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Extended Conditions for the Consequences of a Perfect Isometry

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Abstract

In an important paper [1], M. Broué defines the fundamental concept of a perfect isometry between two p-blocks f and e of finite groups H and G that posits numerical conditions on a generalized character of $G \times H$. Then, in one of the fundamental results of [1], he shows, in [1, Théorème 1.5], that such a perfect isometry implies several equivalences between basic invariants of the blocks f and e. In this paper, we extend the conditions for a perfect isometry (that includes Broué's definition) and demonstrate that all of the conclusions of [1, Théorème 1.5] hold in our more general setting. We seek applications of these results.

Mathematics Subject Classification: 20C20

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1 Introduction and Statements of Results

The purpose of this paper is to present an extension of M. Broué's concept of a perfect isometry between two p-blocks f and e of finite groups H and G that includes Broué's definition and to demonstrate that all of the equivalences between basic invariants of f and e of [1, Théorème 1.5] can be proved under our more general hypotheses using the methods of [1].

This section contains the statements of our results. The required proofs are given in Section 2. In our setting the methods of [1] yield the required proofs.

Throughout this section G will be a finite group, p will be a prime integer and all rings will have identities. Our notation and terminology are standard and tend to follow [1], [2] and [3]. For example, $ccl(G)$ denotes the set of conjugacy classes of G.

Let R be a subring of the commutative ring S (with $1_S \in R$).

We shall use, without explicit reference, the easy to prove:

Lemma 1.1. *Let e be an idempotent of RG. Then $(RG)e = (RG) \cap ((SG)e)$.*

As usual, $F(G, R)$ denotes the set of functions from G to R and via R-linearity we shall identify $F(G, R)$ with $Hom_R(RG, R)$.

For each subset X of G, let $\delta_X \in F(G, R)$ denote the characteristic function of X so that:

$$\delta_X(g) = \begin{cases} 0 \text{ if } g \notin X \\ 1 \text{ if } g \in X. \end{cases}$$

If $g \in G$ and $X = \{g\}$, set $\delta_g = \delta_X$.

Let $A_G\colon RG \to Hom_R(RG, R)$ denote the R-linear homomorphism such that $g \mapsto \delta_{g^{-1}}$ for all $g \in G$ and let $B_G\colon Hom_R(RG, R) \to RG$ denote the R-linear homomorphism such that $\delta_g \mapsto g^{-1}$ for all $g \in G$.

Let $\alpha \in Aut(G)$. Clearly $\alpha \in Aut_R(RG)$ in the usual way and $\alpha \in Aut_R(Hom_R(RG, R))$ where $(\alpha f)(g) = f(\alpha^{-1}(g))$ for all $g \in G$ and all $f \in Hom_R(RG, R)$. Thus $\alpha\delta_g = \delta_{\alpha(g)}$ for all $g \in G$.

As usual, RG is a left RG-module and $Hom_R(RG, R)$ is a left RG-module where, if $\varphi \in RG$, $f \in Hom_R(RG, R)$ and $\gamma \in RG$, then $(\varphi f)(\gamma) = f(\gamma\varphi)$. Thus, if $g, h \in G$, then $g\delta_h = \delta_{hg^{-1}}$.

One can easily prove:

Lemma 1.2. *(a) A_G and B_G are inverse R-linear isomorphisms;*

(b) If $\alpha \in Aut(G)$, $u \in RG$ and $f \in Hom_R(RG, R)$, then $(\alpha \cdot A_G)(u) = A_G(\alpha(u))$ and $B_G(\alpha f) = \alpha B_G(f)$; and

(c) If $u, v \in RG$ and $f \in Hom_R(RG, R)$, then $A_G(vu) = vA_G(u)$ and $B_G(vf) = vB_G(f)$.

Let $CF(G, R) = \{f \in F(G, R) \mid f \text{ is constant on the elements of } \mathcal{C} \text{ for each } \mathcal{C} \in ccl(G)\}$.

