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Prime Ideals in B-Algebras

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Abstract

In this paper, we introduce the definition of ideal in B-algebra and some relate properties. Also, we introduce the definition of prime ideal in B-algebra and we obtain some of its properties.

Keywords: B-algebras, B-subalgebras, ideal, prime ideal

1 Introduction

In 1996, Y. Imai and K. Iseki introduced a new algebraic structure called BCK-algebra. In the same year, K. Iseki introduced the new idea be called BCI-algebra, which is generalization from BCK-algebra. In 2002, J. Neggers and H. S. Kim [9] constructed a new algebraic structure, they took some properties from BCI and BCK-algebra be called B-algebra. A non-empty set X with a binary operation $*$ and a constant 0 satisfying some axioms will construct an algebraic structure be called B-algebra.

The concepts of B-algebra have been disscussed, e.g., *a note on normal subalgebras in B-algebras* by A. Walendziak in 2005, *Direct Product of B-algebras* by Lingcong and Endam in 2016, and *Lagrange's Theorem for B-algebras* by JS. Bantug in 2017. Earlier, in 2010, N. O. Al-Shehrie [1] applied the notion of left-right derivation in BCI-algebra to B-algebra and obtained some related properties. Then, in 2012, the new definition of prime ideal was introduced by R.A. Borzooei and O. Zahiri in this paper entitled *"Prime Ideals in BCI and BCK-algebras"*. They found a new definition of prime ideal in BCI-algebra and some of its properties. Furthermore, we apply the concept of prime ideals in BCI-algebra to B-algebra and investigate some its properties.

2 Preliminaries

We begin with some definitions and some theorems of B-algebra and BCI-algebra.

Definition 2.1 *[9] A B-algebra is a non-empty set X with a constant 0 as identity element and a binary operation $*$ satisfying the following axioms:*

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B3) $(x * y) * z = x * (z * (0 * y))$,

for all $x, y, z \in X$.

Example 2.2 *[9] Let X={0, 1, 2} be a set with Cayley table as follows:*

Table 1: *Cayley* table for $(X; *, 0)$

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

As known 0 $\in X$ *and* $1, 2 \in X$*, it can be seen from the Table 1 above that* $2 * 0 = 2$, $2 * 2 = 0$ *and* $(1 * 2) * 0 = 1 * (0 * (0 * 2)) = 2$ *that satisfying the all axioms B-algebra. Then,* $(X; *, 0)$ *is a B-algebra.*

Definition 2.3 *[2]A nonempty subset S of B-algebra X is called a subalgebra (B-subalgebra) of X if $0 \in S$ and $a * b \in S$, for all $a, b \in S$.*

Definition 2.4 *[9] A B-algebra $(X; *, 0)$ is said to be commutative B-algebra if $a * (0 * b) = b * (0 * a)$, for any $a, b \in X$.*

Example 2.5 *Let $X = \{0, 1, 2\}$ be a set with the table on Example 2.2. Then $(X; *, 0)$ is a commutative B-algebra (see [9]).*

Definition 2.6 *[5] A BCI-algebra is an algebra $(X; *, 0)$ satisfying the following axioms:*

(BCI1) $((x * y) * (x * z) * (z * y)) = 0$,

(BCI2) $(x * (x * y)) * y = 0$,

(BCI3) $x * 0 = x$,

(BCI4) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,

for all $x, y, z \in X$.

BCI-algebra X called BCK-algebra if satisfying $0 * x = 0$, for all $x \in X$. A non-empty subset S of BCI-algebra $(X; *, 0)$ is called a subalgebra of X if $x * y \in S$, for any $x, y \in S$. The set P=$\{x \in X \mid 0 * (0 * x) = x\}$ is called P-semisimple part of BCI-algebra X and X is called a P-semisimple BCI-algebra if $P = X$ (see [3]).

Definition 2.7 *[3] Let* I *be a nonempty subset of* BCI*-algebra* X *containing 0.* I *is called an ideal of* X *if* $y * x \in I$ *and* $x \in I$ *imply* $y \in I$*, for any* x,$y \in X$.

Definition 2.8 *[3] A proper ideal* I *of BCI-algebra* X *is called prime if* $A \cap B \subseteq I$ *implies* $A \subseteq I$ *or* $B \subseteq I$*, for all ideals* A *and* B *of X.*

Theorem 2.9 *[6] Every a commutative B-algebra is a BCI-algebra.*

The converse of this theorem may not true in general.

