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Soft *BCL*-Algebras of the Power Sets

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Abstract

In this work, at the beginning we attempted to introduce the notion of $BCL-$ algebra of the power set and some other new concepts connected to it are introduced and studied in this work like $BCL-$subalgebra of the power set, soft $BCL-$algebra of the power set and soft $BCL-$subalgebra of the power set. Then some binary operations between two soft $BCL-$algebras of the power set are studied. Moreover, we find relations between soft $BCL-$algebra of the power set and soft $BCK/d/d^*/\rho-$ algebra of the power set. Moreover, several examples are given to illustrate the concepts introduced in this paper.

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1 Introduction

Y. Imai and K. Iseki introduced two classes of abstract algebras: $BCK-$algebras and *BCI*-algebras ([5], [6]). It is known that the class of *BCK*-algebras is a proper subclass of the class of BCI-algebras. After then the notion of d-algebras, which is another useful generalization of *BCK* –algebras are introduced (see [2], [3], [19]). Next, the notion of $\rho-$algebra is introduced and discussed [9]. In [17] the basic notions of soft sets theory are introduced by Molodtsov to deal with uncertainties when solving problems in practice as in engineering, environment, social science and economics. This notion is convenient and easy to apply as it is free from the difficulties that appear when using other mathematical tools as theory of theory of rough sets, theory of vague sets and fuzzy sets etc. Next, many studies on soft sets theory and their applications are discussed (see [10]-[15]). On the other hand, many authors applied the notion of soft set on several classes of algebras like soft $BCK/BCL-$algebras [7] and soft $\rho-$algebras [16]. In 2017, for any finite set X, the notion of $d-$algebra of the power set, $BCK-$algebra of the power set, d^*-algebra of the power set, soft $d-$algebra of the power set, soft $BCK-$algebra of the power set, soft d^*-algebra of the power set, soft edge $d-$algebra of the power set, soft edge $BCK-$algebra of the power set, soft edge d^*-algebra of the power set are investigated. The aim of this work is to introduce new branch of the pure algebra it's called BCI-algebra of the power set of X (see [16]). Then some binary operations between two soft $BCL-$algebras of the power set are studied. Moreover, we study the relations between soft $BCL-$algebra of the power set and soft $BCK/d/d^*/\rho-$ algebra of the power set. Moreover, several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries

In this section we recall the basic background needed in our present work.

Definition 2.1: ([19]) A $d-$algebra is a non-empty set X with a constant 0 and a binary operation* satisfying the following axioms:

(i)- $x*x=0$

(ii)- $0*x=0$

(iii)- $x*y=0$ and $y*x=0$ imply that $y=x$ for all x, y in X.

Definition 2.2: ([18]) A $d-$algebra $(X,*,0)$ is called $BCK-$algebra if X satisfies the following additional axioms:

(1). $((x*y)*(x*z))*(z*y)=0$,

(2). $(x*(x*y))*y=0$, for all $x,y\in X$.

Definition 2.3 ([8]) A $\rho-$algebra $(X,*,f)$ is a non-empty set X with a constant $f\in X$ and a binary operation* satisfying the following axioms:

(i)- $x*x=f$,

(ii)- $f*x=f$,

(iii)- $x*y=f=y*x$ imply that $y=x$,

(iv)- For all $y\neq x\in X-\{f\}$ imply that $x*y=y*x\neq f$.

Definition 2.4: ([20]) A $BCL-$algebra is a non-empty set X with a constant 0 and a binary operation* satisfying the following axioms:

(i)- $x*x=0$,

(ii)- $x*y=0$ and $y*x=0$ imply that $y=x$,
(iii)- $((x*y)*z)*((x*z)*y)*((z*y)*x)=0$, for all *x, y ,z* in *X*.

Definition 2.5: ([17]) Let X be an initial universe set and let E be a set of parameters. The power set of X is denoted by $P(X)$. Let K be a subset of E. A pair (F,K) is called a soft set over X if F is a mapping of K into the set of all subsets of the set X.