Hypothesis C. For the finite group G, let K be a finite extension of the rational numbers Q in the complex numbers $\mathbb{C}$ such that K is "large enough" for all subgroups of G and such that K is invariant under complex conjugation. Let $\mathcal{I}$ denote the integral closure of the integers Z in K and let M be a maximal ideal of $\mathcal{I}$ such that $p\mathcal{I} \leq M$. Thus $Z \cap M = pZ$ and $k = \mathcal{I}/M$ is a field of characteristic p. Set $\mathcal{O} = \{\alpha\beta^{-1} \mid \alpha \in \mathcal{I} \text{ and } \beta \in \mathcal{I} - M\}$. Thus $\mathcal{O}$ is a local subring of K, $\mathcal{I} \leq \mathcal{O} \leq \mathrm{K}$, $\mathcal{M} = \{\alpha\beta^{-1} \mid \alpha \in M \text{ and } \beta \in \mathcal{I} - M\}$ is the unique maximal ideal of $\mathcal{O}$. Also $\mathcal{I} \cap \mathcal{M} = M$, $Z \cap \mathcal{M} = pZ$, and K is the field of fractions of $\mathcal{I}$ and of $\mathcal{O}$. As in [3, Chapter 15], the natural ring epimorphism $*\colon \mathcal{I} \to k = \mathcal{I}/M$ extends to the natural ring epimorphism $*\colon \mathcal{O} \to k$ by defining $(\alpha\beta^{-1})^* = \alpha^*(\beta^*)^{-1}$ for all $\alpha \in \mathcal{I}$ and all $\beta \in \mathcal{I} - M$.

Let e be an idempotent of $Z(\mathcal{O}G)$; possibly $e = 1$. Let $\mathcal{I}(G)$ denote the set of irreducible K-characters of G and let $\mathcal{I}(e) = \{\chi \in \mathcal{I}(G) \mid e\chi = \chi\}$. Also let $\mathcal{I}Br(G)$ denote the set of irreducible Brauer characters of G (cf. [3, Chapter 15]) (canonically extended to take the value zero on p-singular elements and let $\mathcal{I}Br(e) = \{\varphi \in \mathcal{I}Br(G) \mid e\varphi = \varphi\}$. Also let $F_{p'}(G,\mathcal{O}) = \{f \in F(G,\mathcal{O}) \mid f(G - G_{p'}) = 0\}$ and similarly let $F_{p'}(G,\mathrm{K}) = \{f \in F(G,\mathrm{K}) \mid f(G - G_{p'}) = 0\}$. As usual $CF(G,\mathcal{O}) = \{f \in F(G,\mathcal{O}) \mid f$ is constant on each $\mathcal{C} \in ccl(G)\}$ and $CF(G,\mathrm{K}) = \{f \in F(G,\mathrm{K}) \mid f$ is constant on each $\mathcal{C} \in ccl(G)$. Set $CF_{p'}(G,\mathcal{O}) = CF(G,\mathcal{O}) \cap F_{p'}(G,\mathcal{O})$ and $CF_{p'}(G,\mathrm{K}) = CF(G,\mathrm{K}) \cap F_{p'}(G,\mathrm{K})$.

For the remainder of this section we adapt the inner product of [3, pp. 20–21] on $CF(G,\mathrm{K})$.

Clearly:

(1.1) $$F(G,\mathcal{O}) = Hom_{\mathcal{O}}(\mathcal{O}G,\mathcal{O}) = \bigoplus_{g\in G}(\mathcal{O}\delta_g) \text{ in } \mathcal{O}\text{-mod};$$

and

$$F(G,\mathrm{K}) = Hom_{\mathrm{K}}(\mathrm{K}G,\mathrm{K}) = \bigoplus_{g\in G}(\mathrm{K}\delta_g).$$

Set $F(G,e,\mathcal{O}) = eHom_{\mathcal{O}}(\mathcal{O}G,\mathcal{O})$, $F(G,e,\mathrm{K}) = eHom_{\mathrm{K}}(\mathrm{K}G,\mathrm{K})$. Similarly set $CF(G,e,\mathcal{O}) = eCF(G,\mathcal{O})$ and $CF(G,e,\mathrm{K}) = eCF(G,\mathrm{K})$.

Thus:

(1.2) $$CF(G,\mathcal{O}) = \bigoplus_{\mathcal{C}\in ccl(G)}(\mathcal{O}\delta_{\mathcal{C}}) \text{ in } \mathcal{O}\text{-mod},\ CF(G,\mathrm{K}) = \bigoplus_{\mathcal{C}\in ccl(G)}(\mathrm{K}\delta_{\mathcal{C}}) = \bigoplus_{\chi\in\mathcal{I}(G)}(\mathrm{K}\chi)$$

and

$$CF(G,e,\mathrm{K}) = \bigoplus_{\chi\in\mathcal{I}(e)}(\mathrm{K}\chi) \text{ in K-mod}.$$

Also set $R_{\mathcal{O}}(G) = \bigoplus_{\chi \in \mathcal{I}(G)} (\mathcal{O}\chi)$ in $\mathcal{O}$-mod and $R_{\mathcal{O}}(G,e) = \bigoplus_{\chi \in \mathcal{I}(e)} (\mathcal{O}\chi)$ in $\mathcal{O}$-mod.

