\vartheta)} : n \to n$ as follows:

$$\begin{aligned}
&\mu_{(\xi,\theta,\zeta,\vartheta)}(A_{i,i+1}) = k_i B_{12} + l_i B_{13},\ 4 \le i \le n-1,\\
&\mu_{(\xi,\theta,\zeta,\vartheta)}(B_{n-1,n}) = p_2 B_{12} + p_3 B_{13},\\
&\mu_{(\xi,\theta,\zeta,\vartheta)}(A_{j,j+2}) = k'_j B_{12} + l'_j B_{13},\ 1 \le j \le n-2,\\
&\mu_{(\xi,\theta,\zeta,\vartheta)}(B_{n-2,n}) = q_2 B_{12} + q_3 B_{13},\\
&\mu_{(\xi,\theta,\zeta,\vartheta)}(A_{pq}) = \mu_{(\xi,\theta,\zeta,\vartheta)}(B_{pq}) = 0, \quad \text{otherwise.}
\end{aligned}$$

It is easy to verify that $\mu_{(\xi,\theta,\zeta,\vartheta)}$ is a Lie triple derivation and

$$\mu_{(\xi,\theta,\zeta,\vartheta)} = \mu_{(\xi,0,0,0)} + \mu_{(0,\theta,0,0)} + \mu_{(0,0,\zeta,0)} + \mu_{(0,0,0,\vartheta)}.$$

(4) Extremal triple derivations:
Let $m_1, m_2, m_3 \in C$. We define a linear mapping $\rho_3 : n \to n$ as follows:

$$\begin{aligned}
&\rho_3(A_{12}) = m_1 B_{12},\\
&\rho_3(A_{23}) = m_2 B_{13},\\
&\rho_3(A_{34}) = m_3 B_{12},\\
&\rho_3(A_{pq}) = \rho_3(B_{pq}) = 0, \quad \text{otherwise.}
\end{aligned}$$

It is clear that ρ_3 is Lie triple derivation.

Theorem 2.1 *For $m \ge 6$, every Lie triple derivation ϕ of n can be uniquely expressed as:*

$$\phi = adx_0 + \eta_{y_0} + \mu_{(\xi,\theta,\zeta,\vartheta)} + \rho_3,$$

where $adx_0, \eta_{y_0}, \mu_{(\xi,\theta,\zeta,\vartheta)}$ and ρ_3 are inner, diagonal, central and extremal triple derivations, respectively.

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