Definition 2.6:([11]) Let (F,A) and (G,B) be two soft sets over X, then their union is the soft set (H,C), where $C=A\bigcup B$ and for all $e\in C$, $H(e)=F(e)$ if $e\in A-B$, $G(e)$ if $e\in B-A$, $F(e)\bigcup G(e)$ if $e\in A\bigcap B$. We write $(F,A)\coprod(G,B)=(H,C)$. Further, [4] for any two soft sets (F,A) and (G,B) over X their intersection is the soft set (H,C) over X, and we write $(H,C)=(F,A)\prod(G,B)$, where $C=A\bigcap B$, and $H(e)=F(e)\bigcap G(e)$ for all $e\in C$.

Definition 2.7: ([8]) Let (F,K) be a soft set over X. Then (F,K) is called a soft $d-$algebra over X if $(F(x),*,0)$ is a $d-$algebra for all $x\in K$.

Definition 2.8: ([7]) Let (F,K) be a soft set over X. Then (F,K) is called a soft $BCK-$algebra over X if $(F(x),*,0)$ is a $BCK-$algebra for all $x\in K$.

Definition 2.9: ([7]) Let (F,K) be a soft set over X. Then (F,K) is called a soft $BCL-$algebra over X if $(F(x),*,0)$ is a $BCL-$algebra for all $x\in K$.

Definition 2.10: ([16]) Let (F,K) be a soft set over X. Then (F,K) is called a soft $\rho-$algebra over X if $(F(x),*,0)$ is a $\rho-$algebra for all $x\in K$.

Definition 2.11: ([16]) Let X be non-empty set and $P(X)$ be a power set of X. Then $(P(X),*,A)$ with a constant A and a binary operation* is called $d-$algebra of the power set of X if $P(X)$ satisfying the following axioms:
(i)- $B*B=A$
(ii)- $A*B=A$
(iii)- $B*C=A$ and $C*B=A$ imply that $B=C$ for all $B,C\in P(X)$.

Definition 2.12: ([16]) Let $(P(X),*,A)$ be a $d-$algebra of the power set of X. Then $P(X)$ is called $BCK-$algebra of the power set of X if it satisfies the following additional axioms:
(1). $((B*C)*(B*D))*(D*C)=A$,
(2). $(B*(B*C))*C=A$, for all $B,C\in P(X)$.

Definition: 2.13: ([16]) Let $(P(X),*,A)$ be a $d-$algebra of the power set of X. Then $P(X)$ is called a ρ −algebra of the power set of X if it satisfies the identity $(B*C)=(C*B)\neq A$, for all $B\neq C\in P(X)-A$.

Definition 2.14: ([16]) Let $(P(X),*,A)$ be a $d-$algebra of the power set of X. Then $P(X)$ is called a $d^{*}-$algebra of the power set of X if it satisfies the identity $(B*C)*B=A$, for all $B,C\in P(X)$.

Definition 2.15: ([16]) Let $(P(X),*,A)$ be a $d-$algebra (ρ −algebra, $BCK-$algebra, $d^{*}-$algebra) of the power set of X and let $H=\{h_i\}_{i\in I}\subseteq P(X)$ be a collection *of some random* subsets of X. Then H is called $d-$subalgebra (resp. ρ −subalgebra, $BCK-$subalgebra, $d^{*}-$subalgebra) of the power set of X, if $h_m*h_k\in H$, for any $h_m,h_k\in H$.

Definition 2.16:([16]) Let $(P(X),*,A)$ be a $d-$algebra (resp. $\rho-$algebra, $BCK-$algebra, $d^{*}-$algebra) of the power set of X and let $F:H\rightarrow P(X)$, be a set valued function , where H is a collection *of some random* subsets of X defined by $F(h)=\{q\in P(X)\mid h\approx q\}$ for all $h\in H$ where $\approx$ is an arbitrary binary operation from H to $P(X)$. Then the pair (F,H) is a soft set over X. Further, (F,H) is called a soft $d-$algebra (resp. soft ρ-algebra, soft $BCK-$algebra, soft $d^{*}-$algebra) of the power set of X, if $(F(h),*,A)$ is a $d-$subalgebra (resp. $\rho-$subalgebra, $BCK-$subalgebra, $d^{*}-$subalgebra) of the power set of X for all $h\in H$.

3. Soft *BCL*-algebra of the power sets

In this section we introduce the notion of soft BCL-algebra of the power set and soft $BCK-$subalgebra of the power set. We will illustrate the definitions with examples.