For each $\varphi \in \mathcal{I}Br(G)$, let Φ_φ denote the associated projective indecomposable characters of φ. Thus

(1.3) $$CF_{p'}(G, \mathrm{K}) = \bigoplus_{\varphi \in \mathcal{I}Br(G)} (\mathrm{K}\varphi) = \bigoplus_{\varphi \in \mathcal{I}Br(G)} (\mathrm{K}\Phi_\varphi) \text{ in K-mod, and}$$
$$CF_{p'}(G, e, \mathrm{K}) = \bigoplus_{\varphi \in \mathcal{I}Br(e)} (\mathrm{K}\varphi) = \bigoplus_{\varphi \in \mathcal{I}Br(e)} (\mathrm{K}\Phi_\varphi) \text{ in K-mod.}$$

(cf. [2, IV, Lemma 3.4]).

Lemma 1.3.

$$CF_{p'}(G, \mathcal{O}) = \bigoplus_{\varphi \in \mathcal{I}Br(G)} (\mathcal{O}\varphi) \text{ in } \mathcal{O}\text{-mod},$$

and

$$CF_{p'}(G, e, \mathcal{O}) = \bigoplus_{\varphi \in \mathcal{I}Br(e)} (\mathcal{O}\varphi) \text{ in } \mathcal{O}\text{-mod}.$$

Definition 1.4. $CF^{pr}(G,e,\mathcal{O}) = \{\pi \in CF(G,e,\mathrm{K}) \mid [\pi,\alpha]_G \in \mathcal{O}$ for all $\alpha \in CF(G,e,\mathcal{O})\}$ and $CF^{pr}_{p'}(G,e,\mathcal{O}) = CF^{pr}(G,e,\mathcal{O}) \cap F_{p'}(G,\mathrm{K})$.

Lemma 1.5. *Let $\pi \in CF(G,e,\mathrm{K})$. Then the following four conditions are equivalent:*

(a) $\pi \in CF^{pr}(G,e,\mathcal{O})$;

(b) $[\pi,\alpha]_G \in \mathcal{O}$ for all $\alpha \in CF(G,\mathcal{O})$;

(c) $[\pi,\delta_{\mathcal{C}}] \in \mathcal{O}$ for all $\mathcal{C} \in ccl(G)$; and

(d) $\dfrac{\pi(g))}{|(C_G(g)|} \in \mathcal{O}$ for all $g \in G$.

Consequently $CF^{pr}(G,e,\mathcal{O})$ is an $\mathcal{O}$-submodule of $CF(G,e,\mathcal{O})$ and $CF^{pr}_{p'}(G,e,\mathcal{O})$ is an $\mathcal{O}$-submodule of $CF_{p'}(G,e,\mathcal{O})$.

Lemma 1.6. $CF^{pr}_{p'}(G,e,\mathcal{O}) = \bigoplus_{\varphi \in \mathcal{I}Br(e)} (\mathcal{O}\Phi_\varphi)$.

Let H also be a finite group.

For the remainder of this section, assume Hypothesis C for $H \times G$, let f be and idemotent of $Z(\mathcal{O}H)$ and let e be an idempotent of $Z(\mathcal{O}G)$. Thus $f \otimes_{\mathcal{O}} e$

is an idempotent of $\mathcal{O}(H \times G)$, $\mathcal{I}(f \otimes e) = \{\psi\chi \mid \psi \in \mathcal{I}(f) \text{ and } \chi \in \mathcal{I}(e)\}$ and the results above hold for $H \times G$, K, $\mathcal{O}$, $f \otimes_{\mathcal{O}} e$, etc.

As in [1, Section 1], each

$$\mu \in CF(H \times G, f \otimes_{\mathcal{O}} e, \mathrm{K}) = \bigoplus_{\substack{\chi \in \mathcal{I}(f) \\ \chi \in \mathcal{I}(e)}} (\mathrm{K}\psi\chi)$$

determines unique K-linear homomorphisms

$$I_\mu \colon CF(H, f, \mathrm{K}) \to CF(G, e, \mathrm{K})$$

and $R_\mu \colon CF(G, e, \mathrm{K}) \to CF(H, f, \mathrm{K})$ where if $\psi \in \mathcal{I}(f)$, then

$$I_\mu(\psi)(g) = \frac{1}{|H|} \sum_{h \in H} (\mu(g, h^{-1})\psi(h))$$

for all $g \in G$ and if $\chi \in \mathcal{I}(e)$, then

$$R_\mu(\chi)(h) = \frac{1}{|G|} \sum_{g \in G} (\mu(g^{-1}, h)\chi(g))$$

for all $h \in H$.