Theorem 2.10 *[6] Every commutative B-algebra is a P-semisimple BCI-algebra.*

The converse of this theorem is true in general.

3 Main Result

R.A. Borzooei and O. Zahiri have been discussed a new definition of prime ideal in BCI and BCK-algebra. By the similar way, we obtain the definition and some theorems of ideal and prime ideal in B-algebra. Some other similar properties from BCI-algebra as a base of this definition.

Definition 3.1 *A non-empty subset* S *of* B*-algebra* X *is called ideal of* X *if*

(i) $0 \in S$*, and*

(ii) $a \in S$ *and* $b * a \in S$*, implies* $b \in S$*, for any* $a, b \in X$.

Table 2: *Cayley* table for $(X; *, 0)$

*	0	1	a	b	c	d
0	0	a	1	b	c	d
1	1	0	a	c	d	b
a	a	1	0	d	b	c
b	b	c	d	0	a	1
c	c	d	b	1	0	a
d	d	b	c	a	1	0

Clearly, $\{0\}$ is an ideal of B-algebra X. An ideal S called *proper* ideal if $S \neq X$ and is called *closed* ideal if $a * b \in S$, for all $a, b \in S$. The least ideal of X containing I, the *generated* ideal of X by I and is denoted by $\langle I \rangle$.

Example 3.2 *Let* $X = \{0, 1, a, b, c, d\}$ *is* B*-algebra with Cayley table as follows:*

Let $S = \{0, 1, a\}$, *so* $0 \in S$. *From the Table 2, we have* $1 \in S$ *and* $a*1 = 1 \in S$, *so* $a \in S$. *Hence,* S *is an ideal of* X *and since* $1 * a = a \in S$, *then* S *a* B*-subalgebra.*

In the next example, we will be checked, are every ideals in B-algebra is B-subalgebra?.

Example 3.3 *Let* $X = (Z; -, 0)$ *with "*$-$*" subtraction operation of integers* Z. *Then, it is easy to prove that* X *is* B*-algebra. Let* $I = Z^+ \cup \{0\}$ *is a subset of* X *with* Z^+ *is positive integers, so* I *is an ideal of* X *and* I *is not a* B*-subalgebra of* X.

So, every ideal in B-algebra is not always B-subalgebra. From the definition of B-subalgebra that is closed of binary operation $*$ and the closed ideal in B-algebra, we obtain that every B-subalgebra is closed ideal in B-algebra.

Definition 3.4 *Let* X *is a* B*-algebra. A proper ideal* M *of* X *is called a maximal ideal of* X *if* $\langle M \cup \{x\} \rangle = X$, *for any* $x \in X$, *where* $\langle M \cup \{x\} \rangle$ *is an ideal generated by* $M \cup \{x\}$. M *is an maximal ideal of* X *if and only if* $M \subseteq A \subseteq X$ *implies that* $M = A$ *or* $A = X$, *for any ideal* A *of* X.

Example 3.5 *Let* $X = \{0, a, b, c\}$ *be a* B*-algebra with Cayley table as follows:*

Table 3: *Cayley* table for $(X;*,0)$

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

We obtain the set of all proper ideal of X is $\{\{0\},\{0,a\},\{0,b\}\}$. Let A=$\langle a\rangle$= $\{0,a\}$ and B=$\langle b\rangle$= $\{0,b\}$, then A and B are maximal ideals of X, because there are no other ideal of X that containing A and B rather than itself.
Let A is an ideal of B-algebra X. Then relation θ is defined $(x,y)\in\theta \Longleftrightarrow x*y, y*x\in A$ is a *congruence* relation on X, denoted A_x for $[x]$=$\{y\in X \mid (x,y)\in\theta\}$. So, A_0 is a closed ideal of B-algebra X. Let X/A=$\{A_x \mid x\in X\}$, then $(X/A;*,A_0)$ is a B-algebra with $A_x * A_y = A_{(x*y)}$, for all $x,y\in X$ (see [3]).

Lemma 3.6 *Let A and B ideals of B-algebra X such that $A\subseteq B$. Denote B/A=$\{A_x|\ x\in B\}$.Then*

(i) $x\in B$ if and only if $A_x\in B/A$, for any $x\in X$,

(ii) B/A=$\{A_x \mid x\in B\}$ is an ideal from X/A.