Definition 3.1 Let X be non-empty set and $P(X)$ be a power set of X. Then $(P(X),*,A)$ with a constant A and a binary operation ($*$) is called $BCL-$algebra of the power set of X if $P(X)$ satisfying the following axioms:

(i)- $B*B=A$,

(ii)- $B*C=A$ and $C*B=A$ imply that $B=C$ for all $B,C\in P(X)$,

(iii)- $((B*C)*D)*((B*D)*C)*((D*C)*B)=0$, for all $B,C,D\in P(X)$.

Definition 3.2 Let $(P(X),*,A)$ be a $BCL-$algebra of the power set of X and let $H=\{h_i\}_{i\in I}\subseteq P(X)$ be a collection *of some random* subsets of X. Then H is called $BCL-$subalgebra of the power set of X, if $h_m * h_k \in H$, for any $h_m, h_k \in H$.

Example 3.3 Let $X = \{1,2,3\}$ and let $/ : P(X) \times P(X) \to P(X)$ be a binary operation defined by $/(B,C) = B/C = \{x \in X \mid x \in B \,\&\, x \notin C\} = B \bigcap C^c$, for all $B, C \in P(X)$. Then $(P(X), /, \phi)$ is a $BCL-$algebra of the power set of X with the following table:

/	ϕ	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	X
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{1}	{1}	ϕ	{1}	{1}	ϕ	ϕ	{1}	ϕ
{2}	{2}	{2}	ϕ	{2}	ϕ	{2}	ϕ	ϕ
{3}	{3}	{3}	{3}	ϕ	{3}	ϕ	ϕ	ϕ
{1,2}	{1,2}	{2}	{1}	{1,2}	ϕ	{2}	{1}	ϕ
{1,3}	{1,3}	{3}	{1,3}	{1}	{3}	ϕ	{1}	ϕ
{2,3}	{2,3}	{2,3}	{3}	{2}	{3}	{2}	ϕ	ϕ
X	X	{2,3}	{1,3}	{1,2}	{3}	{2}	{1}	ϕ

Table (1)

On the other hand, $(P(X), /, \phi)$ is a $BCK/d/d^*-$algebra of the power set of X. Further, $\{1\}, \{2\} \in P(X) - \{\phi\}$, but $\{1\}/\{2\} \neq \{2\}/\{1\}$. Then $(P(X), /, \phi)$ is not ρ-algebra. Also, for example, $q_1 = \{\phi\}$, $q_2 = \{\phi, \{1\}\}$, $q_3 = \{\phi, \{2\}\}$, $q_4 = \{\phi, \{3\}\}$, $q_5 = \{\phi, \{1\}, \{2\}\}$, $q_6 = \{\phi, \{1\}, \{3\}\}$, and $q_7 = \{\phi, \{2\}, \{3\}\}$ are $BCL-$subalgebra and $BCK/d/d^*-$ subalgebra of the power set of X. However, $K = \{\phi, \{1,3\}, \{2,3\}\}$ is not $BCL/BCK/d/d^*-$subalgebra of the power set of X. Moreover, if X is a non-empty set contains exactly two elements. Then for any $\phi \neq A \subset X$ we have $(P(X), /, \phi)$ is a $BCK/d/d^*-$algebra of the power set of X with the following table:

/	ϕ	A	A^c	X
ϕ	ϕ	ϕ	ϕ	ϕ
A^c	A	ϕ	A	A
A^c	A^c	A^c	ϕ	A^c
X	X	X	X	ϕ

Table (2)

Example 3.4: Let $X = \{1,2,3\}$ and let $\oplus : P(X) \times P(X) \to P(X)$ be a binary operation defined by $\oplus(A,B) = A \oplus B = \begin{cases} A \cup B, & if \quad B \neq A \neq \phi, \\ \\ \phi, & Otherwise. \end{cases}$, for all $A, B \in P(X)$. Then $(P(X),\oplus,\phi)$ is a $BCL-$algebra of the power set of X with the following table:

$\oplus$	ϕ	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	X
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{1}	{1}	ϕ	{1,2}	{1,3}	{1,2}	{1,3}	X	X
{2}	{2}	{1,2}	ϕ	{2,3}	{1,2}	X	{2,3}	X
{3}	{3}	{1,3}	{2,3}	ϕ	X	{1,3}	{2,3}	X
{1,2}	{1,2}	{1,2}	{1,2}	X	ϕ	X	X	X
{1,3}	{1,3}	{1,3}	X	{1,3}	X	ϕ	X	X
{2,3}	{2,3}	X	{2,3}	{2,3}	X	X	ϕ	X
X	X	X	X	X	X	X	X	ϕ