Analogously, each $\mu \in CF(H \times G, f \otimes_{\mathcal{O}} e, \mathrm{K})$ determines unique K-linear homomorphisms

$$\begin{aligned} I^0_\mu &\colon Z((\mathrm{K}H)f) \to Z((\mathrm{K}G)e) \text{ and} \\ R^0_\mu &\colon Z((\mathrm{K}G)e) \to Z((\mathrm{K}H)f) \text{ such that} \end{aligned}$$

if $\alpha = \sum_{h \in H} y(h)h \in Z((\mathrm{K}H)f)$, then

$$I^0_\mu(\alpha) = \sum_{g \in G} \left(\frac{1}{|H|} \left(\sum_{h \in H} \mu(g^{-1}, h) y(h) \right) g \right)$$

and if $\beta = \sum_{g \in G} x(g) g \in Z((\mathrm{K}G)e)$, then

$$R^0_\mu(\beta) = \sum_{h \in H} \left(\frac{1}{|G|} \left(\sum_{g \in G} \mu(g, h^{-1}) x(g) \right) h \right).$$

The K-linear isomorphism $A_H \colon F(H, \mathrm{K}) \to \mathrm{K}\mathrm{K}$ such that $\delta_h \mapsto h^{-1}$ where δ_h is the characteristic function of $\{h\}$ on H for all $h \in H$ induces a K-linear isomorphism $A_H \colon CF(H, f, \mathrm{K}) \to Z((\mathrm{K}H)f)$ and similarly we have the K-linear isomorphism $A_G \colon CF(G, e, \mathrm{K}) \to Z((\mathrm{K}G)e)$.

It is easy to see that the following diagram of K-linear homomorphisms commutes:

$$(1.4)\qquad \begin{array}{ccc} CF(H,f,\mathrm{K}) & \xrightarrow{A_H} & Z((\mathrm{K}H)f) \\ {\scriptstyle I_\mu}\downarrow & & \downarrow{\scriptstyle I^0_\mu} \\ CF(G,e,\mathrm{K}) & \xrightarrow{A_G} & Z((\mathrm{K}G)e) \\ {\scriptstyle R_\mu}\downarrow & & \downarrow{\scriptstyle R^0_\mu} \\ CF(H,f,\mathrm{K}) & \xrightarrow{A_H} & Z((\mathrm{K}H)f) \end{array}$$

The next two lemmas restate [1, Proposition 4.1]:

Lemma 1.7. *The following two conditions are equivalent:*

(a) $I_\mu(CF(H,f,\mathcal{O})) \leq CF(G,e,\mathcal{O})$*; and*

(b) $\dfrac{\mu(g,h)}{|C_H(h)|} \in \mathcal{O}$ *for all* $g \in G$ *and* $h \in H$.

Lemma 1.8. *The following two conditions are equivalent:*

(a) $I_\mu(CF_{p'}(H,f,\mathrm{K})) \leq CF_{p'}(G,e,\mathrm{K})$*; and*

(b) if $g \in G$, $h \in H_{p'}$ *and* $\mu(g,h) \neq 0$, *then* $g \in G_{p'}$.

For the remainder of this section we assume that J is a bijection $J\colon \mathcal{I}(f) \to \mathcal{I}(e)$ and that for each $\psi \in \mathcal{I}(f)$ an element $\epsilon_\psi \in \mathcal{O}^\times$ has been chosen such that $\epsilon_\psi\overline{\epsilon}_\psi = 1$.

Set $\mu = \sum_{x\in\mathcal{I}(f)} \epsilon_\psi(\psi J(\psi)$ and $\mu = \sum_{\psi\in\mathcal{I}(f)} \overline{\epsilon}_\psi \psi J(\psi)$. Thus μ and ω are elements of $R_{\mathcal{O}}(G \times H)$,

$$\begin{aligned} I_\mu &\colon CF(H,f,\mathrm{K}) \to CF(G,e,\mathrm{K}), \\ I_\omega &\colon CF(H,f,\mathrm{K}) \to CF(G,e,\mathrm{K}), \\ R_\mu &\colon CF(G,e,\mathrm{K}) \to CF(H,f,\mathrm{K}), \text{ and} \\ R_\omega &\colon CF(G,e,\mathrm{K}) \to CF(H,f,\mathrm{K}) \end{aligned}$$

are K-linear isomorphisms that are isometries such that $I_\mu(\psi) = \epsilon_\psi J(e)$, $I_\omega(\psi) = \overline{\epsilon}_\psi J(\psi)$, $R_\mu(J(\psi)) = \epsilon_\psi\psi$ and $R_\omega(J(\psi)) = \overline{\epsilon}_\psi\psi$ for all $\psi \in \mathcal{I}(f)$.