Definition 3.7 *A proper ideal S of B-algebra X called irreducible ideal of X if $A\cap B=S$ implies $A=S$ or $B=S$, for any ideals A and B of X.*

Definition 3.8 *A proper ideal S of B-algebra X called prime ideal of X if $A\cap B\subseteq S$ implies $A\subseteq S$ or $B\subseteq S$, for all ideals A and B of X.*

Theorem 3.9 *Let S be an ideal of B-algebra X. Then S is a prime ideal of X if and only if $\langle x\rangle\cap\langle y\rangle\subseteq S$ implies $x\in S$ or $y\in S$, for any $x,y\in X$.*

Proof Let S be a prime ideal of X. From Definition 3.8 if A and B are two ideals of X and $A\cap B\subseteq S$ implies $A\subseteq S$ or $B\subseteq S$, its mean if $A=\langle x\rangle$ and $B=\langle y\rangle$ then $\langle x\rangle\subseteq S$. So that, for any $x\in A$ then $x\in S$ or $\langle y\rangle\subseteq S$ and for any $y\in B$ then $y\in S$. Hence, we obtain if S be a prime ideal of X then $\langle x\rangle\cap\langle y\rangle\subseteq S$ implies $x\in S$ or $y\in S$, for any $x,y\in X$. Conversely, Let S be an ideal of X and $\langle x\rangle\cap\langle y\rangle\subseteq S$ implies $x\in S$ or $y\in S$, for any $x,y\in X$. Let A and B are two ideals of X so that $A\cap B\subseteq S$. We assuming $A\not\subseteq S$, there exists $x\in A$ and $x\notin S$. Because for any $y\in B$ then $\langle x\rangle\cap\langle y\rangle\subseteq A\cap B\subseteq S$ and $x\notin S$ then $y\in S$, so that $B\subseteq S$. Therefore, S be a prime ideal of X.

Definition 3.10 *A non-empty subset F of X is called a finite $\cap$-structure, if $(\langle x\rangle \cap \langle y\rangle) \cap F \neq \emptyset$, for all $x, y \in F$ and X is called a finite $\cap$-structure if $X - \{0\}$ is a finite $\cap$-structure.*

Let X and Y are B-algebra. The mapping $f : X \to Y$ is called *homomorphism* of B-algebra or *B-homomorphism* if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. A *B-homomorphism* called *B-monomorphism* if f is *one-to-one* and *B-epimorphism* if f is *onto*. A *B-homomorphism* $f : X \to Y$ called *B-isomorphism* if f is *one-to-one* and *onto* (bijection), and labeled by $X \cong Y$. If $f : X \to Y$ *B-isomorphism* so $f^{(-1)} : Y \to X$ also *B-isomorphism* (see [7]).

Theorem 3.11 *Let X and Y are B-algebra and $f : X \to Y$ B-epimorphism. Then*

(i) An ideal A of X is prime if and only if $F = X - A$ is a finite $\cap$-structure.

(ii) Let A a closed ideal of X and B an ideal of X containing A. If B is a prime ideal of X then B/A is a prime ideal of X/A.

Proof

(i) Let A be a prime ideal of X and $F = X - A$, for any $x, y \in F$. We assume $(\langle x\rangle \cap \langle y\rangle) \cap F = \emptyset$, so $\langle x\rangle \cap \langle y\rangle \subseteq A$. Because A be a prime ideal of X, then $x \in A$ or $y \in A$, while $x, y \in F$. This is impossible, hence be must $(\langle x\rangle \cap \langle y\rangle) \cap F \neq \emptyset$, so that $F = X - A$ is a finite $\cap$-structure. Conversely, if $F = X - A$ is a finite $\cap$-structure and $x, y \in X$, so that $\langle x\rangle \cap \langle y\rangle \subseteq A$. If $x \notin A$ and $y \notin A$ then $x, y \in F$ and $(\langle x\rangle \cap \langle y\rangle) \cap F \neq \emptyset$, we obtained $\langle x\rangle \cap \langle y\rangle\, A$. This is impossible. Therefore, it must be $x \in A$ or $y \in A$ and from Theorem 3.9 concluded that A be a prime idal of X.