Table (3)

Remarks 3.5:

(1) It is not necessary every $BCL-$algebra of the power set of X is $BCK/d/d^*/\rho-$ algebra of the power set. In example (3.4), let $A = \{1\}, B = \{2\}$. Then $(P(X),\oplus,\phi)$ is not d^*-algebra of the power set of X, since $(A \oplus B) \oplus A = \{1,2\} \neq \phi$. Also, let $A = \{1\}, B = \{2\}, C = \{3\}$. Then $(P(X),\oplus,\phi)$ is not $BCK-$algebra of the power set of X, since $((A \oplus B) \oplus (A \oplus C)) \oplus (C \oplus B) = X \neq \phi$. Also, $(P(X),\oplus,\phi)$ is a $d/\rho-$algebra of the power set of X. On the other hand, $q_1 = \{\phi\}$, $q_2 = \{\phi,\{1\}\}$, $q_3 = \{\phi,\{2\}\}$ and $q_4 = \{\phi,\{3\}\}$ are $\rho-$subalgebras. However, $K_1 = \{\phi,\{1\},\{2\}\}$, $K_2 = \{\phi,\{1\},\{3\}\}$, and $K_3 = \{\phi,\{2\},\{3\}\}$ are not $\rho-$subalgebras. Further, see example (3.3) $(P(X),/,\phi)$ is $BCL-$algebra of the power set of X, but is not ρ-algebra.

(2) Further, if $(P(X),*,f)$ is a $BCL-$algebra of the power set of X satisfying $f * h = f$ for any $h \in P(X)$. Then $(P(X),*,f)$ is a d-algebra of the power set of X.

Definition 3.6 Let $(P(X),*,A)$ be a $BCL-$algebra of the power set of X and let $F : H \to P(X)$, be a set valued function , where H is a collection *of some random* subsets of X defined by $F(h) = \{q \in P(X) \mid h \approx q\}$ for all $h \in H$ where $\approx$ is an arbitrary binary operation from H to $P(X)$. Then the pair (F,H) is a soft set over X . Further, (F,H) is called a soft $BCL-$algebra of the power set of X , if $(F(h),*,A)$ is a $BCL-$subalgebra of the power set of X for all $h \in H$.

Example: 3.7 Let $(P(X),\oplus,\phi)$ be a $BCL-$algebra of the power set of X with the following table:

$\oplus$	ϕ	$\{1\}$	$\{2\}$	X
ϕ	ϕ	ϕ	ϕ	ϕ
$\{1\}$	$\{1\}$	ϕ	$\{1\}$	ϕ
$\{2\}$	$\{2\}$	$\{2\}$	ϕ	ϕ
X	X	$\{2\}$	$\{1\}$	ϕ

Table (4)

Let (F,H) be a soft set over $X=\{1,2\}$, where $H=\{\phi,\{1\},\{2\}\}$ and $F:H \to P(X)$ is a set valued function defined by $F(h)=\{k \in P(X) |: h \approx k \leftrightarrow k=h^t, t \in N\}$ for all $h \in H$. Then $F(\phi)=\{\phi\}$, $F(\{1\})=\{\phi,\{1\}\}$, $F(\{2\})=\{\phi,\{2\}\}$which are soft $BCL-$subalgebras of the power set of X . Hence (F,H) is a soft $BCL-$algebra of the power set of X .

The next example shows that there exist set-valued functions $G:B \to P(X)$, where (G,B) the soft set is not a soft $BCL-$algebra of the power set of X .

Example 3.8: Consider the $BCL-$algebra in example (3.7) with a set valued function defined by $G(h)=\{k \in P(X) |: h \approx k \leftrightarrow h \oplus k \in \{\phi,\{2\}\}\}$ for all $h \in B=\{\phi,X\}$. We have (G,B) is not a soft $BCL-$algebra of the power set of X , since there exists $X \in B$, but $G(X)=\{\{1\},X\}$ is not soft $BCL-$subalgebras of the power set of X .