Thus we clearly have:

Lemma 1.9. *(a)* (I_μ, R_ω) *and* (I_ω, R_μ) *are adjoint pairs of inverse* K*-linear isometries;*

(b) $I_\mu(CF(H,f,\mathrm{K})) = CF(G,e,\mathrm{K})$,
$R_\mu(CF(G,e,\mathrm{K})) = CF(H,f,\mathrm{K})$,
$I_\omega(CF(H,f,\mathrm{K})) = CF(G,e,\mathrm{K})$ *and*
$R_\omega(CF(G,e,\mathrm{K})) = CF(H,f,\mathrm{K})$; *and*

(c) $I_\mu(R_{\mathcal{O}}(H,f)) = R_{\mathcal{O}}(G,e)$,
$R_\mu(G,e) = R_{\mathcal{O}}(H,f)$,
$I_\omega(R_{\mathcal{O}}(H,f)) = R_{\mathcal{O}}(G,e)$ *and*
$R_\omega(G,e) = R_{\mathcal{O}}(H,f)$.

For the remainder of this section, we assume:

Hypothesis PI. $\{\epsilon_\psi \mid \psi \in \mathcal{I}(f)\}$ has been chosen such that:

$$I_\mu\colon CF(H,f,\mathrm{K}) \to CF(G,e,\mathrm{K}) \text{ and}$$
$$R_\omega\colon CF(G,e,\mathrm{K}) \to CF(H,f,\mathrm{K})$$

satisfy both equivalent conditions of Lemmas 1.7 and 1.8.

Theorem 1.10. *(a)* $I_\mu(CF(H,f,\mathcal{O})) = CF(G,e,\mathcal{O})$ *and*
$R_\omega(CF(G,e,\mathcal{O})) = CF(H,f,\mathcal{O})$;

(b) $I_\mu(CF_{p'}(H,f,\mathrm{K})) = CF_{p'}(G,e,\mathrm{K})$ *and*
$R_\omega(CF_{p'}(G,e,\mathrm{K})) = CF_{p'}(H,f,\mathrm{K})$;

(c) $I_\mu(CF_{p'}(H,f,\mathcal{O})) = CF_{p'}(G,e,\mathcal{O})$ *and*
$R_\omega(CF_{p'}(G,e,\mathcal{O})) = CF_{p'}(H,f,\mathcal{O})$;

(d) $I_\mu(CF^{pr}(H,f,\mathcal{O})) = CF^{pr}(G,e,\mathcal{O})$ *and*
$R_\omega(CF^{pr}(G,e,\mathcal{O})) = CF^{pr}(H,f,\mathcal{O})$; *and*

(e) $I_\mu(CF^{pr}_{p'}(H,f,\mathcal{O})) = CF^{pr}_{p'}(G,e,\mathcal{O})$ *and*
$R_\omega(CF^{pr}_{p'}(G,e,\mathcal{O})) = CF_{p'}(H,f,\mathcal{O})$.

Remark 1.11. If $\epsilon_\psi = \pm 1$ for all $\psi \in \mathcal{I}(f)$, then $\mu = \omega$ and I_μ is a perfect isometry in the sense of [1, Definition 1.4].

Let $I^0_\mu\colon \mathrm{K}H \to \mathrm{K}G$ and $R^0_\omega\colon \mathrm{K}G \to \mathrm{K}H$ be the K-linear maps such that:
$I^0_\mu(h) = \sum_{g\in G}\left(\frac{1}{|H|}\mu(g^{-1},h)g\right)$ for all $h \in H$ and
$R^0_\omega(g) = \sum_{h\in H}\left(\frac{1}{|H|}\omega(g,h^{-1})h\right)$ for all $g \in G$ as above.