(ii) Let A is a closed ideal of X and B is an ideal of X containing A, such that $A \subseteq B$. Since B is a ideal of X, then from Lemma 3.6(ii) B/A is a ideal of X/A. Let I and J are ideals of X/A such that $I \cap J \subseteq B/A$, then there exists K and L be ideals of X, so that $I = K/A$ and $J = L/A$, thus $K/A \cap L/A = (K \cap L)/A \subseteq B/A$. If B is a prime ideals of X then $K \cap L \subseteq B$, so that $K \subseteq B$ or $L \subset B$, then $K/A \subseteq B/A$ or $L/A \subseteq B/A$. Hence, B/A is a prime ideal of X/A.

Corollary 3.12 *Let $x \in X - \{0\}$, such that $x * y = x$, for all $y \in X - \{x\}$. Then, there exists a prime ideal P of X, such that $x \notin P$.*

Proof Let $P = X - \{x\}$, then $0 \in P$. If $b \in P$ and $a * b \in P$, then $a \neq x$ such that $a \in P$. Since, P is a ideal of X. Clearly, that $X - P$ ia a finite $\cap$-structure. From Theorem 3.11(i) obtained P is a prime ideal of X. Hence, there exists a prime ideal P of X, such that $x \notin P$.

Table 4: *Cayley* table for $(X;*,0)$

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	a	0	c
c	c	c	c	0

Example 3.13 *Let $X = \{0, a, b, c\}$. Define the binary operation $*$ of X with following table:*

Then $(X;*,0)$ is B-algebra. Since $c * y = c$, for all $y \in X - \{c\}$, then from Corollary 3.12 obtained $P = X - \{c\}$ is a prime ideal of X, such that $c \notin P$.

Theorem 3.14 *Let A be an ideal of X.*

(i) Let A is a prime ideal of X, then A/A_0 is a prime ideal of X/A_0.

(ii) Let A is a closed prime ideal of X, then A_0 is a closed prime ideal of X/A.

Proof

(i) Let A is an ideal of X, then $A_0 \subseteq A \subseteq X$. Since A_0 is a closed ideal of X, then from Lemma 3.6 (ii) A/A_0 is an ideal of X/A_0. Let P and Q are two ideals of X/A_0, so that $P \cap Q \subseteq A/A_0$. Then there exists S and T be two ideals of X containing A_0. where $P = S/A_0$ and $Q = T/A_0$, so $(S \cap T)/A_0 = S/A_0 \cap T/A_0 = P \cap Q \subseteq A/A_0$. If A be a prime ideal of X obtained $S \cap T \subseteq A$, then $S \subseteq A$ or $T \subseteq A$ and $P \subseteq A/A_0$ or $Q \subseteq A/A_0$. Therefore, A/A_0 is a prime ideal of X/A_0.

(ii) Let A is an ideal of X. If A is closed, then $A = A_0$ and $A/A_0 = A_0$ implies $X/A_0 = X/A$. Since A is a closed prime ideal of X, then from (i) obtained A_0 is a closed prime ideal of X/A.

From definition of prime ideal and irreducible ideal obtained that every prime ideal is an irreducible ideal in B-algebra, but the convers may not true in general. In the following example, we will show that there exists an irreducible ideal which is not prime ideal in B-algebra.

Example 3.15 *Let $X = \{0, 1, 2, a\}$ be a set with the following table:*

Table 5: *Cayley* table for $(X; *, 0)$

$*$	0	1	2	a
0	0	1	2	a
1	1	0	a	2
2	2	a	0	1
a	a	2	1	0

Then $(X; *, 0)$ is a B-algebra and $\{\{0\}, \{0,1\}, \{0,2\}, \{0,a\}\}$ are all proper ideals of X. We obtain $\{0,1\}, \{0,2\}$ and $\{0,a\}$ are irreducible ideals of X. Since $\{0,1\} \cap \{0,2\} \subseteq \{0,a\}$, then $\{0,a\}$ not a prime ideal of X and $\{0\}, \{0,1\}$, and $\{0,2\}$ not also a prime ideal of X. Therefore, X has not any prime ideal. Furthermore, let $A = \{0,1\}$ be an irreducible ideal of X. Since $2, a \in X - A$ and $\langle 2 \cap a \rangle = \{0,2\} \cap \{0,a\} = \{0\}$, then $(\langle 2 \cap a \rangle) \cap (X - A) = \emptyset$. Therefore, $X - A$ not a finite $\cap$-structure.

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