Definition: 3.9 Let (F,H) be a soft $BCL-$algebra of the power set of X , and let $D \subseteq H$, where $F:H \to P(X)$ is defined by $F(h)=\{q \in P(X) | h \approx q\}$ for all $h \in H$. Then $F|_D : D \to P(X)$ is defined by $F|_D(h)=\{q \in P(X) | h \approx q\}$ for all $h \in D$.

Lemma 3.10: If (F,H) is a soft $BCL-$algebra of the power set of X , then $(F|_D, H)$ is a soft $BCL-$algebra of the power set of X , for any $D \subseteq H$.

Proof: Let $(P(X),*,A)$ be a $BCL-$algebra of the power set of X and let (F,H) be a soft $d-$algebra of the power set of X , then $(F(h),*,A)$ is a $d-$subalgebra of the power set of X , for all $h \in H$. Moreover, for all $h \in D \cap H$ we have $F|_D(h)=F(h)$, but $D=D \cap H$ (since $D \subseteq H$). Hence $(F|_D(h),*,A)$ is a $BCL-$

subalgebra of the power set of X, for all $h \in D$. Then $(F|_D, H)$ is a soft $BCL-$ algebra of the power set of X.

Definition 3.11: Let (F, H) be a soft $BCL-$algebra of the power set of X. Then (F, H) is called a null soft $BCL-$algebra of the power set of X if $F(h) = \{\phi\}$ for all $h \in H$. Also, (F, H) is called an absolutely soft $BCL-$algebra of the power set of X if $F(h) = P(X)$ for all $h \in H$.

Example 3.12 Let $(P(X), \oplus, \phi)$ be the $BCL-$algebra of the power set of X in example (3.3) and let $F_1 : H_1 \to P(X)$, $F_2 : H_2 \to P(X)$, where $H_1 = \{\phi, \{2\}\}$, $H_2 = \{\phi, \{1\}, \{3\}\}$ are defined by $F_1(h) = \{k \in P(X) \mid h \approx k \Leftrightarrow h \oplus k \in H_1\}$, $\forall h \in H_1$, and $F_2(h) = \{k \in P(X) \mid h \approx k \Leftrightarrow k = h \oplus X\}, \forall h \in H_2$. Thus, $F_1(\phi) = F_1(\{2\}) = P(X)$, and hence (F_1, H_1) is an absolutely soft $BCL-$algebra of the power set X. Moreover, $F_2(\{\phi\}) = F_2(\{1\}) = F_3(\{3\}) = \{\phi\}$ and hence (F_2, H_2) is a null soft $BCL-$algebra of the power set X.

Lemma: 3.13 If $(P(X), *, L)$ is a $BCK / d / d^* / \rho -$ algebra of the power set of X. Then $(P(X), *, L)$ is a $BCL-$algebra of the power set of X, if $(A * B) * C = (A * C) * B$, for all $A, B, C \in P(X)$.

Proof: Since $(P(X), *, L)$ is a $BCK / d / d^* / \rho -$ algebra of the power set of X. Then $P(X)$ satisfying the following axioms:

(i)- $B * B = L$,

(ii)- $L * B = L$,

(iii)- $B * C = L$ and $C * B = L$ imply that $B = C$ for all $B, C \in P(X)$.

Now, since $(A * B) * C = (A * C) * B$, for all $A, B, C \in P(X)$. Hence, from (i) we consider that $((A * B) * C) * ((A * C) * B) = L$. Further, from (ii) we have $L * ((C * B) * A) = L$ and this implies that $((A * B) * C) * ((A * C) * B)$ $* ((C * B) * A) = L$. Therefore, the following are hold:

(1)- $B * B = L$,

(2)- $B * C = L$ and $C * B = L$ imply that $B = C$,

(3)- $((A * B) * C) * ((A * C) * B) * ((C * B) * A) = L$, for all $A, B, C \in P(X)$. The $(P(X), *, L)$ is a $BCL-$algebra of the power set of X.

Corollary 3.14 If $(P(X), *, L)$ is a $\rho-$algebra of the power set of X. Then $(P(X), *, L)$ is a $BCL-$algebra of the power set of X, if $C * (A * B) = (A * C) * B$, for all $A, B, C \in P(X)$.