Theorem 1.12. *(a)* $I^0_\mu(\mathrm{K}H) \leq Z(\mathrm{K}G)$ *and* $R^0_\omega(\mathrm{K}G) \leq Z(\mathrm{K}H)$;

(b) *If* $h \in H$*, then*

$$\begin{aligned} I_\mu^0(h) &= I_\mu^0(fh) \\ &= \frac{|G|}{|H|} \sum_{\psi \in I(f)} \left(\frac{\epsilon_\psi \psi(h)}{J(\psi)(1)} e_{J(\psi)} \right) \\ &= I_\mu^0(h)e \end{aligned}$$

and if $g \in G$*, then*

$$\begin{aligned} R_\omega^0(g) &= R_\omega^0(eg) \\ &= \frac{|H|}{|G|} \sum_{\psi \in \mathcal{I}(f)} \left(\bar{\epsilon}_\psi \frac{J(\psi)(g)}{\psi(1)} e_\psi \right) \\ &= R_\omega^0(g)f; \end{aligned}$$

(c) *If* $\psi \in \mathcal{I}(f)$*, then*

$$I_\mu^0(e_\psi) = \epsilon_\psi \frac{\psi(1)}{|H|} \frac{|G|}{J(\psi)(1)} e_{J(\psi)}$$

and

$$R_\omega^0(e_{J(\psi)}) = \bar{\epsilon}_\psi \frac{|H|}{\psi(1)} \frac{J(\psi)(1)}{|G|} e_\psi;$$

(d) $I_\mu^0 \colon Z((\mathrm{K}H)f) \to Z((\mathrm{K}G)e)$ *and* $R_\omega^0 \colon Z((\mathrm{K}G)e) \to Z((\mathrm{K}H)f)$ *are* K*-linear inverse maps;*

(e) $I_\mu^0 \colon Z((\mathcal{O}H)f) \to Z((\mathcal{O}G)e)$ *and* $R_\omega^0 \colon Z((\mathcal{O}G)e) \to Z((\mathcal{O}H)f)$ *are* $\mathcal{O}$*-linear inverse maps;*

(f) $\sigma_\mu = R_\omega^0(e)I_\mu^0 \colon Z((\mathrm{K}H)f) \to Z((\mathrm{K}G)e)$ *is a* K*-linear map such that* $\sigma_\mu(e_\psi) = e_{J(\psi)}$ *for all* $\psi \in \mathcal{I}(f)$ *and*

$$\tau_\omega = I_\mu^0(f)R_\omega^0 \colon Z((\mathrm{K}G)e) \to Z(KHf)$$

is a K*-linear map such that* $\tau_\omega(e_{J(\psi)}) = e_\psi$ *for all* $\psi \in \mathcal{I}(f)$*. Thus* σ_μ *and* t_ω *are inverse* K*-algebra isomorphisms; and*

(g) $\sigma_\mu \colon Z((\mathcal{O}H)f) \to Z((\mathcal{O}G)e)$ *and* $I_\omega \colon Z(\mathcal{O}Ge) \to Z((\mathcal{O}H)f)$ *are inverse* $\mathcal{O}$*-algebra isomorphisms.*

Remark 1.13. Let $R \in \{\mathcal{O}, \mathrm{K}\}$. Then the following diagram of R-linear isomorphisms commutes:

$$\begin{array}{ccc}
fCF(H,R) = CF(H,f,R) & \xrightarrow{A_H} & Z((RH)f) = Z(RH)f \\
\downarrow{\scriptstyle I_\mu} & & \downarrow{\scriptstyle I^0_\mu} \\
eCF(G,R) = CF(G,e,R) & \xrightarrow{A_G} & Z((RG)e) = Z(RG)e \\
\downarrow{\scriptstyle R_\omega} & & \downarrow{\scriptstyle R^0_\omega} \\
fCF(H,R) = CF(H,f,R) & \xrightarrow{A_H} & Z((RH)f) = Z(RH)f
\end{array}$$

By Theorem 1.12(g), there is an $\alpha \in Z((\mathcal{O}H)f)$ such that

$$I^0_\mu(\alpha R^0_\omega(e)) = I^0_\mu(f).$$

Thus $\alpha R^0_\omega(e) = f$ and $R^0_\omega(e)$ is an invertible element of $Z((\mathcal{O}H)f)$. Let $\beta \in B\ell((\mathcal{O}H)f)$ and let $\psi \in \mathcal{I}(\beta)$. Then $\omega_\psi \colon Z(\mathcal{O}H) \to \mathcal{O}$ is an $\mathcal{O}$-algebra homomorphism such that $\omega_\psi(f) = 1$. Thus $\omega_\psi(\alpha)\omega_\psi(R^0_\omega(e)) = 1$ where $\omega_\psi(R^0_\omega(e)) = \overline{\epsilon}_\psi \dfrac{|H|}{\psi(1)} \dfrac{J(\psi)(1)}{|G|} \in \mathcal{O}^\times$. Similarly, since $J(\psi) \in \sigma_\mu(\beta) \in B\ell((\mathcal{O}G)e)$, we have $\epsilon_\psi \dfrac{|G|}{J(\psi)(1)} \dfrac{\psi(1)}{|G|} \in \mathcal{O}^\times$.

Lemma 1.14. *Let $\beta \in B\ell((\mathcal{O}H)f)$ and let $\psi \in \mathcal{I}(\beta)$. Then:*

(a) β and $\sigma_\mu(\beta)$ have the same defect;

(b) ψ and $J(\psi)$ have the same height; and

(c) if $\psi^* \in \mathcal{I}(\beta)$ also, then

$$\overline{\epsilon}_\psi \frac{|H|}{\psi(1)} \frac{\overline{J}(\psi)(1)}{|G|} \equiv \overline{\epsilon}_{\psi^*} \frac{|H|}{\psi^*(1)} \frac{(J(\psi^*)(1)}{|G|} \quad (mod\ \mathcal{M})$$

Corollary 1.15. *Let $\beta \in B\ell((\mathcal{O}H)f)$, so that $\sigma_\mu(\beta) \in B\ell((\mathcal{O}G)e)$. Then:*

(a) *if $\psi \in \mathcal{I}(\beta)$, then $J(\psi) \in \mathcal{I}(\sigma_\mu(\beta))$; and*

(b) *if $\varphi \in \mathcal{I}Br(\beta)$ and Φ_φ is the associated indecomposable projective character of φ, then $I_\mu(\varphi) \in CF_{p'}(G, \sigma_\mu(\beta), \mathcal{O})$ and $I_\mu(\Phi_\varphi) \in CF^{pr}_{p'}(G, \sigma_\mu(\beta), \mathcal{O})$.*

From the above, we clearly have:

Corollary 1.16. *All of the conclusions of [1, Théorème 1.5 and Lemme 1.6] hold for f and e.*

2 Required Proofs

A Proof of Lemma 1.3. Clearly

$$\bigoplus_{\varphi\in\mathcal{I}B\ell(G)}(\mathcal{O}\varphi) \le CF_{p'}(G,\mathcal{O}) = \bigoplus_{\substack{\mathcal{C}\in ccl(\mathcal{O})\\ \mathcal{C}\subseteq G_{p'}}}(\mathcal{O}\delta_{\mathcal{C}})$$

$$\le CF_{p'}(G,\mathrm{K}) = \bigoplus_{\varphi\in\mathcal{I}Br(e)}(\mathrm{K}\varphi).$$

Fix $\mathcal{C}\in ccl(G)$ with $\mathcal{C}\subseteq G_{p'}$ and let $g\in\mathcal{C}$. Here $\delta_{\mathcal{C}} = \sum_{\varphi\in\mathcal{I}Br(G)} k_\varphi\varphi$ for unique $k_\varphi\in\mathrm{K}$ for all $\varphi\in\mathcal{I}Br((G)$. Thus $k_\varphi = [\Phi_\varphi,\delta_{\mathcal{C}}]_G = \frac{1}{|G|}\Phi_\varphi(g)|\mathcal{C}| = \frac{\Phi_\varphi(g)}{|C_G(g)|}\in \mathcal{O}$ and we are done.

A Proof of Lemma 1.5. Let $\pi\in CF(G,e,\mathrm{K})$. If $\mathcal{C}\in ccl(G)$ and $g\in\mathcal{C}$, then $[\pi,\delta_{\mathcal{C}}]_G = \frac{\pi(g)}{|C_G(g)|}$. Thus (b), (c) and (d) are equivalent and (b) implies (a). Assume (a) and let $\alpha\in CF(G,\mathcal{O})$. Then $\alpha = e\alpha + (1-e)\alpha$ and so $[\pi,\alpha]_G = [\pi,e\alpha]_G\in\mathcal{O}$. Thus (b) holds and we are done.

A Proof of Lemma 1.6. Clearly $\bigoplus_{\varphi\in\mathcal{I}Br(e)}(\mathcal{O}\Phi_\varphi)\le CF^{pr}_{p'}(G,e,\mathcal{O})$. Let $\pi\in CF^{pr}_{p'}(G,e,\mathcal{O})$. Thus $\pi = \sum_{\varphi\in\mathcal{I}Br(e)} k_\varphi\Phi_\varphi$ for unique $k_\varphi\in\mathrm{K}$ for each $\varphi\in\mathcal{I}Br(e)$. Consequently for $\varphi\in\mathcal{I}Br(e)$, $[\pi,\varphi]_G = k_\varphi\in\mathcal{O}$ and we are done.