Proof: Since $(P(X),*,L)$ is a $\rho-$algebra of the power set of X. Then $P(X)$ satisfying the following axioms:
(i)- $B*B=L$,
(ii)- $L*B=L$,
(iii)- $B*C=L$ and $C*B=L$ imply that $B=C$, for all $B,C\in P(X)$.
(iv)- $(B*C)=(C*B)\neq L$, for all $B\neq C\in P(X)-L$, Therefore, we consider only the following are hold:
1)- $B*B=L$,
2)- $B*C=L$ and $C*B=L$ imply that $B=C$,
Then to prove that $(P(X),*,L)$ is a $BCL-$algebra of the power set of X , we need also to prove that $((A*B)*C)*((A*C)*B)*((C*B)*A)=L$, for any $A,B,C\in P(X)$. Thus we have to show that $((A*B)*C)=L$ and hence we have $((A*B)*C)*((A*C)*B)*((C*B)*A)=L*((A*C)*B)*((C*B)*A)=$ $L*((C*B)*A)=L$. For any $A,B,C\in$ $P(X)$ we have the all cases that are cover all probabilities as following:
(1) If $A=B,C\in P(X)\Rightarrow((A*B)*C)=L$.
(2) If $L=A\neq B,C\in P(X)\Rightarrow((A*B)*C)=L$.
(3) If $L=B\neq A,C\in P(X)\Rightarrow B*(A*C)=L$, but $B*(A*C)=(A*B)*C\Rightarrow$ $(A*B)*C=L$.
(4) If $B\neq A=C,B\neq L\neq A\in P(X)$. Then, $B*(A*C)=(A*B)*C\Rightarrow$ $B*L=(A*B)*C\Rightarrow(A*B)*B=(A*B)*(A*B)*C\Rightarrow(A*B)*B=L*C=L$ $\Rightarrow(A*B)*B*B=L*C=L\Rightarrow(A*B)=L\Rightarrow((A*B)*C)=L$.
(5) If $B\neq A\neq C,B\neq L\neq A\in P(X)$. Thus, since $A\neq C\in P(X)-L$ and $(P(X),*,L)$ is a $\rho-$algebra of the power set of X. Then from (iv) we have $(A*C)\neq L$ and this implies that $B*(A*C)=(A*C)*B$ and hence $(A*C)*B=(A*B)*C$. Then by lemma (3.13) we have $(P(X),*,L)$ is a $BCL-$ algebra of the power set of X .

Theorem 3.15: Let (F,A) and (G,B) be two soft $BCL-$algebras over X. If $A\cap B=\phi$, then the union $(H,C)=(F,A)\coprod(G,B)$ is a soft $BCL-$algebra of the power set of X .

Proof: Since $A\cap B=\phi$ and by definition (2.6), we have for all $k\in C$,

$$H(k)=\begin{cases}F(k),\ if\ \ k\in A\setminus B,\\ G(k),\ if\ \ k\in B\setminus A.\end{cases}$$

If $k\in A\setminus B$ then $H(k)=F(k)$ is a $BCL-$subalgebra of the power set of X. Similarly, if $k\in B\setminus A$, then $H(k)=G(k)$ is a $BCL-$subalgebra of the power set of X . Hence $(H,C)=(F,A)\coprod(G,B)$ is a soft $BCL-$algebra of the power set of X . Thus the union of two soft $BCL-$algebras of the power set of X is a soft $BCL-$algebra of the power set of X .

Example 3.16: In example (3.4), let (F,A) and (G,B) be two soft sets over $X=\{1,2,3\}$ where $A=\{\phi,\{1\}\{2,3\}\}$ and $B=\{\{2\},\{1,3\}\}$. Define $F:A\to P(X)$ by $F(h)=\{k\in P(X)\,|\,k\approx h\leftrightarrow k\oplus h\in\{\phi,\{1,2\}\}$ for all $h\in A$ and $G:B\to P(X)$ by $G(h)=\{k\in P(X)\,|\,k\approx h \leftrightarrow k\oplus h\in\{\phi,\{1,3\}\}$ for all $h\in B$. Note that $A\bigcap B=\phi$. Thus, we have $F(\phi)=\{\phi,\{1,2\}\}$, $F(\{1\})=\{\phi,\{1\},\{2\},\{1,2\}\}$, $F(\{2,3\})=\{\phi,\{2,3\}\}$, $G(\{2\})=\{\phi,\{2\}\}$ and $G(\{1,3\})=\{\phi,\{1\},\{3\},\{1,3\}\}$. Then $H(\phi)=F(\phi)=\{\phi,\{1,2\}\}$, $H(\{1\})=F(\{1\})=\{\phi,\{1\},\{2\},\{1,2\}\}$, $H(\{2,3\})=F(\{2,3\})=\{\phi,\{2,3\}\}$, $H(\{2\})=G(\{2\})=\{\phi,\{2\}\}$ and $H(\{1,3\})=G(\{1,3\})=\{\phi,\{1\},\{3\},\{1,3\}\}$ which are $BCL-$subalgebras of the power set of X. Hence, (H,C) is a soft $BCL-$algebra of the power set of X.

Remark 3.17: The condition $A\bigcap B=\phi$ is important as if $A\bigcap B\neq\phi$, then the theorem does not apply. In above example, if $A=\{\phi,\{2\},\{2,3\}\}$ and $B=\{\phi,\{1\}\}$. Then $H(\phi)=F(\phi)\bigcup G(\phi)=\{\phi,\{1,2\},\{1,3\}\}$ which is not a $BCL-$subalgebra of the power set of X. Therefore, is not a soft $BCL-$algebra of the power set of X.

Theorem 3.18: Let (F,A) and (G,B) be two soft $BCL-$algebras over X. If $A\bigcap B=\phi$, then the union $(H,C)=(F,A)\prod(G,B)$ is a soft $BCL-$algebra of the power set of X.

Proof: Since $(H,C)=(F,A)\prod(G,B)$, where $C=A\bigcap B$, and $H(k)=F(k)\bigcap G(k)$ for all $k\in C$ [by definition (2.6)], Note that $H:C\to P(X)$ is a mapping and so (H,C) is a soft set over X. We have, $H(k)=F(k)$ or $H(k)=G(k)$ is a $BCL-$subalgebra of the power set of X. Hence, $(H,C)=(F,A)\prod(G,B)$ is a soft $BCL-$algebra of the power set of X. Therefore, the intersection of two soft $BCL-$algebras is a soft $BCL-$algebra.

Example 3.19: Consider the algebra in example (3.16) with $A=\{\phi,\{2\},\{2,3\}\}$ and $B=\{\phi,\{1\}\}$. Then $H(\phi)=F(\phi)=\{\phi,\{1,2\}\}$ or $H(\phi)=G(\phi)=\{\phi,\{1,3\}\}$. Note that both are $BCL-$subalgebras of the power set of X. Hence, (H,C) is a soft $BCL-$algebras of the power set of X.

Theorem 3.20: Let $(P(X),*,f)$ be a $BCL-$algebra of the power set of X with the condition $f*h=f$ for any $h\in P(X)$. If (F,A) is a soft $BCL-$algebra of the power set of X, then (F,A) is a soft $d-$algebra of the power set of X.

Proof: Straightforward from Definitions [(3.6), (2.7)] and remark [(2)-(3.5)].

Theorem 3.21: If $(P(X),*,f)$ is a d-algebra of the power set of X with the condition $(A*B)*C=(A*C)*B$ and (F,A) is a soft d-algebra of the power set of X. Then (F,A) is a soft $BCL-$algebra of the power set of X.

Proof: Straightforward from Definitions [(3.6) ,(2.7)] and Lemma[(3.13)].
From [14, Theorem 2.1] and [1, Theorem 3.5] we can give the following theorems (the proves are direct).

Theorem 3.22: If $(P(X),*,f)$ is a $BCK-$algebra of the power set of X and (F,A) is a soft $BCK-$algebra of the power set of X. Then (F,A) is a soft $BCL-$algebra of the power set of X.

Theorem 3.23: If $(P(X),*,f)$ is a $BCL-$algebra of the power set of X. If $f*h=f$ and $k*h=h*g$, for any $h,k,g\in P(X)$, then $(P(X),*,f)$ is a $BCK-$algebra of the power set of X.

Theorem 3.24: If $(P(X),*,f)$ is a $BCL-$algebra of the power set of X and (F,A) is a soft $BCL-$algebra of the power set of, then (F,A) is a soft $BCK-$algebra of the power set of X.

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