A Proof of Lemma 1.7. Here $CF(H,f,\mathcal{O}) = f\left(\bigoplus_{\mathcal{C}\in ccl(H)}(\mathcal{O}\delta_{\mathcal{C}})\right)$. Let $h\in\mathcal{C}\in ccl(H)$ and let $g\in G$. Then $I_\mu(\delta_{\mathcal{C}})(g) = \frac{1}{|H|}\mu(g,h^{-1})|\mathcal{C}| = \frac{\mu(g,h^{-1})}{|C_G(h^{-1})|}$ and our proof is complete.

A Proof of Lemma 1.8. Let $g\in G$ and $h\in\mathcal{C}\in ccl(G)$ with $\mathcal{C}\subseteq H_{p'}$. Then $I_\mu(\delta_p)(g) = \frac{\alpha(g,h^{-1})}{|C_H(h^{-1})|}$. Assume (a) and $\mu(g,h^{-1})\ne 0$. Then $g\in G_{p'}$ and (b) holds. The converse is clear and we are done.

A Proof of Theorem 1.10. Here $I_\mu(CF(H,f,\mathcal{O}))\le CF(G,e,\mathcal{O})$. Thus $CF(H,f,\mathcal{O})\le R_\omega(CF(G,e,\mathcal{O}))\le CF(H,f,\mathcal{O})$ and (a) holds. Similarly (b) follows and (c) is a consequence of (a) and (b). Let $\pi\in CF^{pr}(H,f,\mathcal{O})$ and let $\alpha\in CF(G,e,\mathcal{O})$. Then

$$[I_\mu(\pi),\alpha]_G = [\pi,R_\omega(\alpha)]_H\in\mathcal{O}$$

and so $I_\mu(CF^{pr}(H,f,\mathcal{O}))\le CF^{pr}(G,e,\mathcal{O})$. Thus (d) is proved, (e) follows from (c) and (d) and we are done.

A Proof of Theorem 1.12. Here (a) is clear and if $h \in H$,

$$I^0_\mu(h) = \sum_{g\in G} \frac{1}{|H|} \sum_{\psi\in\mathcal{I}(f)} (\epsilon_\psi \psi(h) J(\psi)(g^{-1})g) = \frac{|G|}{|H|} \sum_{\psi\in\mathcal{I}(f)} \left(\frac{\epsilon_\psi\psi(h)}{J(\psi)(1)} e_{J(\psi)}\right).$$

Thus (b) is proved and (c) follows similarly. Clearly (c) implies (d). If $h \in \mathcal{C} \in ccl(H)$, then

$$\begin{aligned} I^0_\mu(\sum_{u\in\mathcal{C}} u) &= \frac{1}{|H|} \sum_{g\in G} (\mu(g^{-1},h)|\mathcal{C}|g) \\ &= \sum_{g\in G} \left(\frac{\mu(g^{-1},h)}{|C_H(h)|} g\right) \in Z((\mathcal{O}G)e). \end{aligned}$$

Thus (e), (f) and (g) hold and we are done.

A Proof of Lemma 1.14. Here (a) and (b) follow as on [1, page 65] and, since $\omega_\psi(\alpha) \equiv \omega_{\psi^*}(\alpha) \pmod{\mathcal{M}}$, (c) follows.

A Proof of Corollary 1.15. Let $\psi \in \mathcal{I}(\beta)$. Then

$$A_H(\psi) = A_H(\beta\psi) = \beta A_H(\psi) = \frac{|H|}{\psi(1)} e_\psi = \beta \frac{|H|}{\psi(1)} e_\psi.$$

Thus

$$\sigma_\mu(A_H(\psi)) = \sigma_\mu(\beta)\sigma_\mu(A_H(\psi)) = \frac{|H|}{\psi(1)} e_{J(\psi)} = \sigma_\mu(\beta) \frac{|H|}{\psi(1)} e_{J(\psi)}.$$

Thus $\sigma_\mu(\beta) e_{J(\psi)} = e_{J(\psi)}$ and (a) follows. Let $\varphi \in \mathcal{I}Br(\beta)$. Thus $I_\mu(\Phi_\varphi) \in CF^{pr}_{p'}(G, \sigma_\mu(\beta), \mathcal{O})$ and $\Phi_\varphi = \sum_{\psi\in\mathcal{I}Br(\beta)} c_\psi \psi$ for unique $c_\psi \in Z$ with $c_\psi \geq 0$ for all $\psi \in \mathcal{I}Br(\beta)$. Since the Cartan matrix of β is invertible, $I_\mu(\varphi) \in CF_{p'}(G, e, \mathcal{O}) \cap CF_{p'}(G, \sigma_\mu(\beta), \mathrm{K}) = CF_{p'}(G, \sigma_\mu(\beta), \mathcal{O})$ and we are done.

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